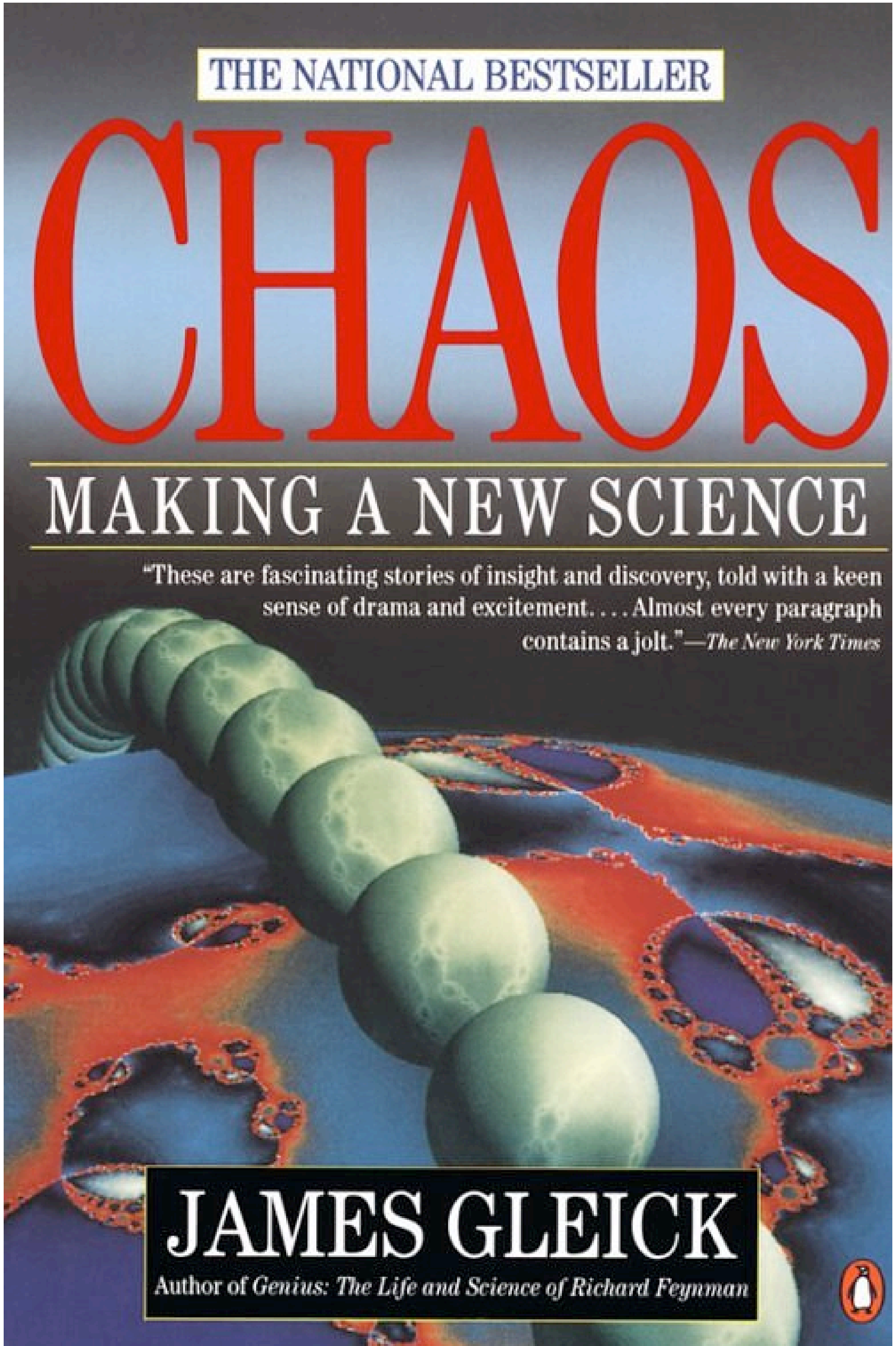
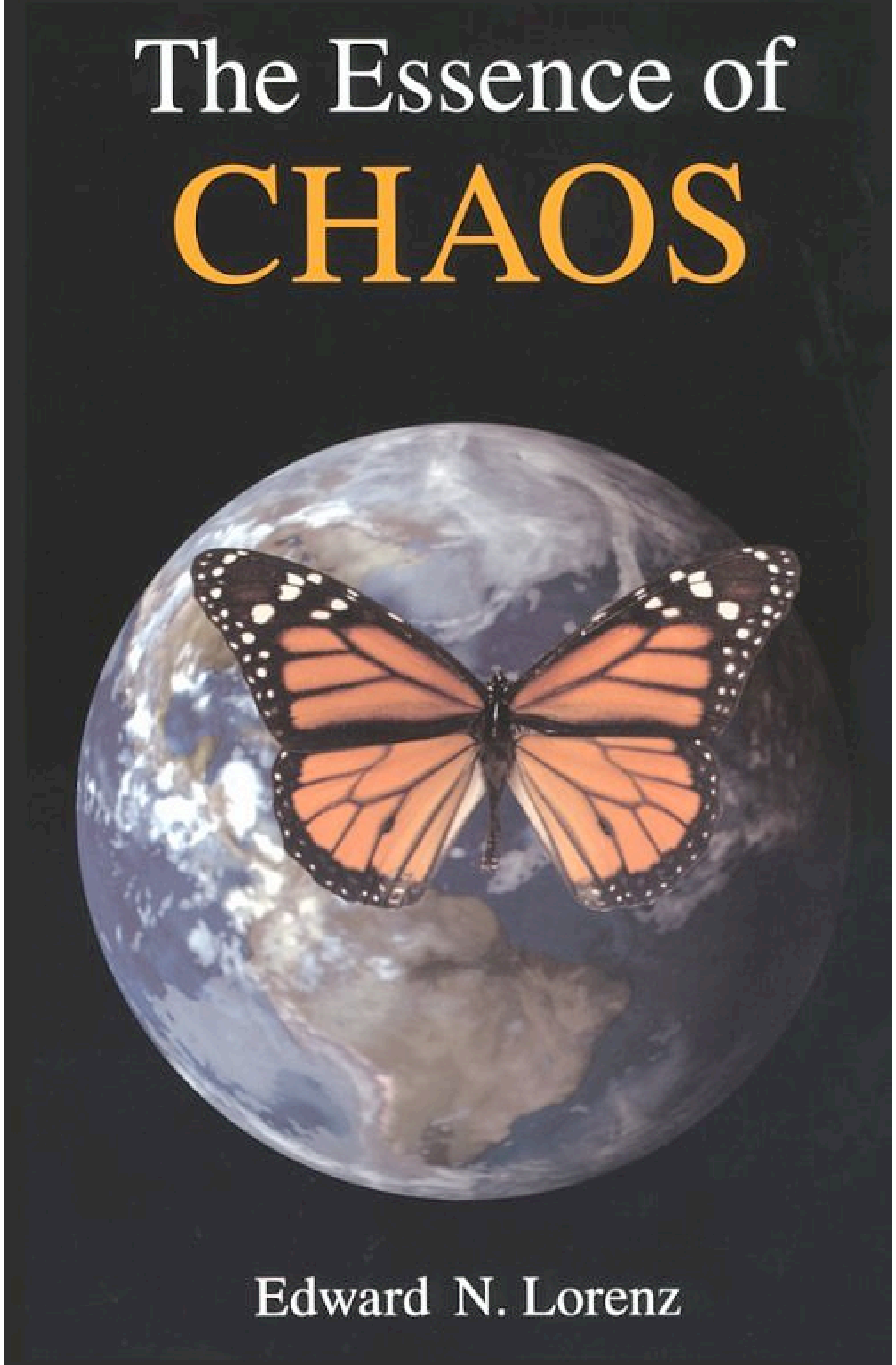
Lecture 1

What is Chaos?



