HW 7 Math 563 Due Monday, Nov. 25, 2019

1. Let $X \sim U[0,1]$, and η be a r.v. given by

$$\eta = \begin{cases} X & X < 1/2 \\ 1/2 & X \geq 1/2. \end{cases}$$

Find $E[X|\eta]$.

2. Let (X,Y) be a real random vector with joint density $e^{-y}1(0 < x < y)$. Compute $E[Y|X]$, $\mathrm{Var}(Y|X)$ and $\mathrm{Var}(Y)$. More generally, in a probability space $(\Omega, \mathcal{B}, P)$, where X is a random variable (with $E[X^2] < \infty$) and $\mathcal{F}$ a sub-sigma field, show that

$$\mathrm{Var}(X) = E[\mathrm{Var}(X|\mathcal{F})] + \mathrm{Var}(E[X|\mathcal{F}]).$$

(see Durrett 5.1.9)

3. Let $\{Y_n\}$ be independent, positive r.v.'s with mean $E[Y_n] = 1$ for all $n \geq 1$. Let $X_n = \prod_{i=1}^n Y_i$.

(a) Show that $\{X_n\}$ is a martingale with respect to $\mathcal{F}_n = \sigma(Y_1, \ldots, Y_n)$.

(b) Show that X_n converges a.s. to a r.v. X where $E[X] < \infty$.

(c) Suppose specifically now that $\{Y_n\}$ are iid with $P(Y_n = 1/2) = P(Y_n = 3/2) = 1/2$. Identify the distribution of X.