HW 3 Math 563 Due Wednesday, Oct 2, 2019

1. (Durrett 2.1.18) If we want an infinite sequence of coin tossings, we do not have to use Kolmogorov's theorem. Let Ω be the unit interval $(0,1)$ equipped with the Borel sets $\mathcal{F}$ and Lebesgue measure P. Let $Y_n(\omega) = 1$ if $[2^n\omega]$ is odd and 0 if $[2^n\omega]$ is even. Show that $Y_1, Y_2, \ldots$ are independent with $P(Y_k = 0) = P(Y_k = 1) = 1/2$.

2. (Durrett 2.2.3 I've modified the assumption on f here) "Monte Carlo Integration" Let f be a measurable function on $[0,1]$ with $\int_0^1 |f(x)|^2 dx < \infty$. Let $U_1, U_2, \ldots$ be independent and uniformly distributed r.v.'s on $[0,1]$. Let

$$I_n = \frac{1}{n}\sum_{i=1}^{n} f(U_i).$$

(i) Show that $I_n \to I \equiv \int_0^1 f(x)dx$ in probability.
(ii) Use Chebychev's inequality to estimate $P(|I_n - I| > a/n^{1/2})$.

3. (Durrett 2.3.1) Prove that

$$P(\liminf A_n) \le \liminf P(A_n) \le \limsup P(A_n) \le P(\limsup A_n).$$

4. Let $\{X_n : n \ge 1\}$ be identically distributed.
(i) Show that $X_n/n \to 0$ in probability.
(ii) If $E|X_1| < \infty$, then $X_n/n \to 0$ a.s.
(iii) If $\{X_n : n \ge 1\}$ are independent, and $X_n/n \to 0$ a.s., then $E|X_1| < \infty$.
Therefore, we conclude that when $\{X_n : n \ge 1\}$ are iid we have that

$$\frac{X_n}{n} \to 0 \text{ a.s.} \quad \Leftrightarrow \quad E|X_1| < \infty.$$

We may conclude further that if $\{X_n : n \ge 1\}$ are iid and $E|X_1| = \infty$, then we have an example that $X_n/n \to 0$ in probability but not almost surely.