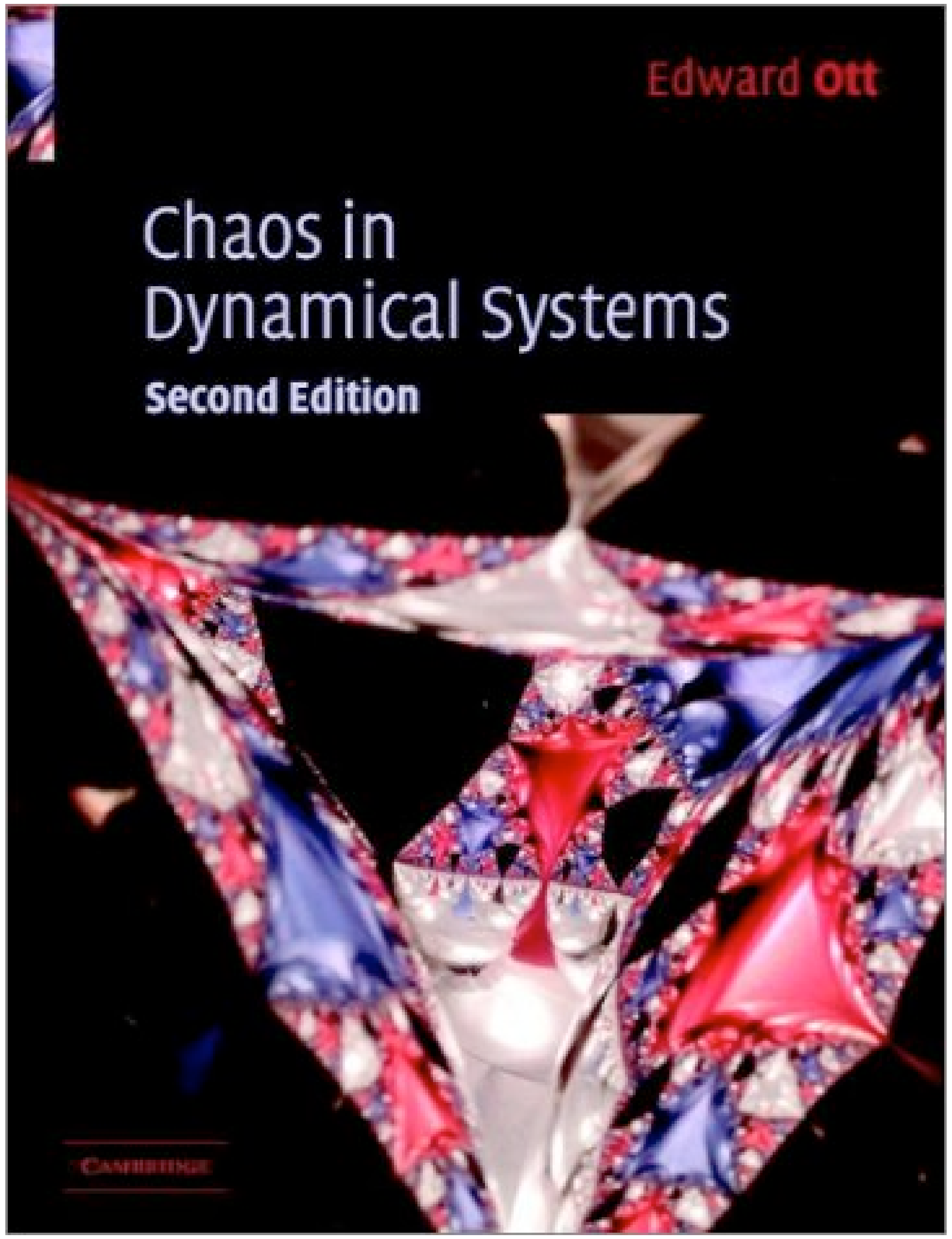
Chaos

James Gleick



Edward Lorenz

The “discoverer” of Chaos



Chaos in
Dynamical Systems

Edward Ott

The Scientific Method

- Observe natural phenomena and classify them.
- Deduce regularities and rules from the observations.
- Use the deductions to make predictions.
- Compare the predictions with reality.

The role of Mathematics in the Physical Sciences

Mathematics is the language of the Physical sciences.

- Numbers are needed for observation of many natural phenomena.
- The rules that are deduced from the observations are often expressed as mathematical equations.
- These equations enable us to make precise predictions that can be compared with reality.

" The unreasonable effectiveness of Mathematics in the Natural sciences"

- Eugene Wigner

"Mathematics allows us to replace words by exact outcomes which we can examine dispassionately."

- Leo Kadanoff

Systems

A system is something of interest that we are trying to describe or predict.

Examples of systems

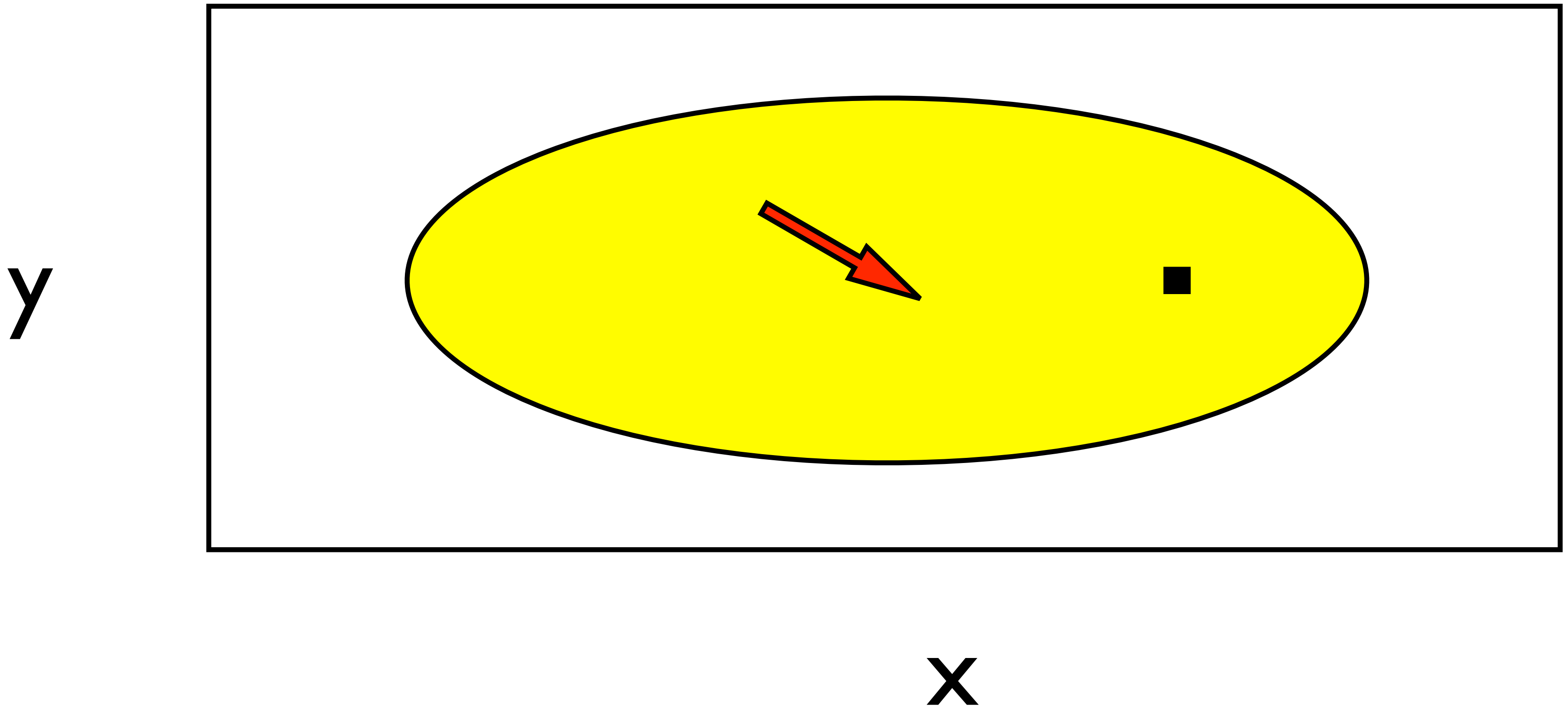
- A wristwatch.
- The Solar system.
- An ecosystem, *e.g.* Yellowstone National Park.
- The Stock market.
- The weather in Alpbach.

Phase Space

The Phase space : An abstraction where the system is replaced by a representation of the space of the possible states the system can be in.

The phase space is a way to use *numbers* (Mathematical variables) to represent the *state* of the system.

The number of variables needed to represent the system is called its dimensionality.



Examples of Phase space

Your bank account :

Your Bank Account can be represented by two variables, C, the Checking account balance and S, the Savings account balance. The Phase space in this case is the set of two numbers (C, S).

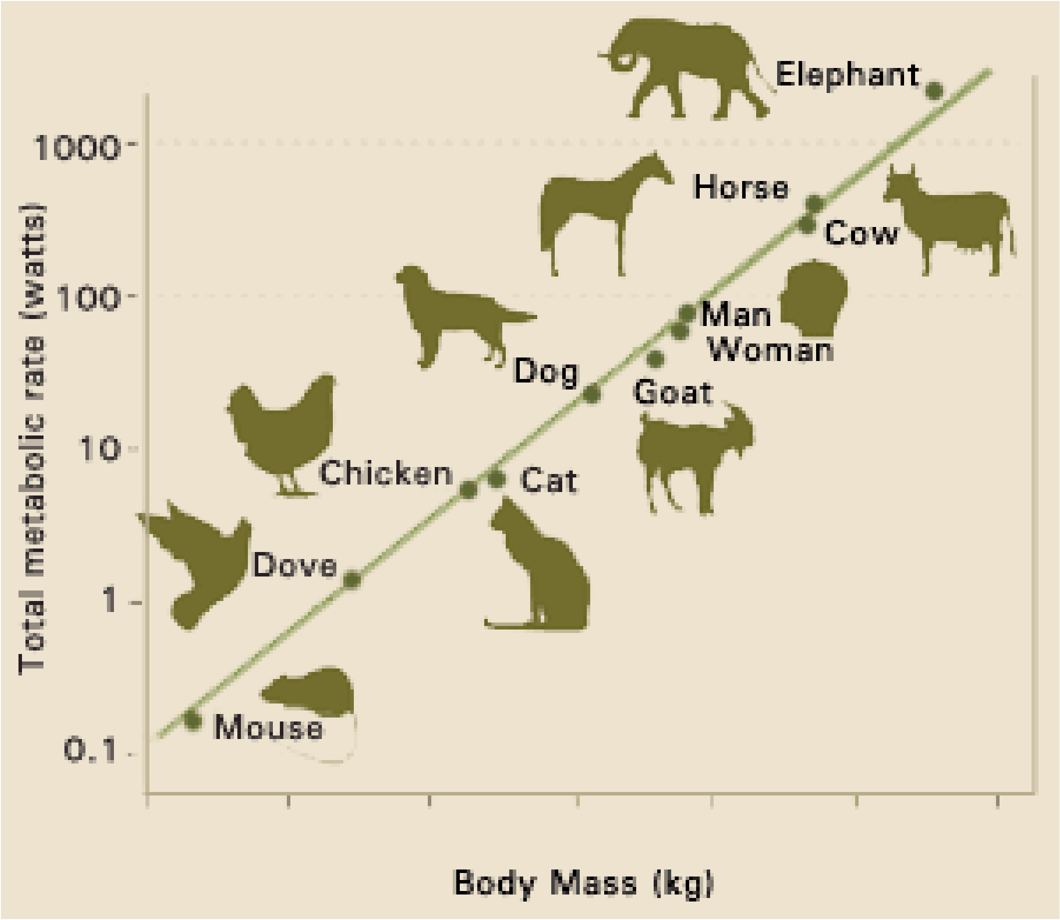
A simple pendulum :

The state of a simple pendulum can be represented by two variables, θ, the angle between the pendulum and the vertical, and v, the velocity of the pendulum. The phase space is again two dimensional, but it is *now the surface of a cylinder*.

