Order of Results in Class

Definitions
Definition: Markov property, and its implications
Computations: Chapman-Kolmogorov property, Matrix formulation, \( n \)-step distribution
Example: Two state Markov chain; asymptotics
Lemma: \( p^{(n)}(x, y) = \sum_{m=1}^{n} P_x(T_y = m)p^{(n-m)}(y, y) \) for \( n \geq 1 \).

Recurrence/Transience
Definition: \( x \rightarrow y \) if \( \rho_{x,y} > 0 \). Set \( \rho_{x,y} \) is the first time \( x \) hits \( y \).
Lemma: \( \rho_{x,y} > 0 \iff \exists k \ s.t. \ p^{(k)}(x, y) > 0 \).
Definition: \( x \) is recurrent if \( \rho_{x,x} = 1 \), and transient otherwise, if \( \rho_{x,x} < 1 \).
Proposition: If \( x \rightarrow y \) and \( y \not\rightarrow y \), then \( x \) is transient.
Computation: Distribution of \( N(y) \), number of returns to \( y \). 
\[ G(x, y) = E_x[N(y)] = \sum_{n=1}^{\infty} p^{(n)}(x, y). \]

Theorem [Characterization in terms of expected number of returns]: Let \( y \) be a transient state. Then \( P_x(N(y) < \infty) = 1 \) and \( G(x, y) = \rho_{x,y}(1 - \rho_{y,y})^{-1} < \infty \) for all \( x, y \in \Sigma \). On the other hand, let \( y \) be a recurrent state. Then, \( P_y(N(y) = \infty) = 1 \) and \( G(y, y) = \infty \). Also, \( P_x(N(y) = \infty) = \rho_{x,y} \). If \( \rho_{x,y} = 0 \), then \( G(x, y) = 0 \), otherwise if \( \rho_{x,y} > 0 \), then \( G(x, y) = \infty \).
Lemma: For a finite state Markov chain, there is at least one recurrent state.
Proposition: Let \( x \) be recurrent, and \( x \rightarrow y \). Then, \( y \) is recurrent and \( \rho_{x,y} = \rho_{y,x} = 1 \).

Decomposition of State space
Definition: a closed set \( C \) is a set of states such that \( \rho_{x,y} = 0 \) if \( x \in C \) and \( y \not\in C \).
Lemma: \( C \) is closed if and only if \( p(x, y) = 0 \) when \( x \in C \) and \( y \not\in C \).
Definition: an irreducible closed set \( C \) is a closed set such that \( x \rightarrow y \) for all choices \( x, y \in C \). An irreducible Markov chain is one where \( x \rightarrow y \) for all \( x, y \in \Sigma \).
Theorem: In an irreducible closed set, either all states are transient or all states are recurrent.
Corollary: In an irreducible closed finite set, all states are recurrent.
Theorem [Decomposition]: \( \Sigma = \Sigma_T \cup \Sigma_R \) where \( \Sigma_T \) are the transient states and \( \Sigma_R \) are the recurrent states. If \( \Sigma_R \neq \emptyset \), then \( \Sigma_R = \bigcup_i C_i \) where \( \{C_i\} \) are disjoint closed irreducible sets of recurrent states.
Example: Birth-Death Markov chains.

Absorption Probabilities
Definition: Let \( u_{i,k} \) be the probability starting from a transient state \( i \) that \( k \) is the first recurrent state hit. That is, when absorbed into \( \Sigma_R \), \( k \) is the first recurrent state hit.
Definition: Let \( s_{i,j} \) be the mean time in transient state \( j \), when starting from transient state \( i \), up to the absorption into \( \Sigma_R \).
Computation: \( U = (I - Q)^{-1}R \) and \( S = (I - Q)^{-1} \)

Stationary distributions
Definition: \( \pi = \)stationary distribution
Lemma: Distribution of $X_n$ is independent of $n \Leftrightarrow$ Initial distribution is a stationary distribution.

Definition: Reversible distribution.

Lemma: A reversible distribution is a stationary distribution.

Example: Birth-Death Markov chains

**Frequency Computation**

Theorem: $N_n(y)/n \rightarrow 1_{T_y<\infty}/E_y[T_y]$ with probability 1. Also, $G_n(x,y)/n \rightarrow \rho_{xy}/E_y[T_y]$.

Corollary: Let $C$ be an irreducible closed set of recurrent states. Then, $G_n(x,y)/n \rightarrow 1/E_y[T_y]$ for $x, y \in C$. Also, if $P(X_0 \in C) = 1$, then $N_n(y)/n \rightarrow 1/E_y[T_y]$ for $y \in C$ with probability 1.

**Positive and Null Recurrence**

Definition: Positive and Null recurrence.

Theorem: If $y$ is transient or null recurrent, then $G_n(y,y)/n \rightarrow 0$. But, if $y$ is positive recurrent, then $G_n(y,y)/n \rightarrow 1/E_y[T_y] > 0$.

Theorem: If $x$ is positive recurrent and $x \rightarrow y$, then $y$ is positive recurrent.

Corollary: An irreducible chain is all transient, all null recurrent or all positive recurrent.

Theorem: If $C$ is a finite closed set of states, then there is at least one positive recurrent state.

**Transience, Null and Positive Recurrence and Stationary Distributions**

Theorem: Let $\pi$ be a stationary distribution. Then $\pi(x) = 0$ for $x$ which is transient or null recurrent.

Corollary: A stationary distribution $\pi$ is supported on positive recurrent states. Or, stationary distributions do not exist for chains without any positive recurrent states.

**Frequency Interpretation of Stationary Distributions**

Theorem: An irreducible positive recurrent Markov chain has a unique stationary distribution given by $\pi(x) = 1/E_x[T_x]$.

Corollary: An irreducible chain is positive recurrent $\Leftrightarrow$ it has a stationary distribution.

Corollary: An irreducible finite state space chain has a unique stationary distribution.

Corollary: For an irreducible positive recurrent chain, $N_n(x)/n \rightarrow \pi(x)(= 1/E_x[T_x])$ with probability 1.

Example: Birth-Death Markov chains

**Reducibility**

Theorem: Let $\Sigma_p$ be the positive recurrent states. Then, (1) if $\Sigma_p = \emptyset$, there is no stationary distribution. (2) If $\Sigma_p$ is non-empty and irreducible, then there exists a unique stationary distribution. (3) If $\Sigma_p$ is non-empty but reducible, then there exists an infinite number of stationary distributions.
Periodicity and Convergence

Theorem: For an irreducible, positive recurrent and aperiodic chain, \( p^{(n)}(x, y) \to \pi(y) \). Also, for an irreducible positive recurrent chain with period \( d > 1 \), we have, for \( x \) and \( y \), that there is \( 0 \leq r < d \) such that \( p^{(n)}(x, y) = 0 \) unless \( n \) is of the form \( n = md + r \) for \( m \geq 0 \), and then \( \lim_{m \to \infty} p^{(md+r)}(x, y) = d\pi(y) \).

Examples: Two state-chain, etc.