Europäisches Forum Alpbach 16 August, 2003

Fractals Everywhere

Scales in the description of systems

A Scale is the level of detail at which we are looking at a system.

Examples:

- The Economy: Personal finances (Scale is each person) to Global economy (Scale is the whole world).
- The Physics of the Universe : Subatomic particles to the entire universe.
- The Human body : Molecules to the complete person.

The laws governing a system are usually different at different scales.

Self-similarity

A system is Self similar if it looks like itself at different scales.

Leibniz : "A drop of water contained a whole teeming universe, containing in turn, water drops and new universes within."
- James Gleick in "Chaos".

The idea of a *Homunculus*.

These ideas died in the face of scientific discoveries that each smaller scale had different and "more fundamental" laws.

Reductionism

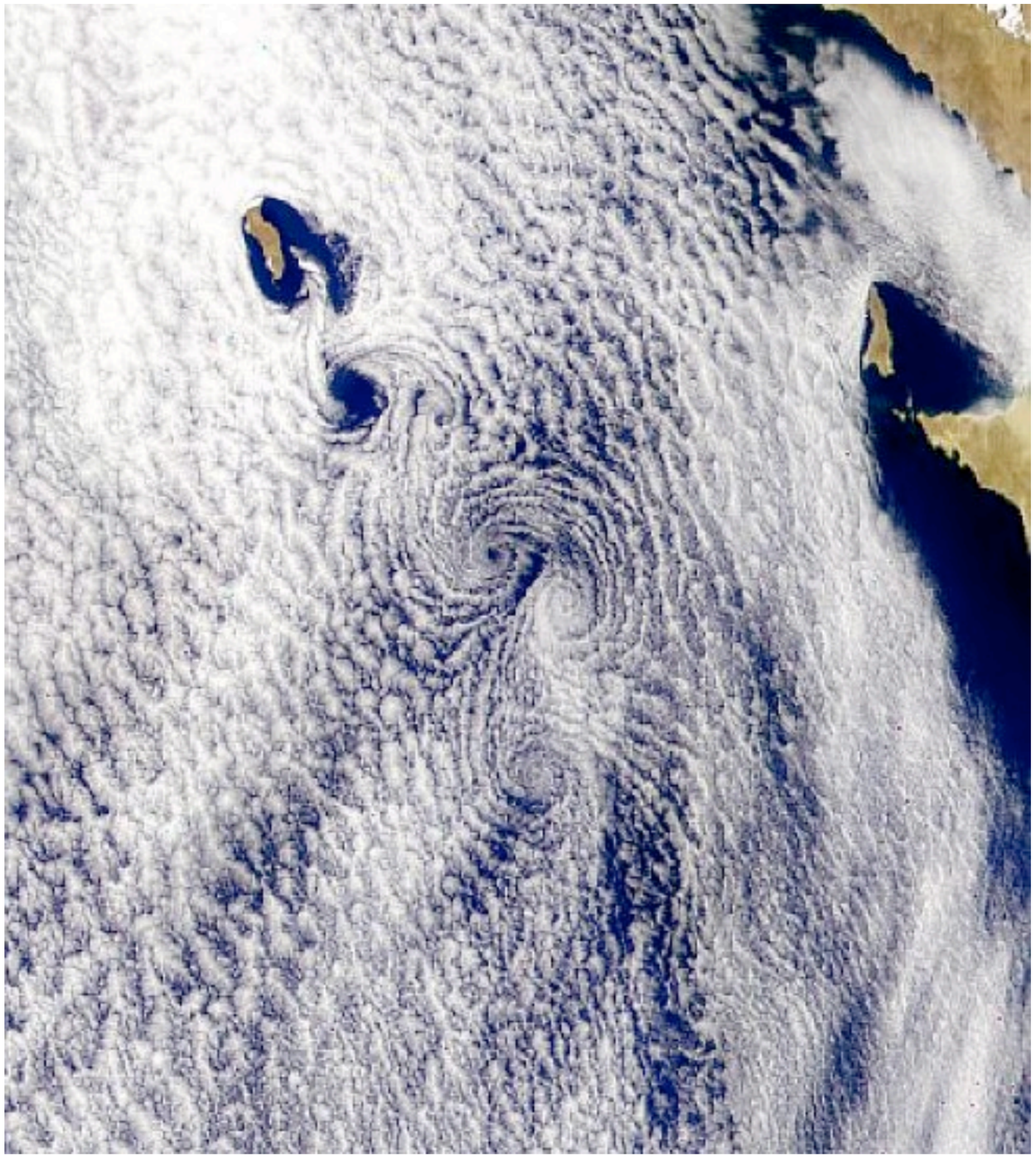
Self Similarity

Scaling Laws.

Self similarity implies the lack of a preferred scale. Scaling laws arise when there is no preferred scale.

Physical laws, *e.g.*, gravity, are universal, and are valid independent of the scale of the objects (classically).

This can be used to show that similar behavior has to occur across many scales – *Dimensional Analysis*.



Vortex street
in the atmosphere

Guadalupe Island,
Aug 20, 1999

Image Courtesy
NASA Goddard

Shetland Islands

Falkland Islands

Images Courtesy NASA Goddard

Flow in a Soap Film

Experiments by

Maarten Rutgers,
Ohio State University

Scaling laws II

The behavior of systems that is *scale-independent*, is governed by *Scaling laws*

$$\boxed{Y \sim X^{\alpha}.}$$

A straight line on a log-log plot.