Graphs

Graphs are a visual way of representing the information in mathematical equations.

We will also use pictures to represent phase space.



Dynamical Systems

The world is not static and systems of interest change with time - Dynamics.

As a system changes, the numbers representing the state of the system in the phase space also change.

A Dynamical System is the phase space along with the rules governing how the numbers representing the state evolve.

The path traced out in the phase space by the evolution is called an orbit.

For a system to be a dynamical system by the above definition, we need that the future state of the system should be completely determined by the current state of the system.

Maps

Systems can change at discrete times, for example many insects have a life cycle of a year, so that to study the population of these insects, we only need to look at the system once every year.

A discrete time dynamical system is also called a Map.

The dynamics is then given by a list of numbers. For example

$$x_0 = 125, x_1 = 250, x_2 = 500, x_3 = 1000, \ldots$$

x_n represents the state variable x at the nth time instant.

A map is then given by

$$x_{n+1} = F(x_n)$$

where $F(x_n)$ is the mathematical rule (function) governing the evolution of the system.

Equations of this form are called Difference Equations.

Flows

Time is continuous and the system evolves continuously.

A continuous time dynamical system is called a Flow.

The study of these systems requires *Calculus*.

Leibniz and Newton.

Let $x(t)$ represent a generic state variable x at the time t.

The flow is given by

$$\frac{dx(t)}{dt} = F(x(t))$$

where $F(x(t))$ is the function governing the evolution of the system.

Equations of this form are called Ordinary Differential Equations.

Linear Systems

A linear process is one in which, if a change in any variable at some initial time produces a change in the same or some other variable at some later time, twice as large a change at the same initial time will produce twice as large a change at the same latter time. You can substitute "half" or "five times" or "a hundred times" for "twice" and the description remains valid.

- Edward Lorenz in The Essence of Chaos.

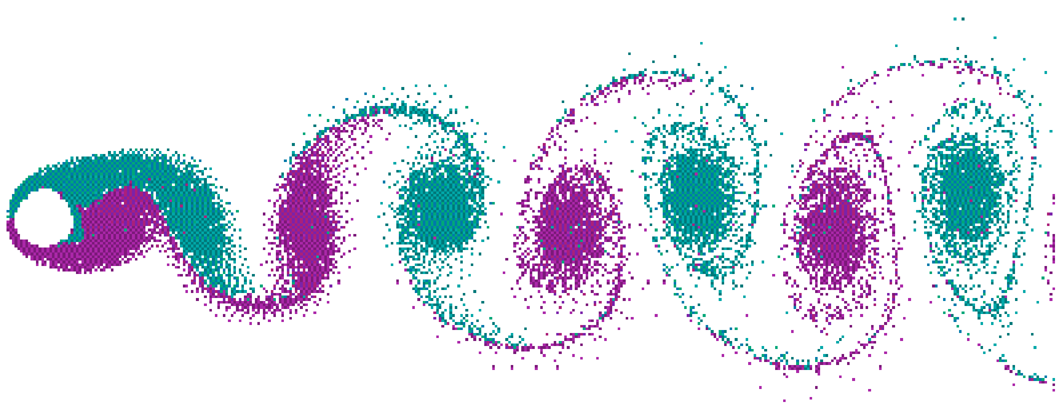
A Linear system is a dynamical system whose evolution is a linear process.

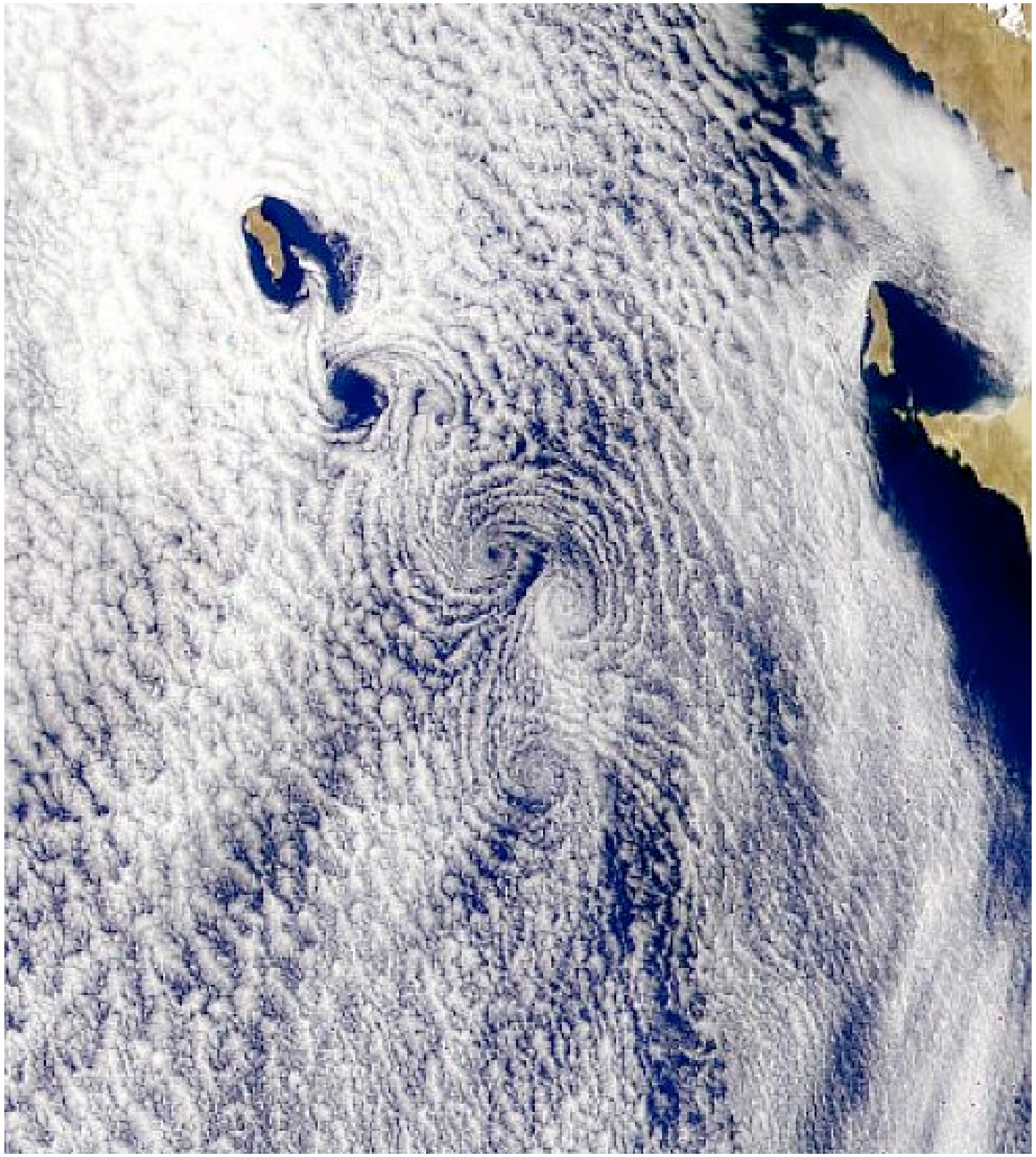
Nonlinear systems

All systems that are not linear are called Nonlinear systems. In these systems, the change in a variable at an initial time can lead to a change in the same or a different variable at a later time, that is not proportional to the change at the initial time.

Examples:

- Fluid flows.





Vortex street
in the atmosphere

Guadalupe Island,
Aug 20, 1999

Image Courtesy
NASA Goddard

Shetland Islands

Falkland Islands

Images Courtesy NASA Goddard

Linear vs. Non-linear systems

For a linear system, we can combine two solutions, and the result is also a solution for the system. This is not true for nonlinear systems.

The above property is called linearity and it makes the linear systems mathematically tractable.

We can break up a linear problem into little pieces, solve each piece separately and put them back together to make the complete solution.

Nonlinear systems on the other hand cannot be broken up into little pieces and solved separately. They have to be dealt with in their full complexity.

Nonlinear Science

The study of nonlinear dynamical systems is called nonlinear science.

Nature is intrinsically nonlinear and nonlinearity is the rule rather than the exception.

"It does not say in the Bible that all laws of nature are expressible linearly."

- Enrico Fermi.

"Using a term like nonlinear science is like referring to the bulk of zoology as the study of non-elephant animals."

- Stanislaw Ulam.

If linear systems are an exception, why were they studied for such a long time ?

- All linear problems are “solvable”.
- Many nonlinear systems of interest are approximately linear for small perturbations about points of equilibrium.
- Nonlinear problems are seldom exactly solvable. Before the advent of computers, almost nothing could be said about the behavior of nonlinear systems.

What do we now know about nonlinear systems?

- They are ubiquitous.
- The behavior of nonlinear systems can differ qualitatively from the behavior of linear systems and one cannot use the solutions of linear equations as a guide to understand the behavior of many real world systems.
- Nonlinear systems can display a variety of behaviors including chaos. This has profound consequences in all of the sciences. It has also altered our view on the principle of scientific determinism.

For all these reasons, the study of nonlinear systems is now at the forefront of research in many disciplines including

- Mathematics
- Biology
- Physics
- Chemistry
- Meteorology
- Economics
- Computer Science

What is Chaos?

chaos (k̀eios) *n.* 1(Usu. cap.) The disordered formless matter supposed to have existed before the ordered universe.

