# On Professor S. R. S. Varadhan's contributions to hydrodynamic limits

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One of the principal problems in statistical physics is to connect the dynamics of fluid on the microscopic scale to that on the macroscopic scale: How do we derive large-scale conservation laws, in other words the Euler equations, say, from individual particle considerations given by classical mechanics? The mathematical difficulty stems in part from making certain physical approximations precise. Granted that a very large system of differential equations can be conceived to govern the motion, but this large system has proved intractable to analysis. So, what can be done? The guiding idea of recent years has been not to solve approximately this exact problem, but to try to solve exactly an approximate tractable problem. One way to go about this is to introduce a little randomness to the microscopic particle motion, making the system a true stochastic process. If enough noise is incorporated, the system evolution can be made to depend on this noise so that the need for an eternal memory is removed which was a prime difficulty for the original deterministic problem. In this approximate stochastic system, one recovers the macroscopic law by averaging out the randomness in a suitable space–time scaling limit. Briefly, the intuition is that particles move and interact quite rapidly on the microscopic scale but only a small fraction of this interaction acts locally to contribute to the macroscopic flow. Therefore, by linking the space and time variables in terms of a scale parameter and then taking limits on this parameter, one passes from the microscopic to the macroscopic. These limits, called 'hydrodynamic limits', of course need to be explained and justified, and this is the crux of the problem.

The history of the scaling approach goes back to Boltzmann and Maxwell at the time when the atomic structure of matter was being formulated. In the context of the Boltzmann equation, these hydrodynamic limits have been studied in part by Grad (1958), Kac (1959) and Lanford (1975) to some success. With respect to Euler equations, such limits were initiated by Morrey (1951) followed in part by work by Rost (1981) and others in the context of specific one-dimensional models. We refer to De Masi and Presutti[1] and Spohn[2] for exhaustive surveys.

The seminal contribution of Prof. Varadhan and his colleagues M. Guo and G. Papanicolaou to the subject of hydrodynamic limits is the development in 1988 of a robust method which allows for the rigorous derivation of macroscopic conservation laws in a wide variety of $d \geq 1$ models[3]. Although the passage to Euler equations from the microscopic equations of classical mechanics is still open, Varadhan's work[3,4] is a tour de force in that it provides a robust framework upon which many of the difficulties, such as shock formation, are now better understood; in fact, in ref. 5 the Euler equations are rigorously derived from noisy microscopic relations in the smooth regime. At this point, we remark that many other avenues of research have opened up as a result of this framework, such as large deviation and fluctuation estimates, which have not been understood even in the physics literature.

We now describe more carefully Varadhan's contributions with respect to a basic model. We follow the presentation given in ref. 6. Let $-\infty < x < \infty$ be the basic variable representing, say, charge. Let the size of the system be given by a large integer $N$ which will be the scaling parameter and tends to $\infty$. For each $N$, we have $N$ charges $x_1, x_2, \ldots, x_N$ located at sites $1, 2, \ldots, N$. These sites are placed at a distance $1/N$ apart on the unit interval with end points identified. Denote by $S$ the unit interval with end points identified and let $\theta$ be its typical point. The charges are dynamic in time and we specify the configuration at time $t$, $0 \leq t \leq T$, by $\{x_1(t), x_2(t), \ldots, x_N(t)\}$. The way the charge moves is given by the equations

$$dx_i(t) = dz_{i-1,i} - dz_{i,i+1}.$$

The term $dz_{i,i+1}$ is thought of as the flow of charge from site $i$ to site $i+1$ along the 'bond' which connects them. With this understanding, the equations above represent the conservation law that the net change in charge at any site is the difference between the inflow and outflow at that site. Clearly, the total charge $\Sigma_{i=1}^{N} x_i(t)$ is conserved. We now must describe the law governing the flow of charge along a bond. For a suitable function $\psi$, define

$$dz_{i,i+1} = [\psi(x_i(t)) - \psi(x_{i+1}(t))]\, dt + d\beta_{i,i+1}(t),$$

where the $\beta_{i,i+1}(t)$ are $N$ independent Bronian motions. As an example, note that if $\psi(x) = cx$, for some $c > 0$, then system motion tends to equalize charge at the different sites. With the definition above, the charge configuration evolves as a diffusion on $R^N$ with infinitesimal generator given by

$$\frac{1}{2}\sum_{i=1}^{N}\left(\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_{i+1}}\right)^2 + \sum_{i=1}^{N}[\psi(x_{i-1}) - 2\psi(x_i) + \psi(x_{i+1})]\frac{\partial}{\partial x_i}.$$

Since the space has been scaled by a factor $1/N$, following standard intuition regarding diffusions, we should speed up time by a factor $N^2$, and we can redefine the generator as

$$L_N = \frac{N^2}{2}\sum_{i=1}^{N}\left(\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_{i+1}}\right)^2 + N^2\sum_{i=1}^{N}[\psi(x_{i-1}) - 2\psi(x_i) + \psi(x_{i+1})]\frac{\partial}{\partial x_i}.$$