Slope of the straight line is α.

"Nontrivial" scaling laws also arise in the study of *Phase transitions*.

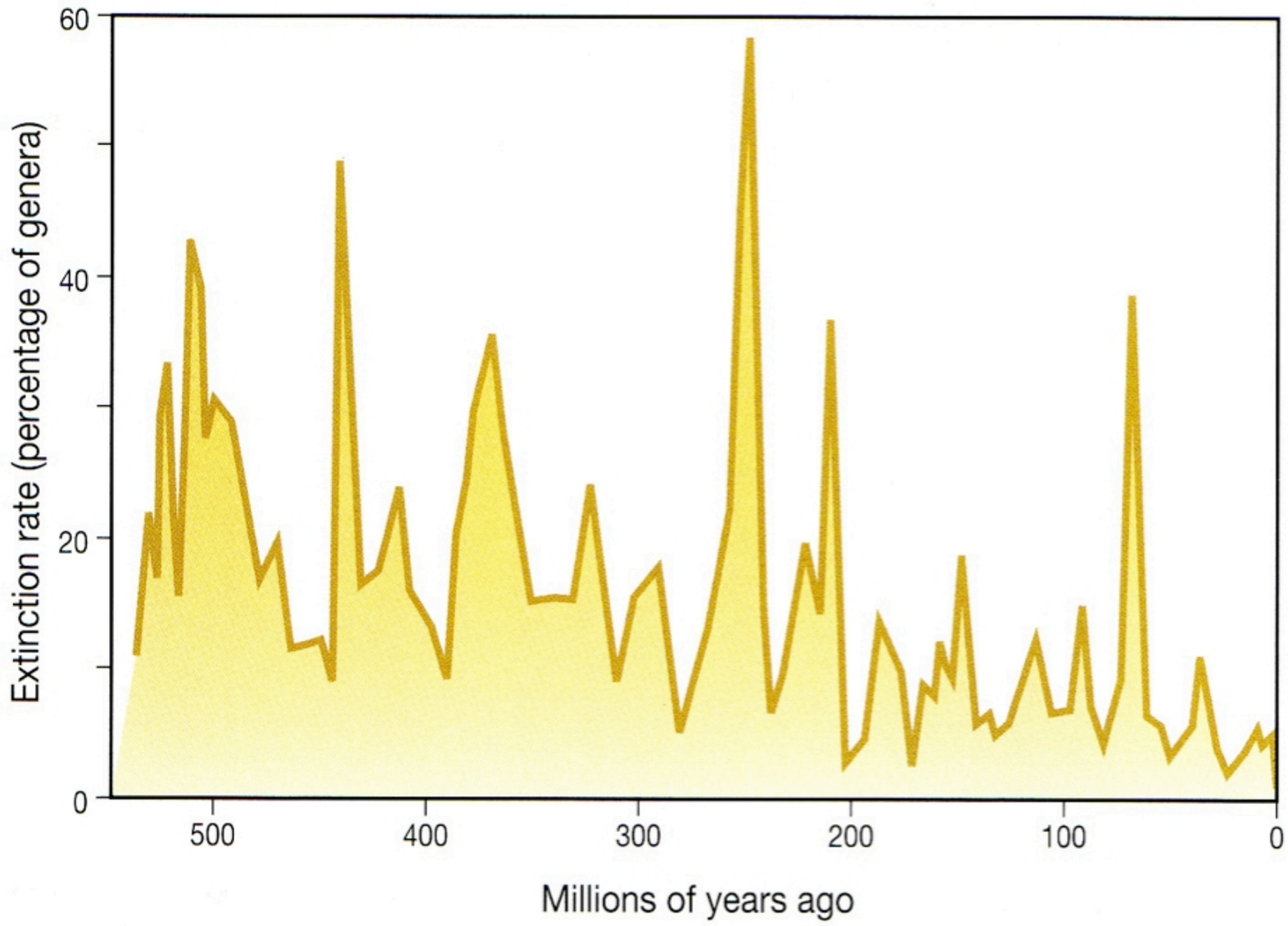
Renormalization and *Universality*.

Scaling laws show up in many surprising contexts

- Frequency of the occurrence of words in written text.
- The number of islands of a given size.
- Distribution of times of error free transmission over a communication link.
- The fluctuations in the price of cotton.
- The frequency of earthquakes a function of their magnitude.
- The bifurcation diagram of the logistic map.

- Scaling laws in Biology.
- The distribution of stock market crashes.
- The distribution of avalanches in a sandpile.
- Disease epidemics.
- Extinctions.

A BRIEF HISTORY OF MASS EXTINCTIONS



This graph charts the changing extinction rate of marine invertebrates over the past 600 million years, revealing five pulses of mass extinction.