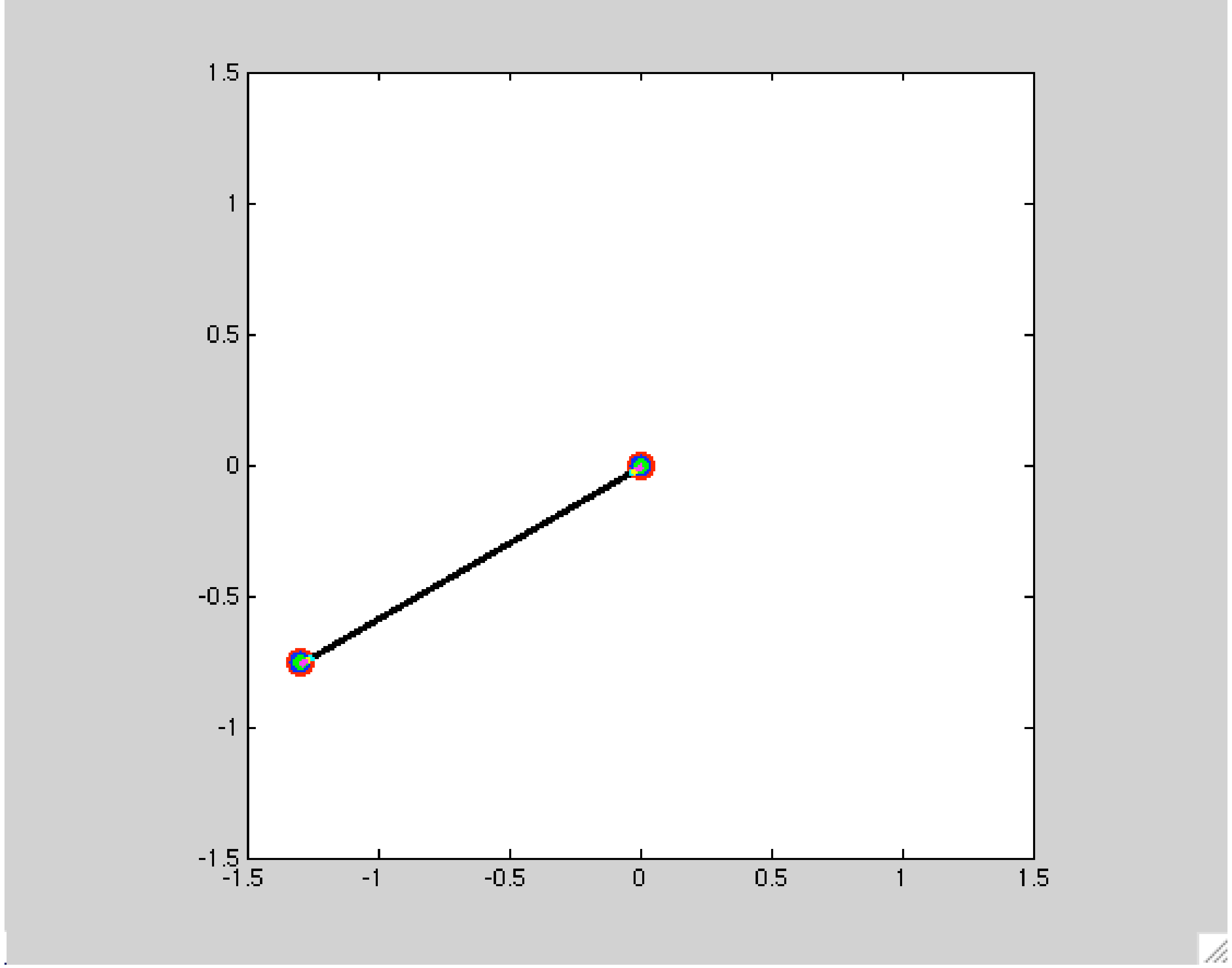
2. Complete disorder, utter confusion.

3.(Math.) Stochastic behavior occurring in a deterministic system.

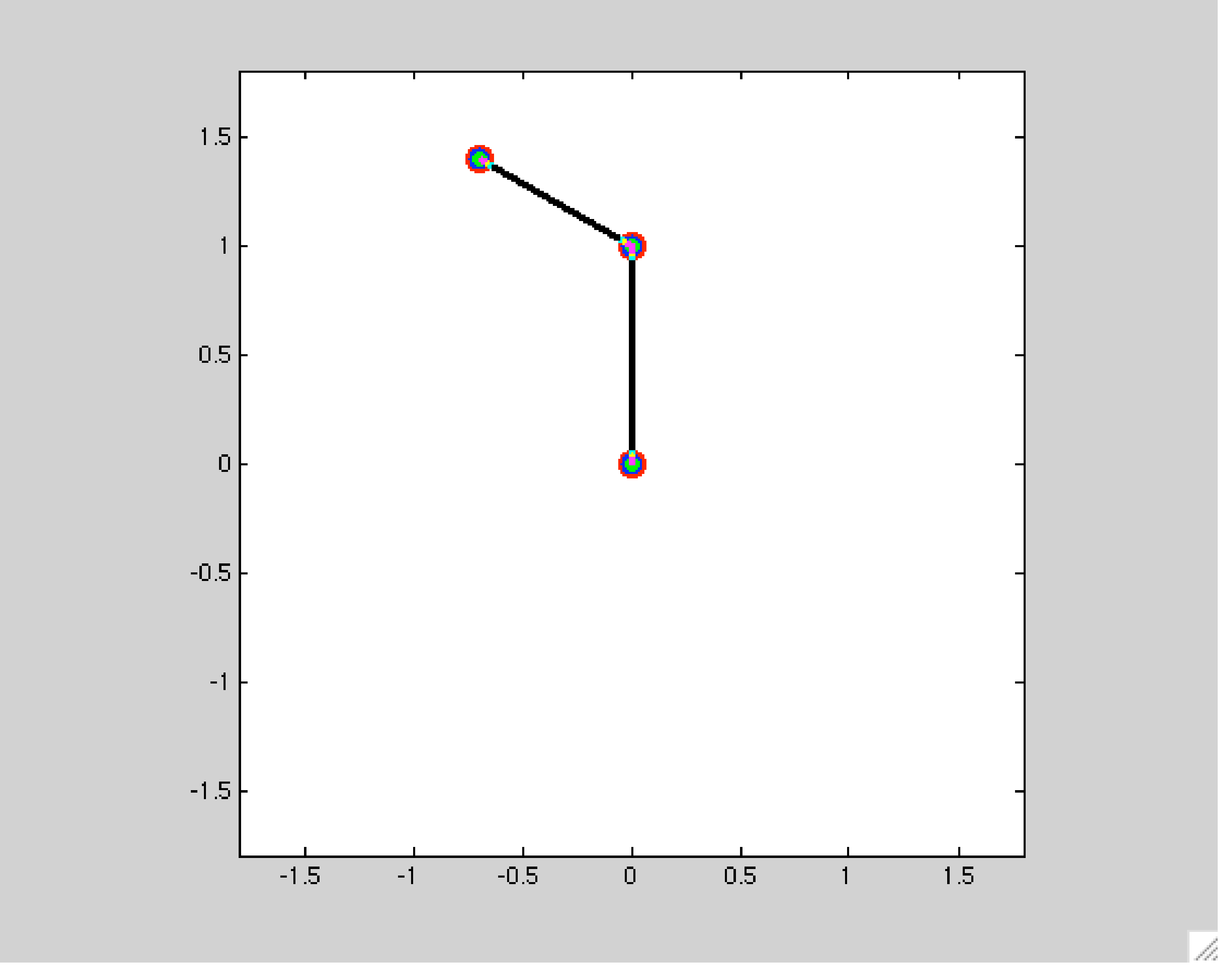
Stochastic = Random.

Deterministic System = A system which is governed by exact rules with no element of chance.

How can a system with no element of chance be random?



A simple pendulum



A double pendulum

Logistic Map

A simple model for population growth (Malthus)

$$x_{n+1} = rx_n.$$

Make it nonlinear by letting the growth rate r depend on x.

We want the growth rate to decrease as x increases.

Choose $r(x) = r(1 - x)$.

This gives the Logisitc Map:

$$x_{n+1} = rx_n(1 - x_n).$$

Insects on an Island

There is a species of insects on an island with a fertility r, that is every insect produces an average of r offspring in a life-cycle.

