HW 8 Math 563
Due Wednesday, December 11, 2019. [We can discuss solutions on this day]

1. (Durrett 4.1.4 and 4.1.6) Suppose S and T are stopping times.
(a) Is $S+T$ a stopping time?
(b) if $S \leq T$, show that $\mathcal{F}_S \subset \mathcal{F}_T$.

2. [optional as we may not get to this section, but one might try for an intuitive argument] (Durrett 5.5.2) Let $Z_1, Z_2, \ldots$ be iid with $E[|Z_1|] < \infty$. Let θ be an independent r.v. with finite mean and $Y_i = Z_i + \theta$, e.g. adding some noise each term. If Z_i are $N(0,1)$, show that $E[\theta|Y_1, \ldots, Y_n] \to \theta$ a.s. [The distribution of θ is the prior distribution and that of θ given $Y_1, \ldots, Y_n$ is the posterior distribution.

3. A population consists of two types of atoms. Those who give rise to 3 atoms (type A), and those who do no propagate (type B). When an atom is born, the chance it is type A is 1/2. Starting from a type A atom, what is the chance the atom's family dies out?

4. Let $f : \mathbb{Z} \to \mathbb{R}$ be a bounded function on the integers. Suppose, for all $x \in \mathbb{Z}$, we have

$$f(x) \;=\; \frac{1}{2}(f(x-1) + f(x+1)).$$

(a) Consider fair RW, $S_n = x_0$ and $S_n = \sum_{i=1}^n \xi_i$ for $n \geq 1$ where $\{\xi_i\}$ are iid with $P(\xi_1 = 1) = P(\xi_1 = -1) = 1/2$. Show that $\{f(S_n)\}$ is a martingale with respect to $\mathcal{F}_n = \sigma(\xi_1, \ldots, \xi_n)$.
(b) Let $T_y = \inf\{n \geq 1 : S_n = y\}$ be the hitting time of $y \in \mathbb{Z}$. It has been argued in class that $T_y < \infty$. Show now that $f(x_0) = f(y)$, and therefore f is constant! [This means bounded, harmonic (w.r.t. fair RW) functions on $\mathbb{Z}$ are constant]

5. (Polya's Urn). An urn contains b black and r red balls. A ball is drawn at random. It is then replaced and moreover c balls of the color drawn are added. Let $X_0 = b/(b+r)$, and let X_n be the proportion of black balls attained at stage n, that is just after the nth draw and replacement. Show that $\{X_n\}$ with respect to the natural σ-fields is a martingale. Explain why it converges a.s.

[This is the classic example of 'reinforcement'; if you have time, one might simulate to see what the limit distribution looks like for some choices of b, r, c (e.g. $b = r = c = 1$)–we already know $X_n(\omega) \to X_\infty(\omega)$ a.s., something much stronger than weak convergence. See also section 5.3.2 Durrett for some calculations, and an answer to what the limit distribution is.]