From "Evolution"
by Carl Zimmer

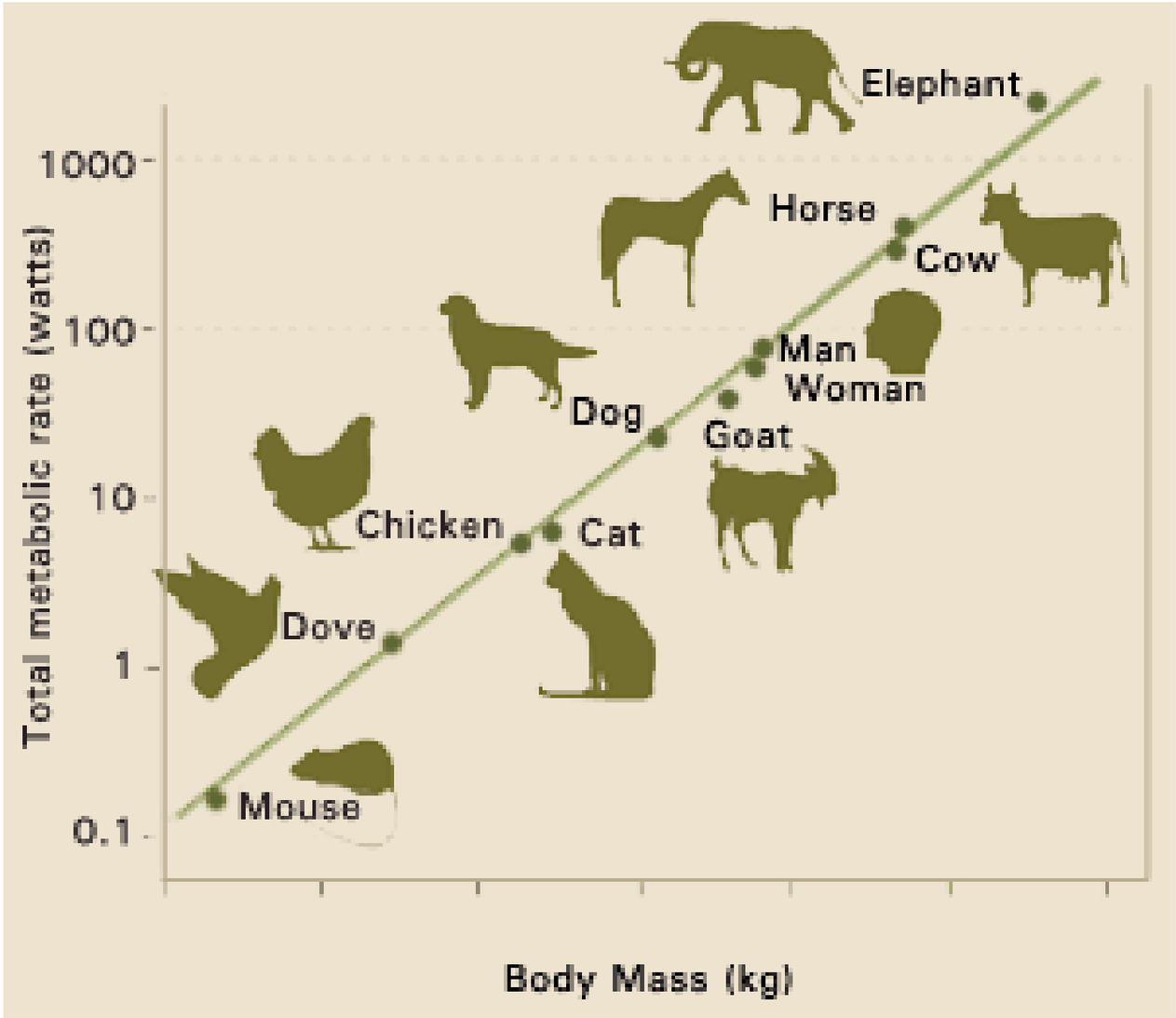
Zipf's Law

$$\boxed{p(n) \sim n^{-1}.}$$

- The frequency of occurrence of words in written text scales with the rank of the word occurrence.
- The population of cities scales with the ranking of the cities by population.
- The number of hits per month of websites, scales with their ranking by poularity.

Lotka's law – $a(n) \sim n^{-2}$, where $a(n)$ is the number of authors who write n scientific papers.