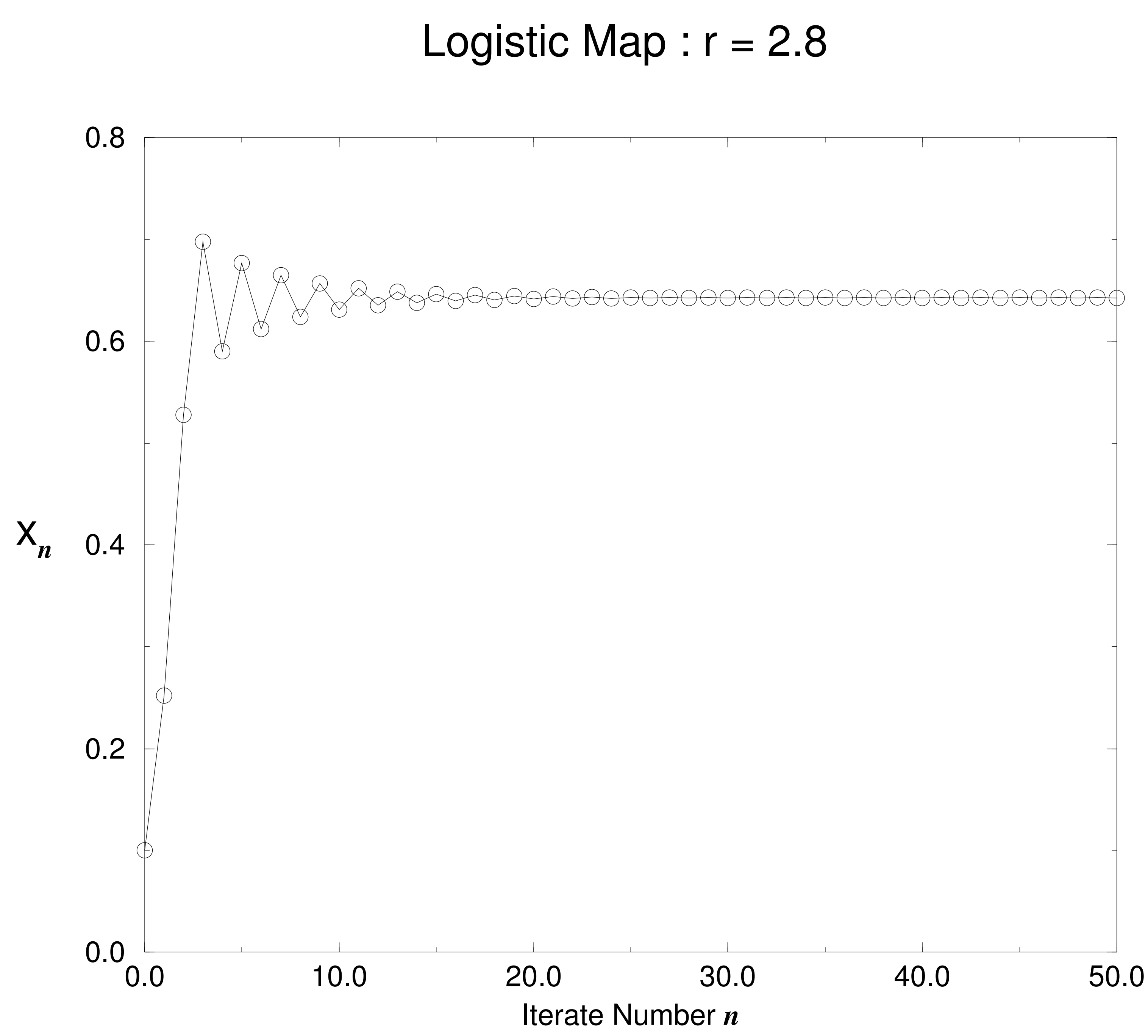
There is a fixed amount of food on the island so that if the population of insects gets large, the average fertility decreases.

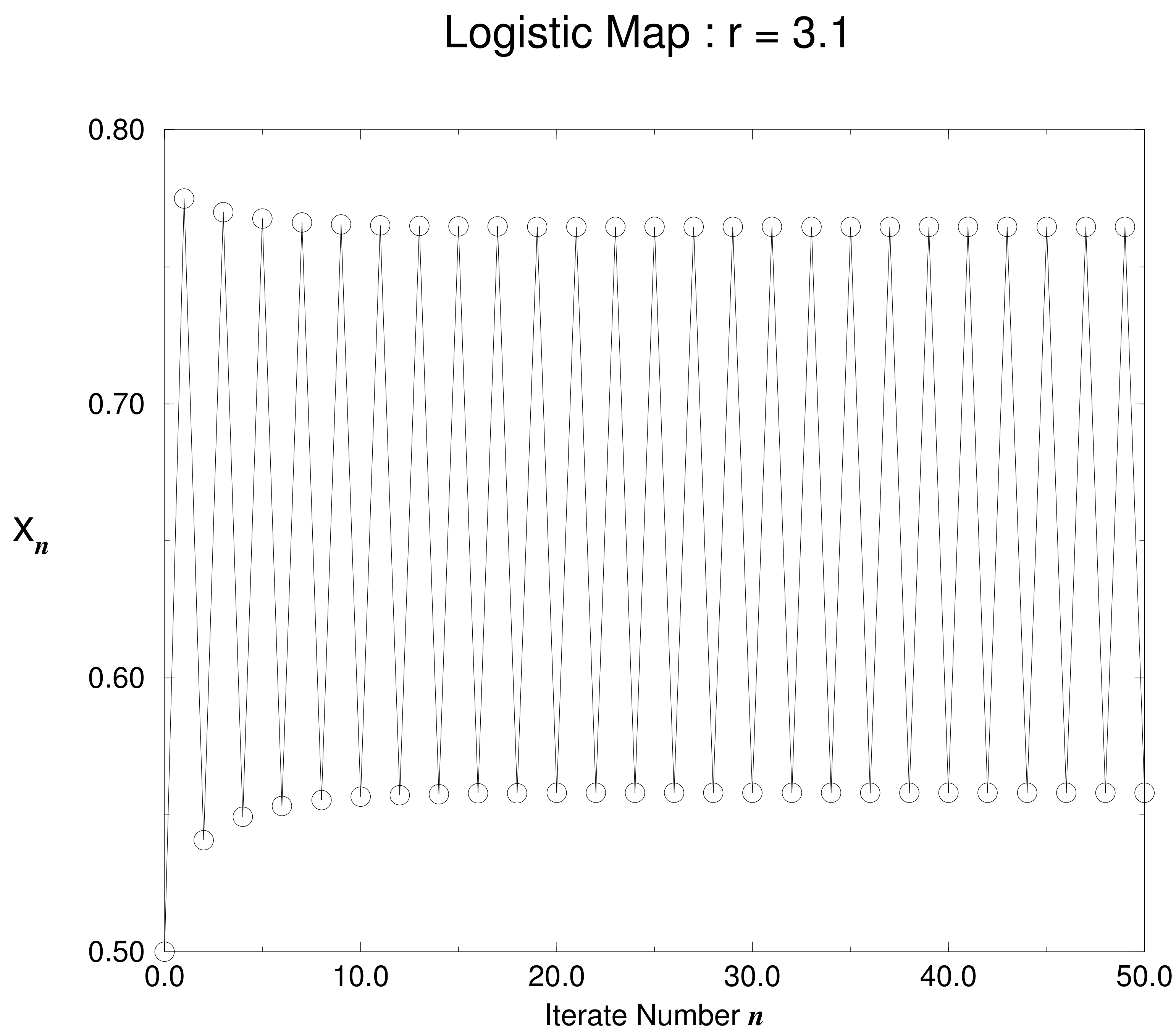
This leads to the Logistic Map

$$x_{n+1} = rx_n(1 - x_n).$$

Robert May (a Biologist.)

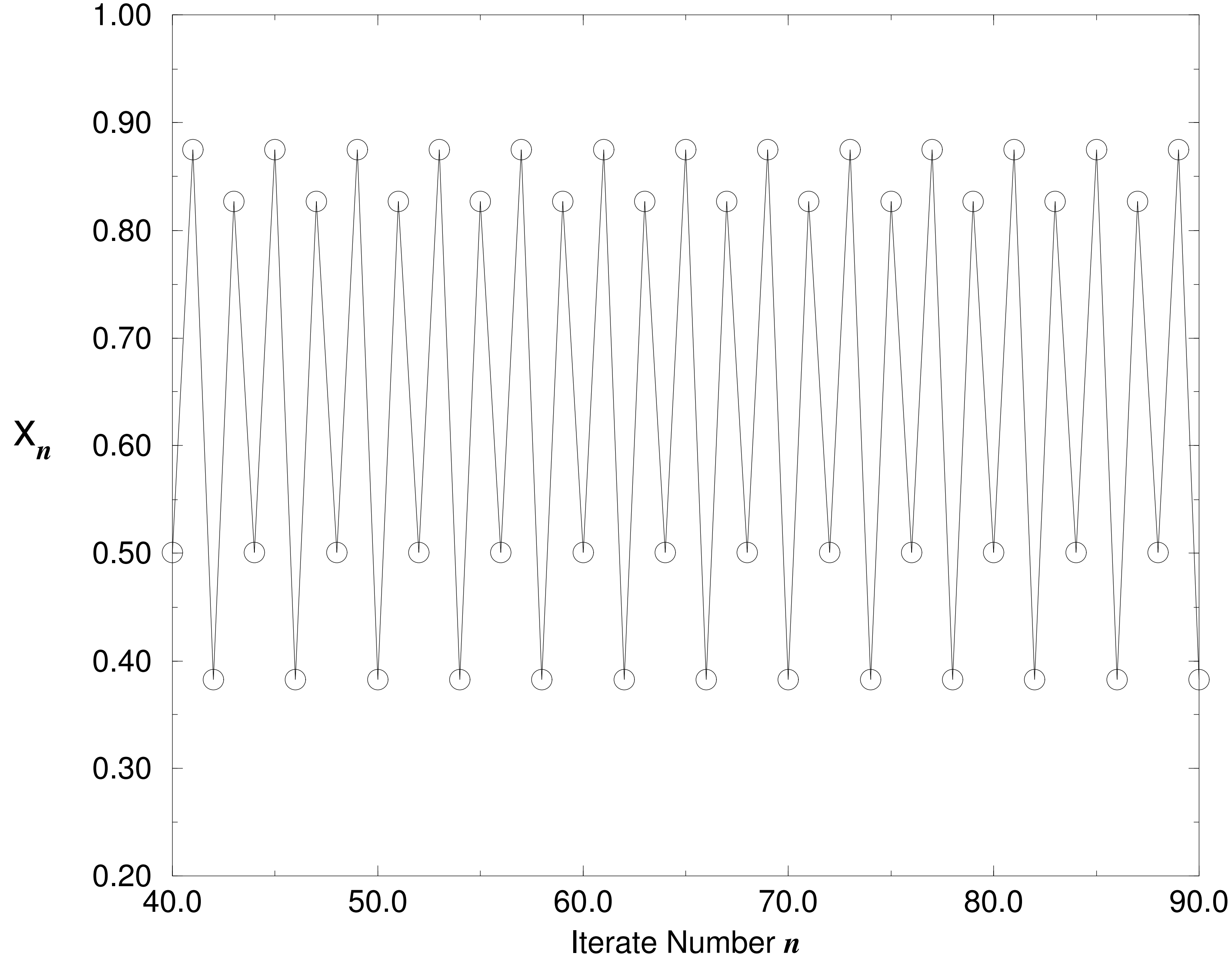
The prevalent notion was that there was a balance in nature, so that the population will increase to an optimum value and remain steady.