Suppose now that the initial charge configuration is randomly distributed. For example, choose a smooth function or 'profile' $a_0(\theta)$ defined on $S$, and define $a_i = a_0(i/N)$. We may take the initial configuration $\{x_1, x_2, \ldots, x_N\}$ to be independent random variables with $x_i$ uniformly distributed in the interval $[a_i - \frac{1}{2}, a_i + \frac{1}{2}]$. We now carry out Varadhan's calculations in mean-value to give the main flavour. Let $J(\theta)$ be a smooth function on $S$ and note that

$$\lim_{N\to\infty} E\left[\frac{1}{N}\sum_{i=1}^{N} x_i J\left(\frac{i}{N}\right)\right] = \lim_{N\to\infty} \frac{1}{N}\sum_{i=1}^{N} a_i J\left(\frac{i}{N}\right)$$

$$= \lim_{N\to\infty} \sum_{i=1}^{N} a_0\left(\frac{i}{N}\right) J\left(\frac{i}{N}\right)$$

$$= \int_s a_0(\theta) J(\theta)\, d\theta.$$

This gives $a_0(\theta)$ as the profile of the macroscopic charge density initially.

To compute the profile, $a(t, \theta)$, at time $t$, we must use the infinitesimal generator $L_N$:

$$\frac{\partial}{\partial t}\int a(t,\theta)\, d\theta = \lim_{N\to\infty} \frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{N} E\left[\sum_{i=1}^{N} J\left(\frac{i}{N}\right) x_i(t)\right]$$

$$= N^2 E\left[\frac{1}{N}\sum_{i=1}^{N} \{\psi(x_{i-1}(t)) - 2\psi(x_i(t)) + \psi(x_{i+1}(t))\} J\left(\frac{i}{N}\right)\right]$$

$$= N^2 E\left[\sum_{i=1}^{N}\left\{J\left(\frac{i-1}{N}\right) - 2J\left(\frac{i}{N}\right) + J\left(\frac{i+1}{N}\right)\right\}\psi(x_i(t))\right]$$

$$\sim E\left[\frac{1}{N}\sum_{i=1}^{N} J''\left(\frac{i}{N}\right)\psi(x_i(t))\right].$$

At this point, unfortunately, one cannot apply the fact that

$$\frac{1}{N}\sum_{i=1}^{N} J\left(\frac{i}{N}\right) x_i(t) \to \int_s J(\theta) a(t,\theta)\, d\theta$$

to compute the limit of

$$\frac{1}{N}\sum_{i=1}^{N} J''\left(\frac{i}{N}\right)\psi(x_i(t)).$$

As $\psi$ is generally nonlinear, one must understand the whole charge distribution $\{x_1(t), x_2(t), \ldots, x_N(t)\}$. However, one has the intuition from ergodic theory that as $\psi(x_i(t))$ fluctuates rapidly we can replace it by its average value for a given macroscopic charge density. If we call this average value, for a given charge density $a$, as $\lambda(a)$, the exact statement is the following:

$$\lim_{\varepsilon\to 0} \limsup_{N\to\infty} E\left[\int_0^T \left|\frac{1}{N}\sum_{i=1}^{N} J''\left(\frac{i}{N}\right)\times \psi(x_i(t)) - \frac{1}{N}\sum_{i=1}^{N} J''\left(\frac{i}{N}\right)\times \lambda\left(\frac{x_{i-N\varepsilon}(s)+\cdots+x_{i+N\varepsilon}(s)}{1+2N\varepsilon}\right)\right| ds\right] = 0.$$

If this can be accomplished for some function $\lambda(\cdot)$, then it follows that the profile $a(t, \theta)$ at time $t$ given by

$$\frac{d}{dt}\int J(\theta) a(t,\theta)\, d\theta = \int J''(\theta)\lambda(a(t,\theta))\, d\theta,$$

or

$$\frac{\partial}{\partial t} a(t,\theta) = \frac{\partial^2}{\partial \theta^2}\lambda(a(t,\theta)),$$

with $a(0, \theta) = a_0(\theta)$.

The problem then reduces to identifying $\lambda(\cdot)$ and proving the limit above. When the current $\psi$ satisfies certain regularity conditions, say a 'gradient' condition,

$$\psi(x) = \frac{1}{2}\frac{d\phi}{dx}$$

for some $\phi$ such that $M(\sigma) = \int e^{\sigma x - \phi(x)}\, dx < \infty$ for all $\sigma$, then these two things can be done with supple and deep ideas. In fact, Varadhan computes, from some large deviation ideas, that

$$\lambda(a) = \tfrac{1}{2} h'(a),$$

where $h(a) = \sup_\sigma\{a\sigma - \rho(\sigma)\}$ and $\rho(\sigma) = \log M(\sigma)$. We refer to refs 6 and 3 for further details, and ref. 4 for significant extensions to 'non-gradient' $\psi$'s.


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