Scaling Laws in Biology



From "Why is Animal size so important"
Knut Schmidt-Nielsen

Animal Size

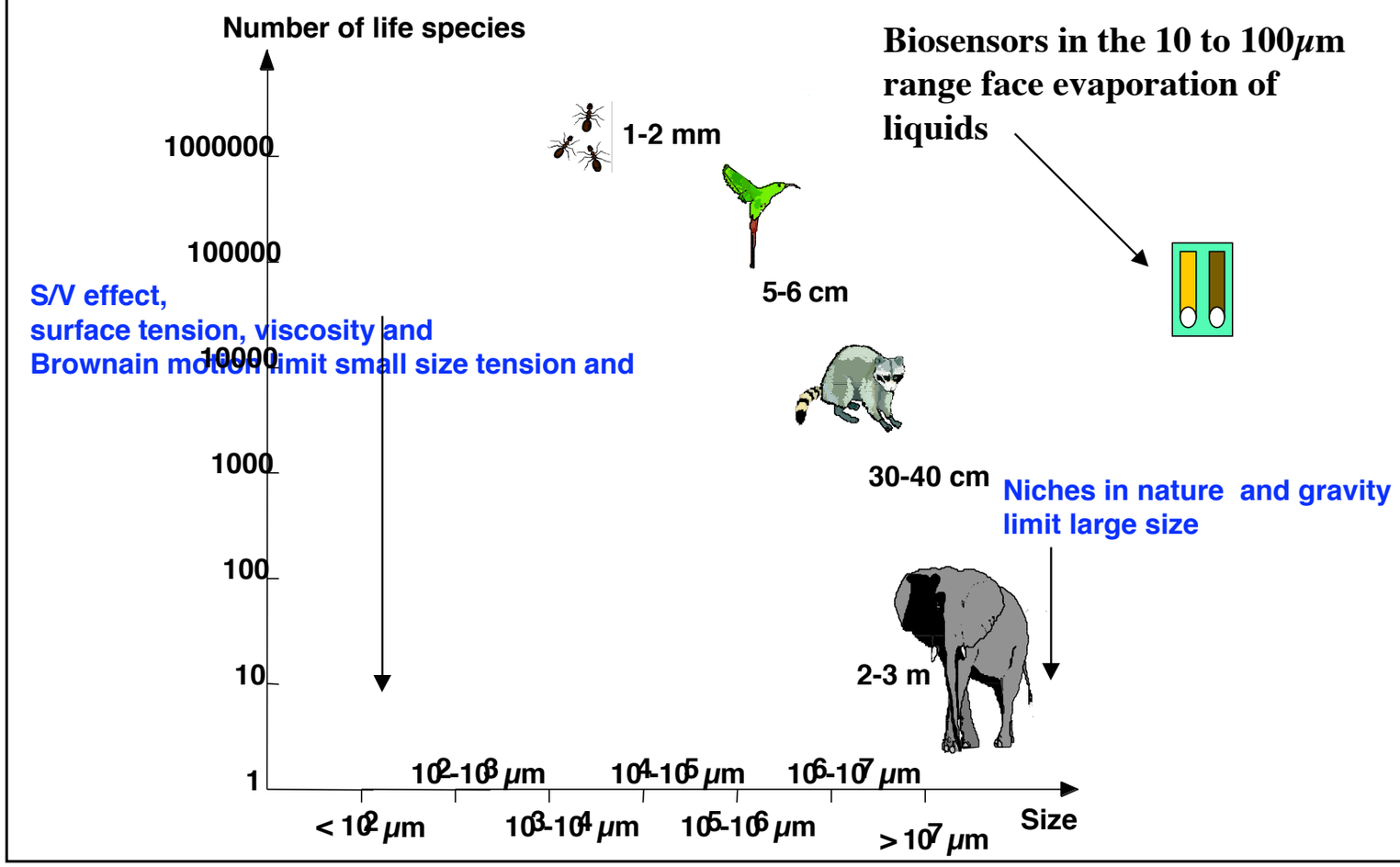


Image courtesy Prof. Marc Madou
University of California, Irvine

Fractals

Benoit Mandelbrot - French mathematician (also economist, physiologist, ...), later worked at IBM.

The "father" of fractal geometry.

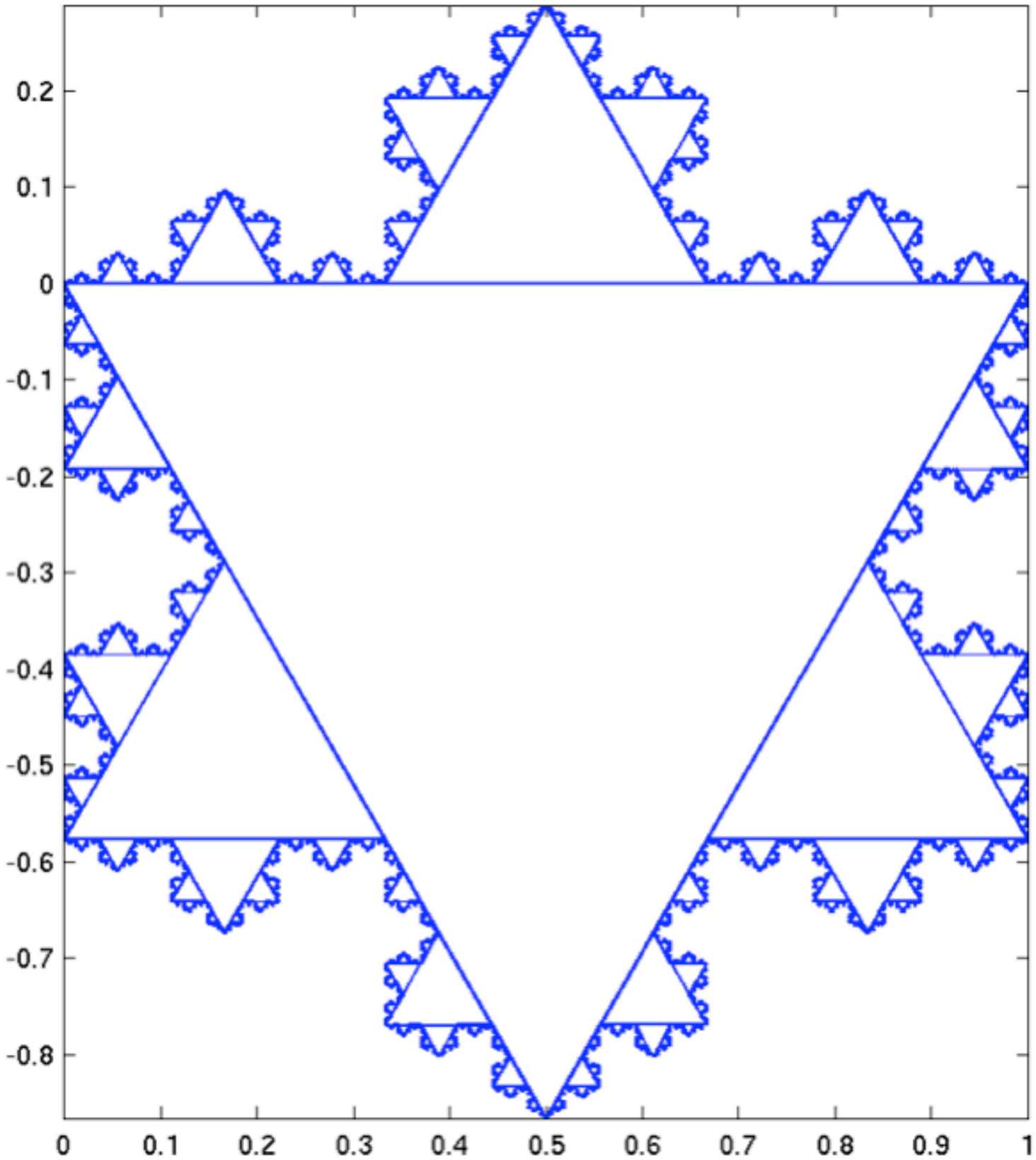
The world around us is not smooth. it possesses structures on all scales.