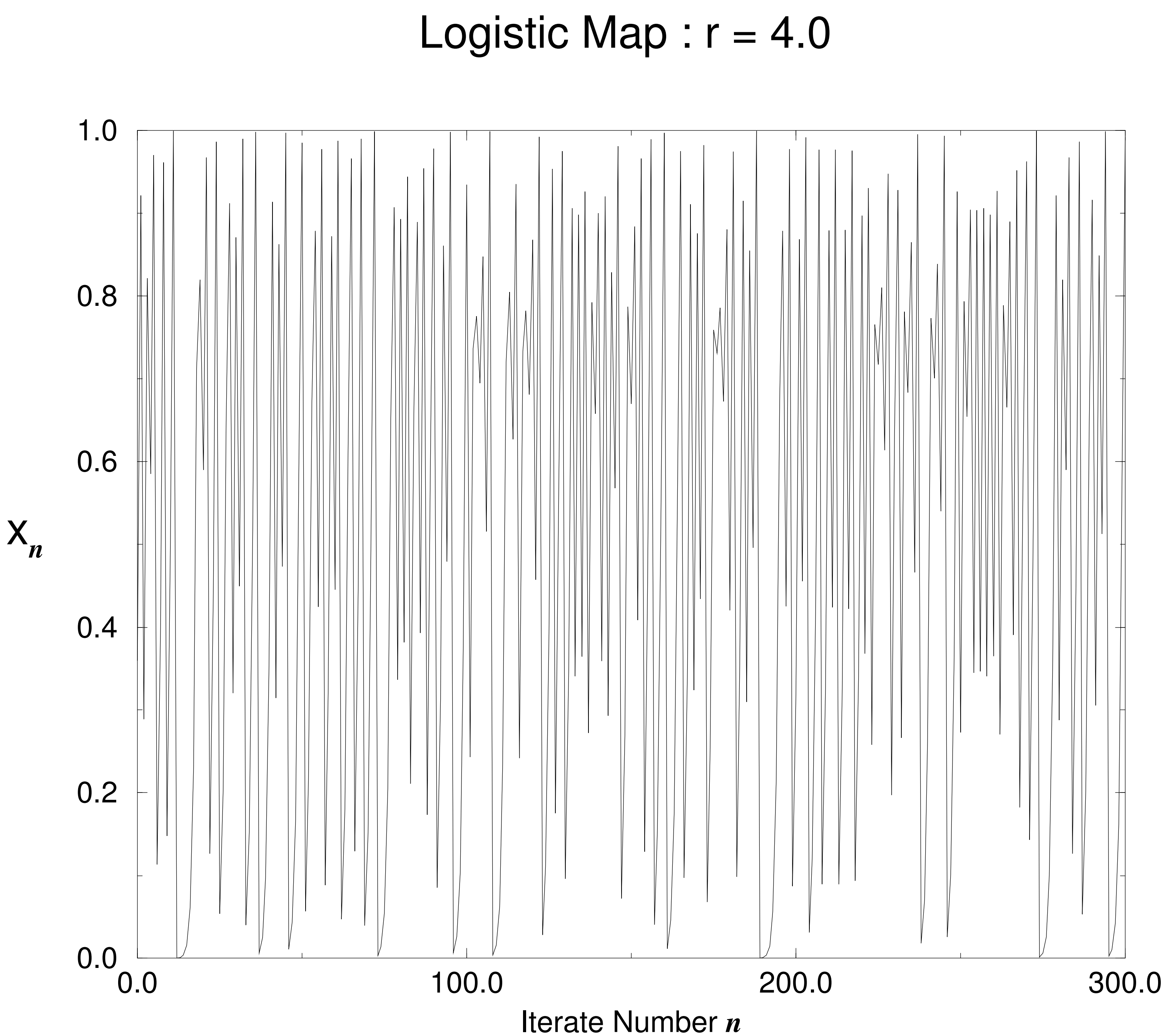

Logistic Map : r = 2.8
X_n
0.0
0.2
0.4
0.6
0.8
0.0
10.0
20.0
30.0
40.0
50.0
Iterate Number n



Logistic Map : r = 3.1
0.80
0.70
0.60
0.50
X_n
0.0
10.0
20.0
30.0
40.0
50.0
Iterate Number n

Logistic Map : r = 3.5





Logistic Map : r = 4.0
1.0
0.8
0.6
0.4
0.2
0.0
X_n
0.0
100.0
200.0
300.0
Iterate Number n

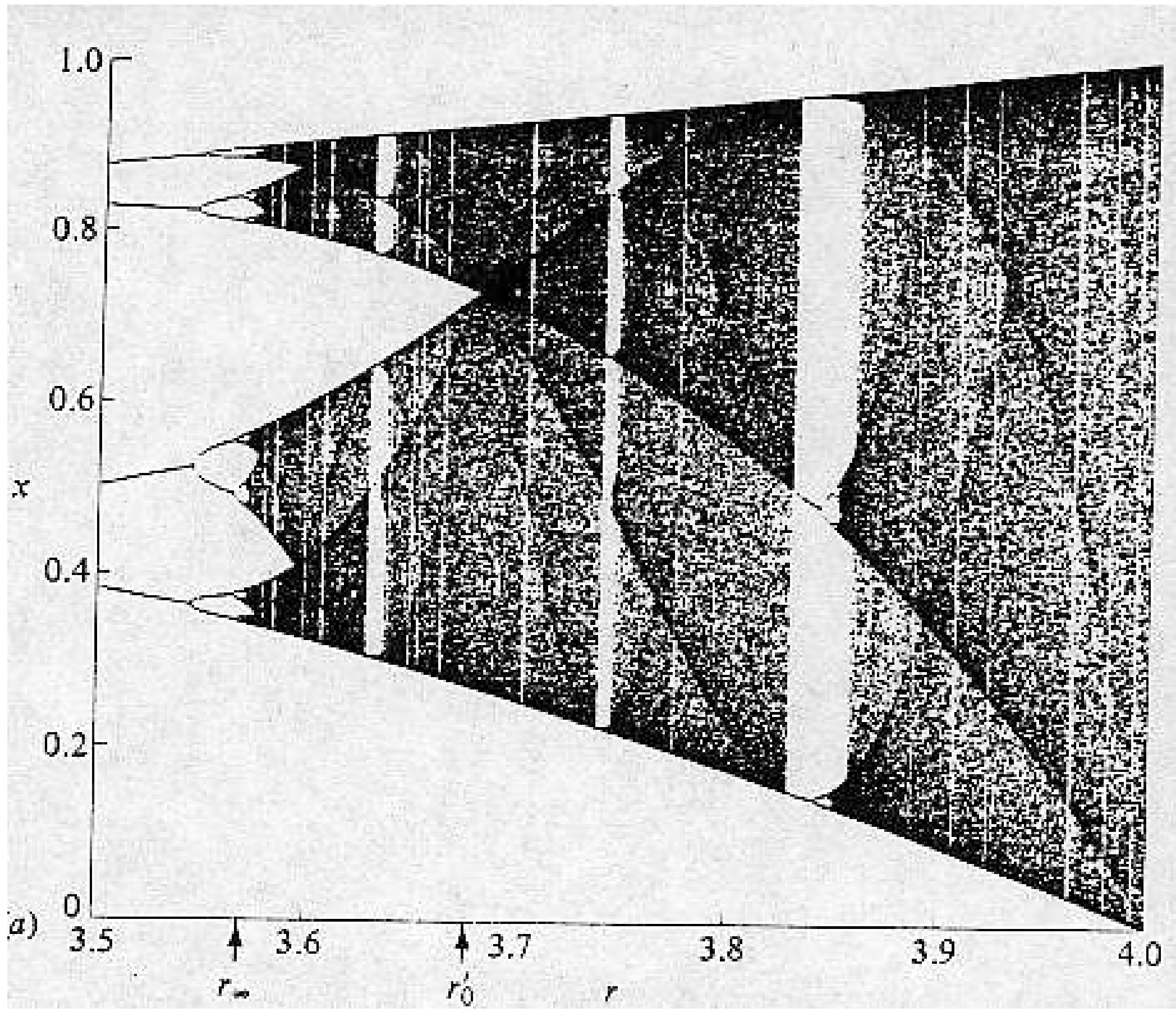
Bifurcations

The logistic map shows a variety of behaviors and it has transitions between these behaviors as we change the parameter r. Such transitions in dynamical systems are called bifurcations.