All the examples described have events on a whole range of scales.

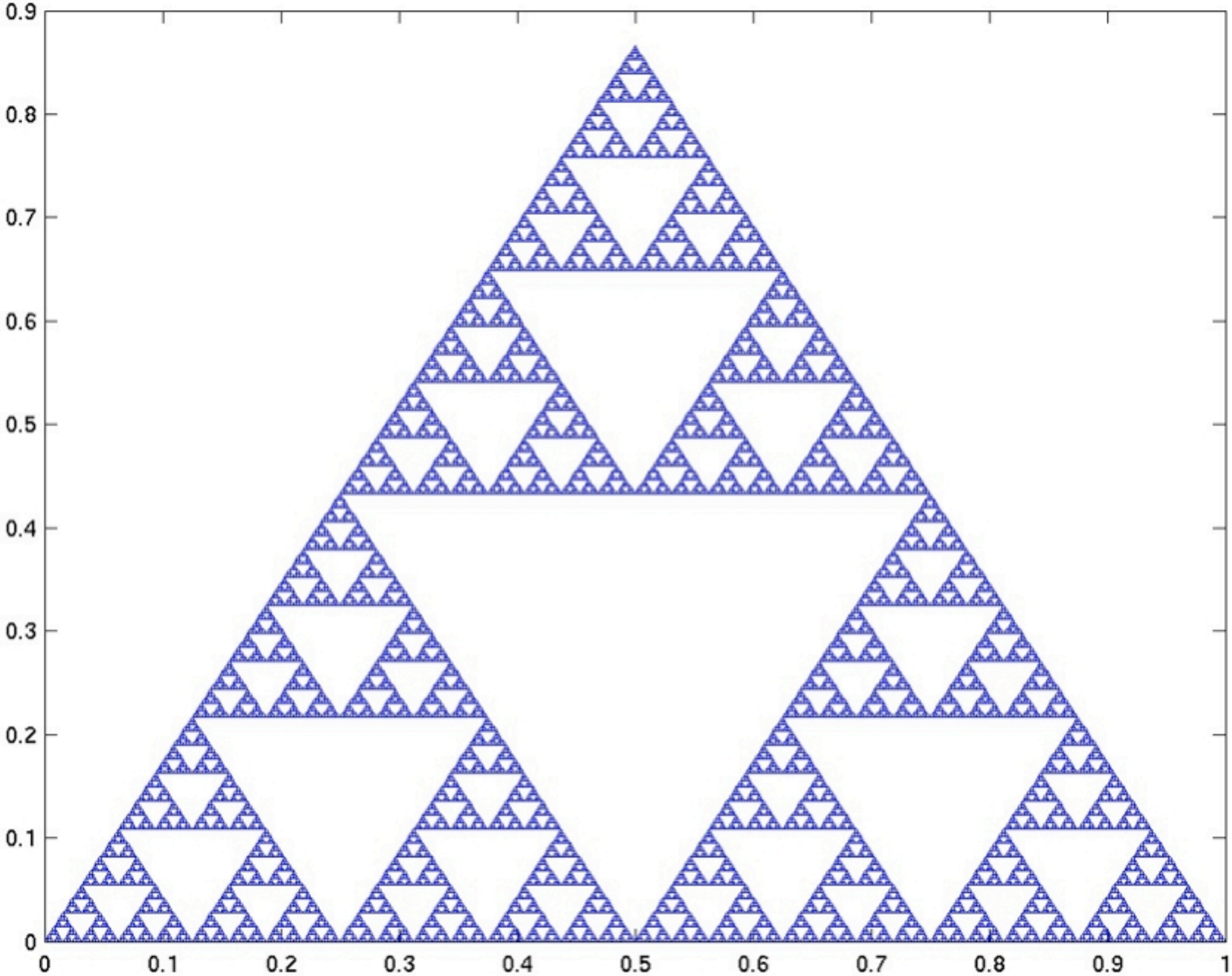
The Geometry of nature.

Geometric structures on all scales - Fractals.