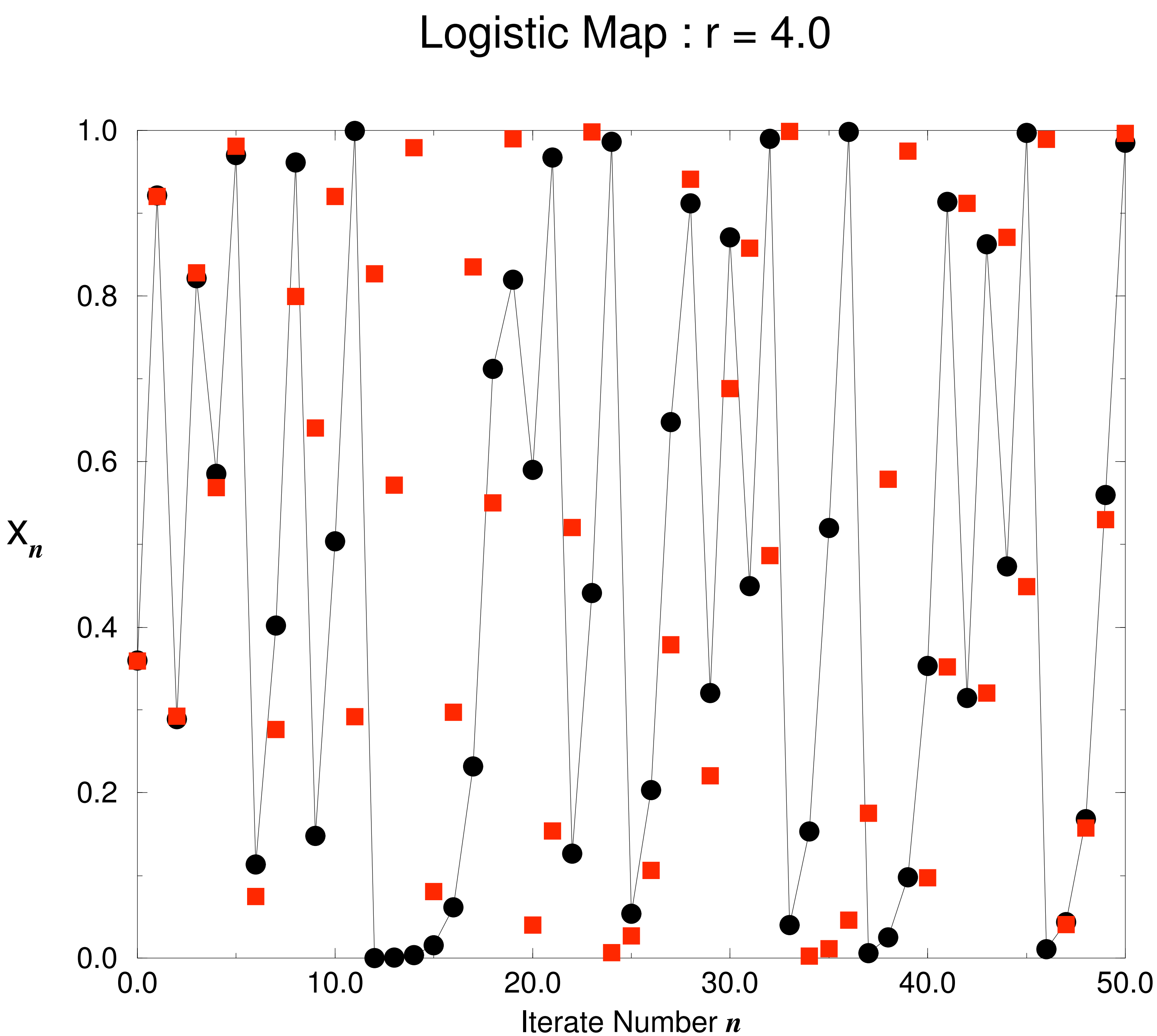
The logisitc map has different kinds of regular behavior and it also has chaotic behavior in contrast to linear systems.

The logisitc map has an infinite sequence of period doublings leading to Chaos.

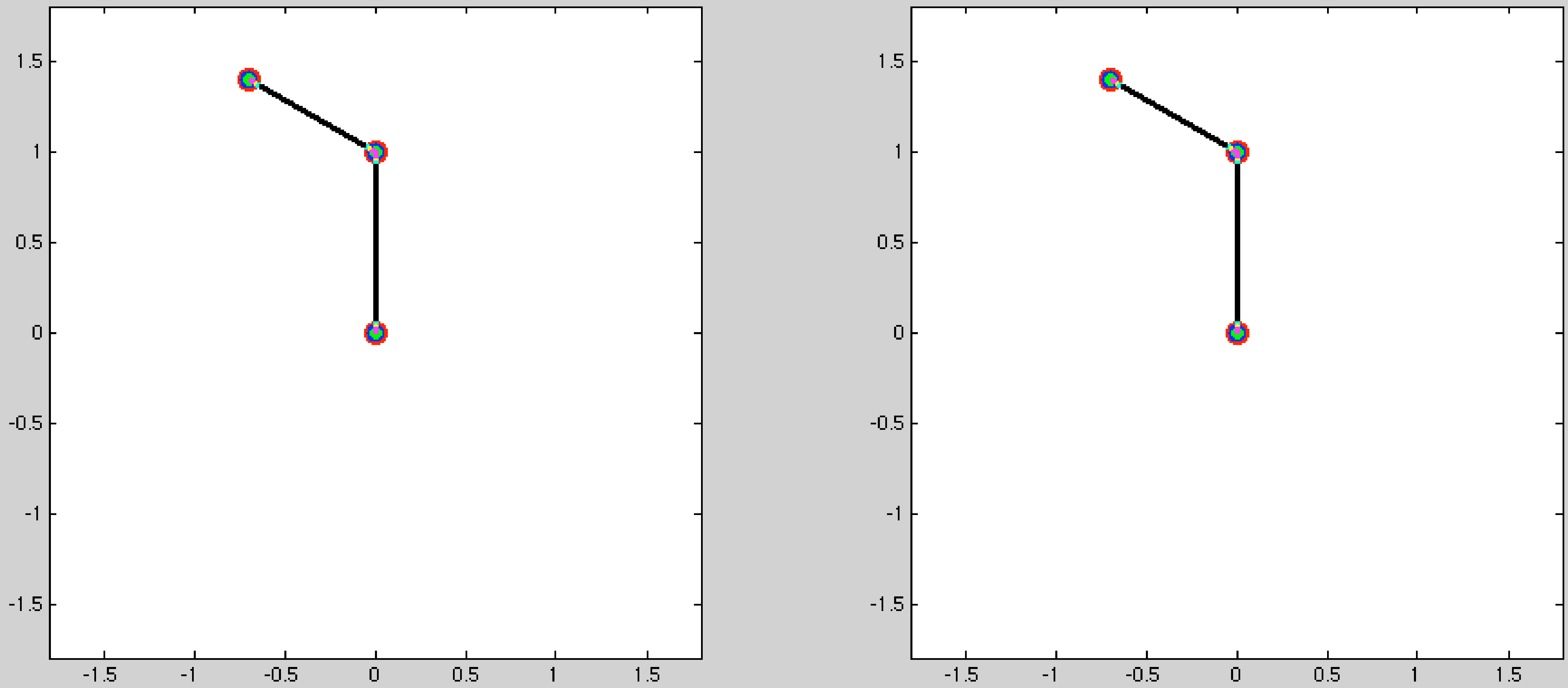
In the chaotic regime, the logisitc map has a dense set of periodic windows, so that regular and chaotic behavior are intermingled on arbitrarily fine scales - fractals.



The bifurcation diagram for the logistic map.
This figure is taken from *Chaos in Dynamical Systems* by Ed Ott.



Logistic Map : r = 4.0
X_n
Iterate Number n
1.0
0.8
0.6
0.4
0.2
0.0
0.0
10.0
20.0
30.0
40.0
50.0



The double pendulum with two slightly different initial conditions

The Butterfly Effect

Sensitive dependence to initial conditions.

Ed Lorenz (a meteorologist at M.I.T) and his toy weather.

A system with twelve variables (a twelve dimensional phase space).

The year 1961.

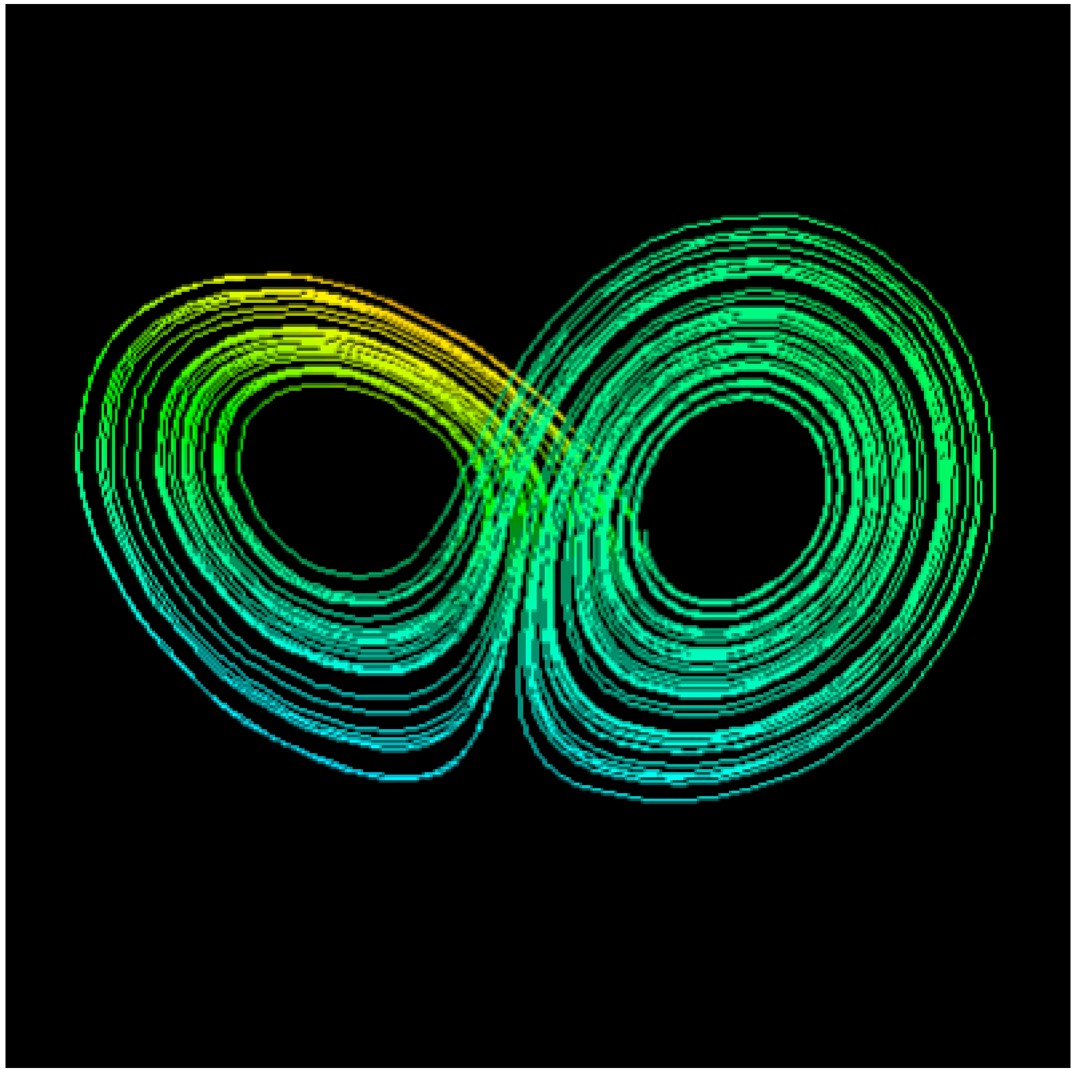
A computer the size of half a room.