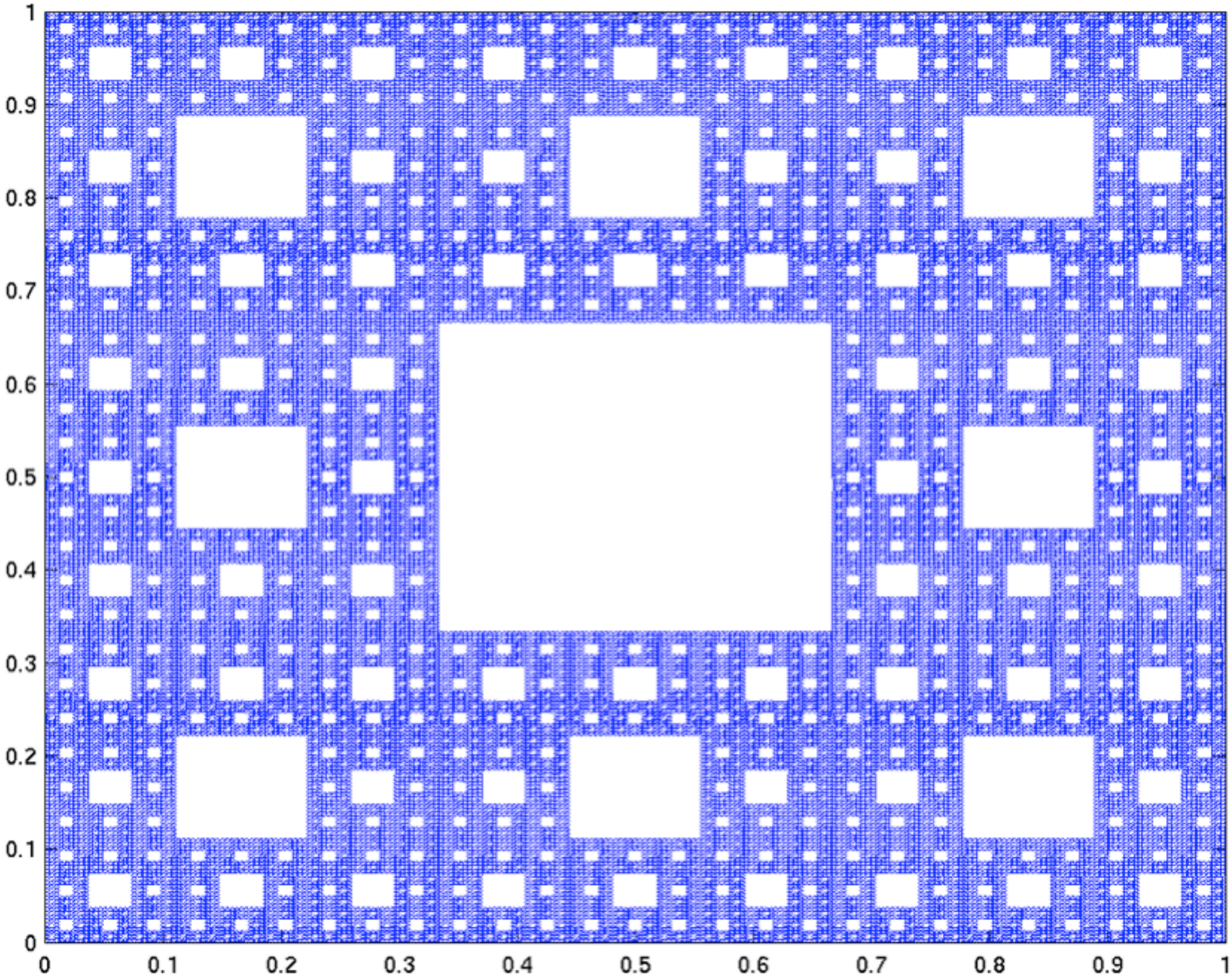
How long is the coastline of Britain ?



The Koch curve



The Sierpinski Triangle



The Sierpinski Carpet

Cantor sets and other "simple" fractals

- Structure on all scales.
- Iteration - Repeated application of a rule.
- The rule is nonlinear.
- Exact Self-similarity.

Questions

- What is the connection to dynamical systems?
- Are fractals just mathematical curiosities or do they occur in the "real" world?

The Mandelbrot Set

Complex numbers : $a + ib$.

$i = \sqrt{-1}$.

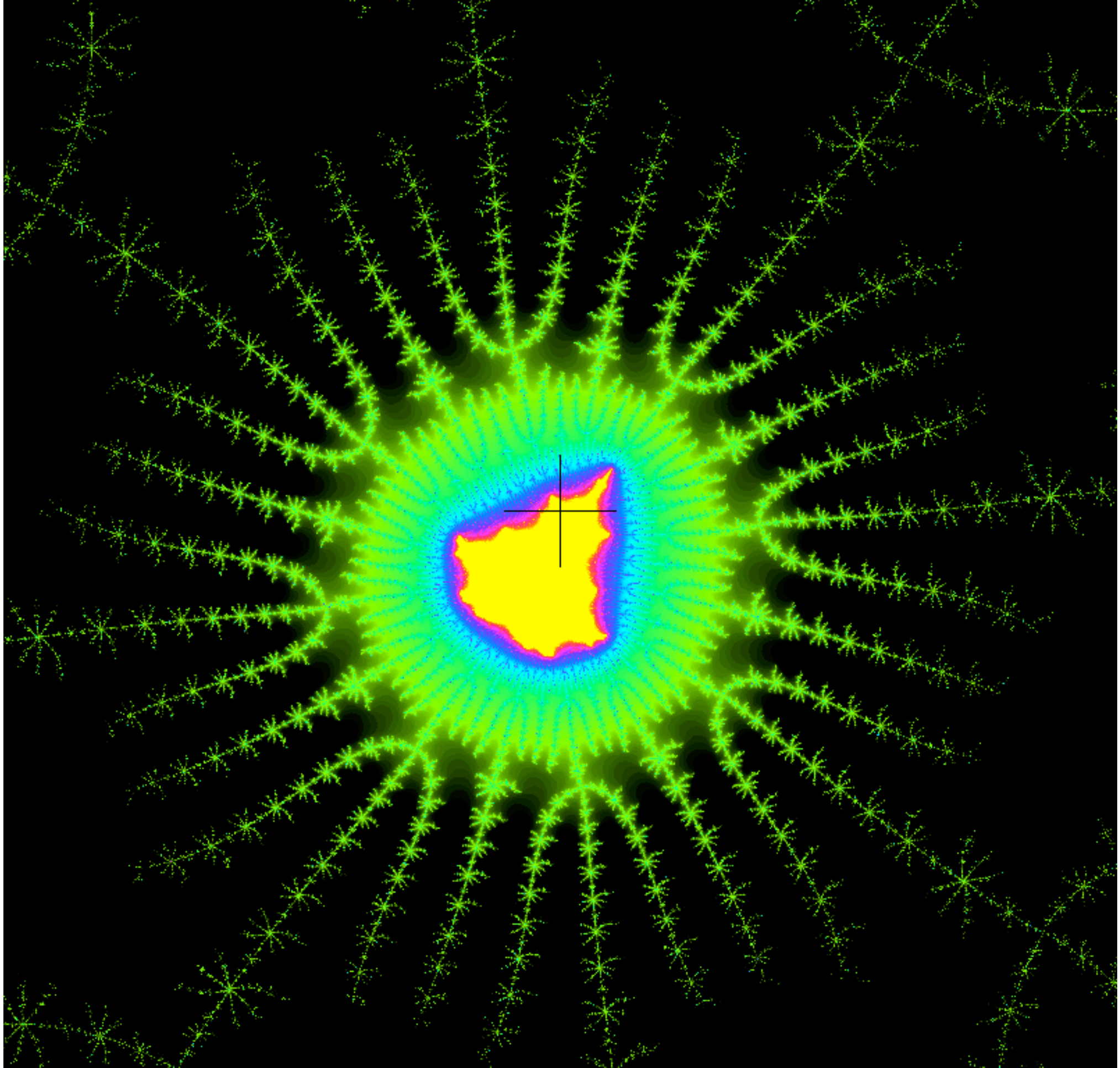
Consider the nonlinear map

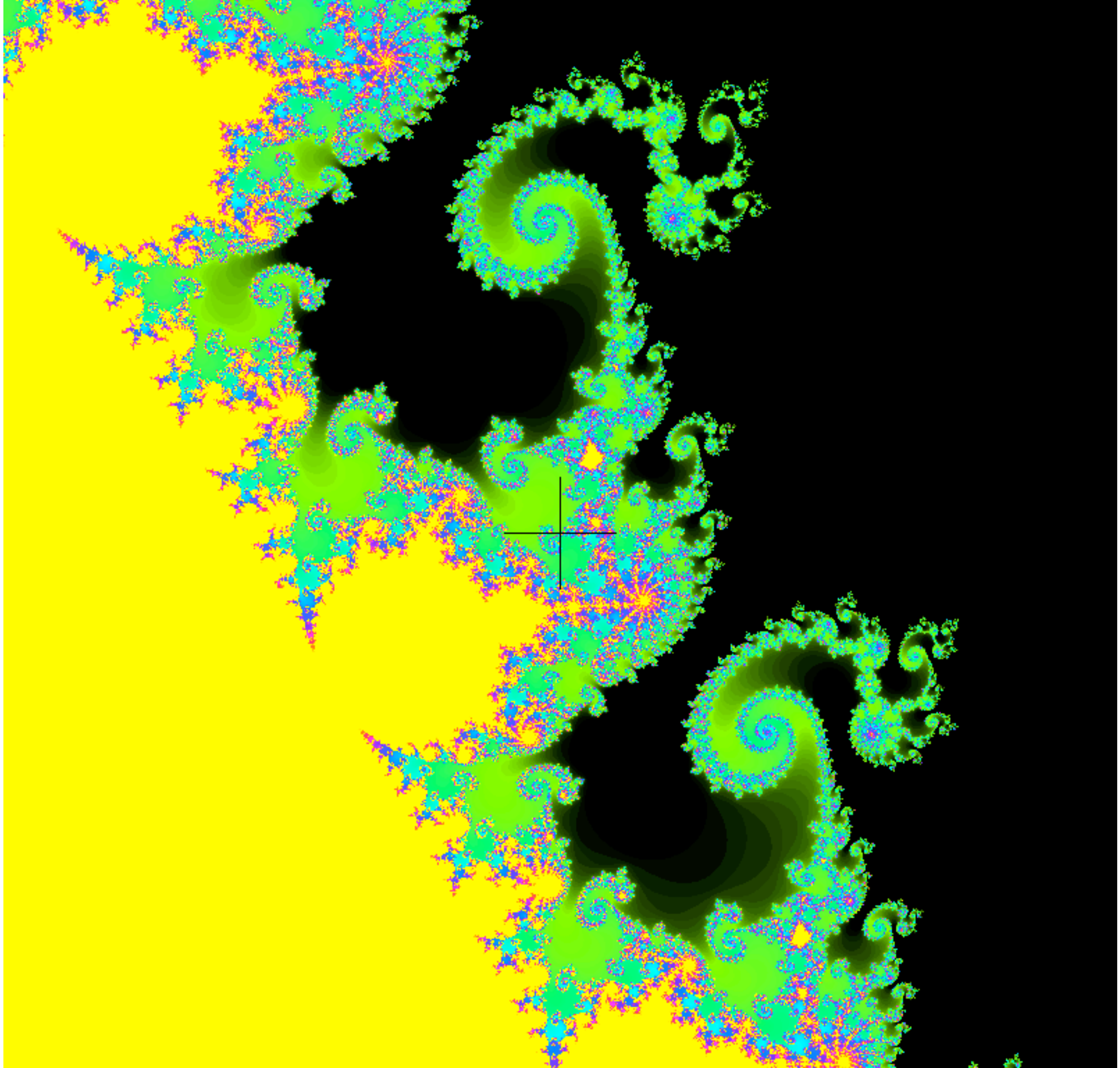