No long-range weather prediction - Ed Lorenz.

Simpler model in a three dimensional phase space.

"Deterministic Nonperiodic flow," *Journal of Atmospheric Sciences* **20** (1963) 130.

"Predictability: Does the Flap of a Butterfly's wings in Brazil, set off a tornado in Texas?" *Address to the annual meeting of the AAAS, 1979.*



The Lorenz Attractor

Generated by the Program "Chaotic Flows"

Thanks to John Lindner, Bryan Prusha and Josh Bozeday

A signature of Chaos

- Motion in a bounded region of phase space.
- An orbit eventually gets close to a point it has been at before.
- If there is no butterfly effect, the orbit will stay close to itself. Therefore, we will have cycles and therefore have regular behavior.
- Because of the butterfly effect, the orbit never comes close to repeating itself.
- This leads to apparently random behavior - Chaos.

Sensitive dependence on initial conditions

“For want of a nail, the shoe was lost;

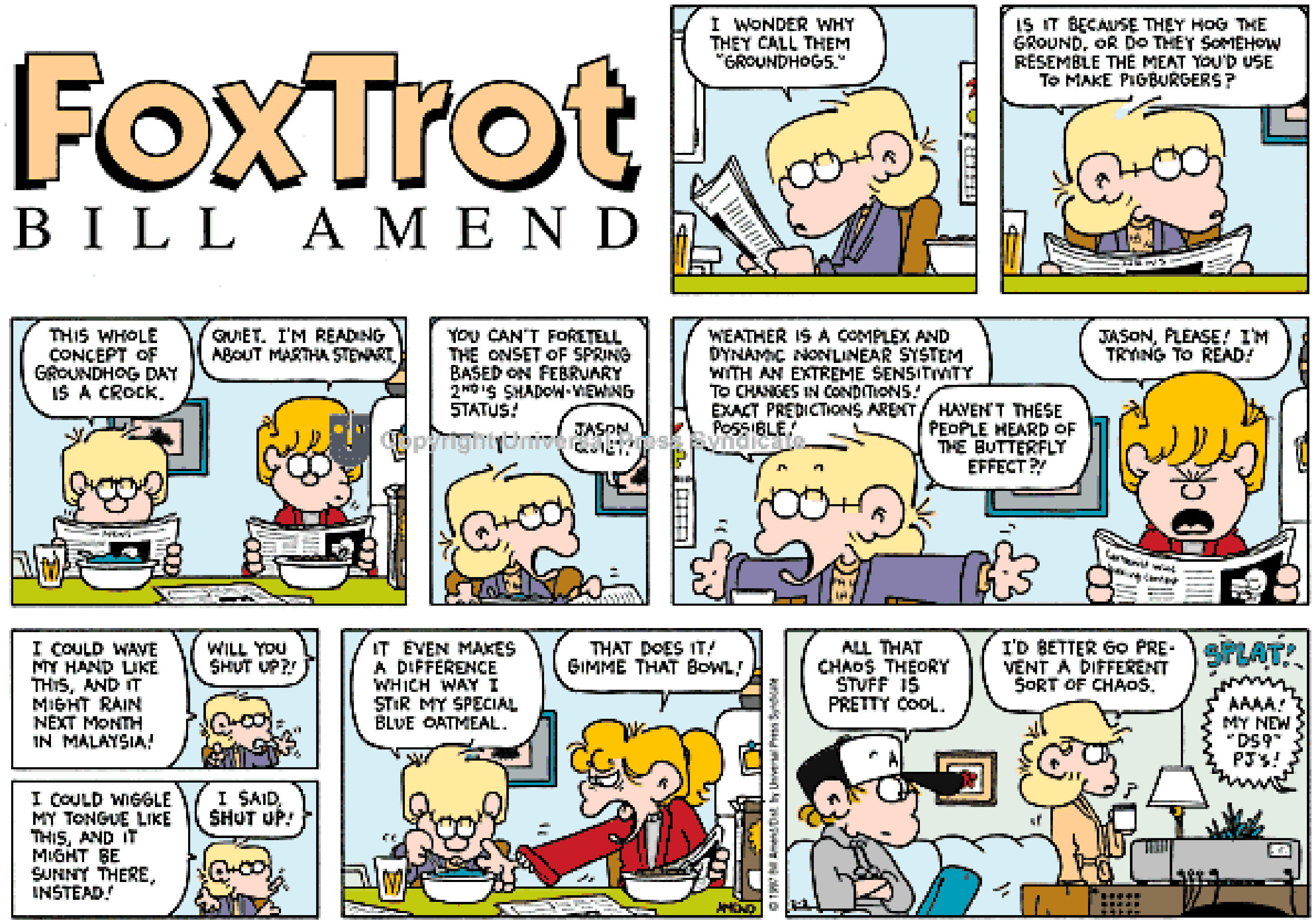
For want of a shoe, the horse was lost;

For want of a horse, the rider was lost;

For want of a rider, the battle was lost;

For want of a battle, the kingdom was lost!”

- A poem in folklore.



Foxtrot Feb 2, 1997

Bill Amend

Limits to prediction

- It is impossible to know the initial conditions exactly (with infinite precision).
- This puts an effective limit to our ability to predict the future state of a chaotic system.
- In a chaotic system, small errors grow exponentially with time.
- This means that, as the prediction time grows arithmetically, the required precision grows geometrically.
- Example: If we need a precision of 0.1 to predict for 1 hour and 0.01 to predict for 2 hours, then we need a precision of 0.0001 to predict for 4 hours.

Weather Prediction

- The global weather system is chaotic.
- There exist very good models for the equations that govern the weather.
- For predicting the weather, the time it takes for a disturbance on the scale of a kilometre to grow to the scale of a global weather pattern is about 2 weeks.
- A week is therefore effectively the limit of our long range weather prediction.
- It is indeed concievable, that a butterfly flapping its wings could cause a hurricane somewhere else on the globe in a few of weeks.

The philosophy of science

- Laplacian Determinism.
- Simple rules imply simple behavior and this simple behavior is robust.
- Complex and unpredictable behavior requires complex rules or outcomes that depend on chance.
- Once we find all the equations that govern nature, science can tell us the answer to everything and we can predict the future exactly.

The Chaos Revolution

- Simple rules can produce complex behavior - Chaos.
- Simple rules can produce behavior that looks random - Chaos.
- Chaotic systems are very sensitive to their initial conditions - the Butterfly effect.
- This sensitivity puts an effective limit on our ability to predict the behavior of chaotic systems over long periods of time.
- We should lose our philosophical prejudice that simple laws lead to simple behavior and examine dispassionately such ideas as a balance in nature or long-range economic planning.

Summary

- A Phase space is a mathematical representation of all the states of a system.
- The number of variables used to describe the system is called its dimensionality.
- A Dynamical system is the mathematical representation of the rules that govern the evolution of a system.
- The trajectory of a system in its phase space is called an orbit.
- A discrete time dynamical system is called a Map and a continuous time dynamical system is called a Flow.

Summary

- Some dynamical systems have the property that sums of two different solutions are also solutions. Such dynamical systems are linear.
- Linear dynamical systems can be solved exactly and they show a limited variety of behaviors.
- Most of the systems we encounter in the real world are nonlinear.
- Nonlinear dynamical systems are often not solvable.
- Nonlinear systems display a rich variety of dynamical behavior.

- A Phase space is a mathematical representation of all the states of a system. It is the arena in which the system evolves.
- A Dynamical system is the set of the rules that govern the evolution of a system.
- The state in which a system starts its evolution is called the initial state and the numbers representing the initial state in the phase space are the initial conditions.
- The trajectory in the phase space as a system evolves starting out from a given set of initial conditions is called an orbit.

Summary

- Deterministic systems can show apparently random behavior - Chaos.
- Nonlinear systems can show a variety of dynamical behaviors and can go form one kind of behavior to another as a parameter is changed - bifurcations.
- Chaotic systems are extremely sensitive to small changes is the state - the butterfly effect.
- This sensitivity leads to an effective limit to our ability to predict the behavior of chaotic systems.
- The realization that simple systems can display chaotic behavior has led us to reconsider the role of determinism.

What is Chaos?

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2. Complete disorder, utter confusion.

3.(Math.) Stochastic behavior occurring in a deterministic system.

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How can a system with no element of chance be random?

Simplicity underlying complex behavior?

Fractals, Attractors
Bifurcations

Physics, Chemistry,
Biology, Meteorology,
Astronomy

Art, Economics
Philosophy