$$z_{n+1} = z_n^2 + c.$$

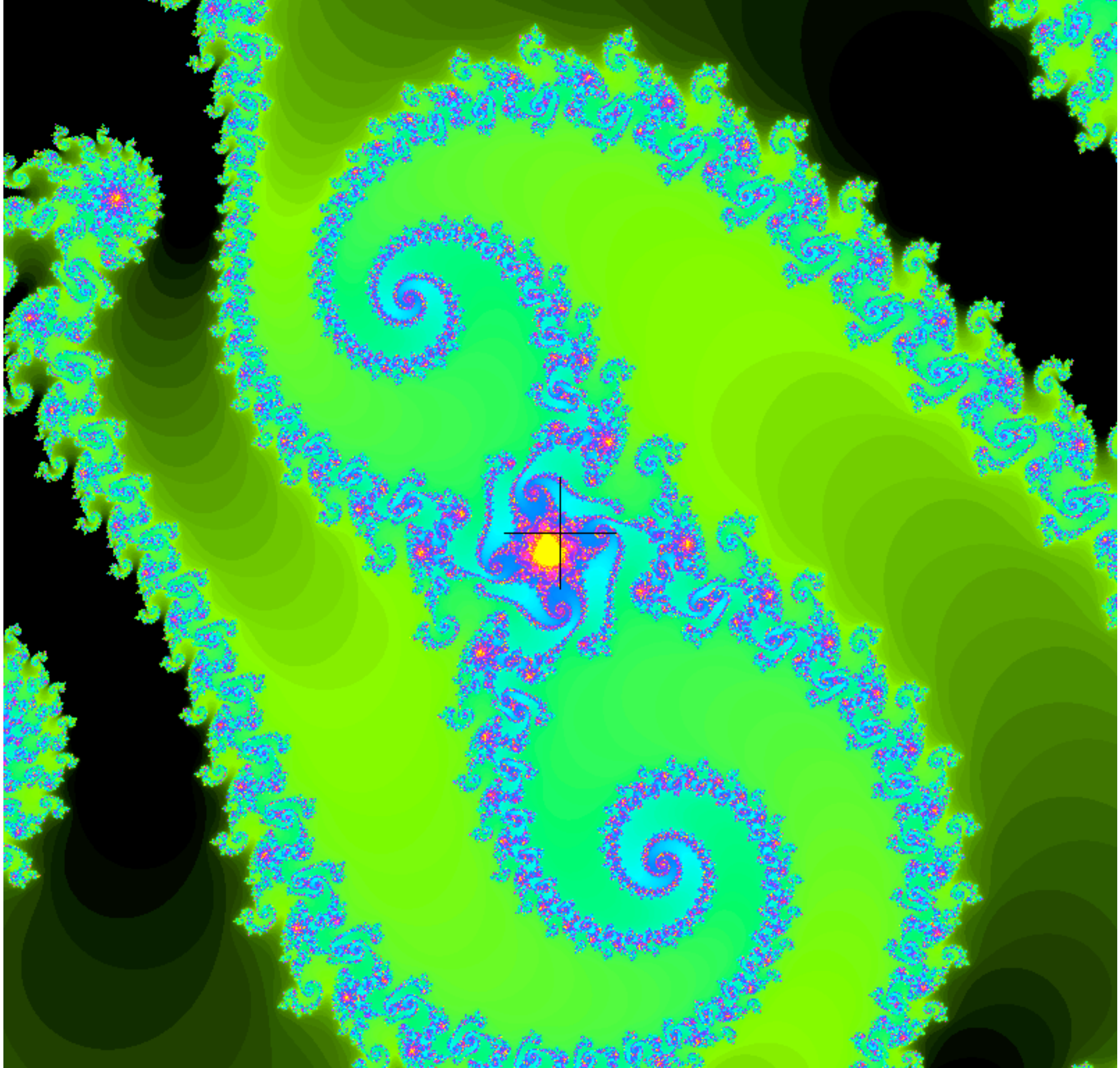
Start at $z_0 = 0$.

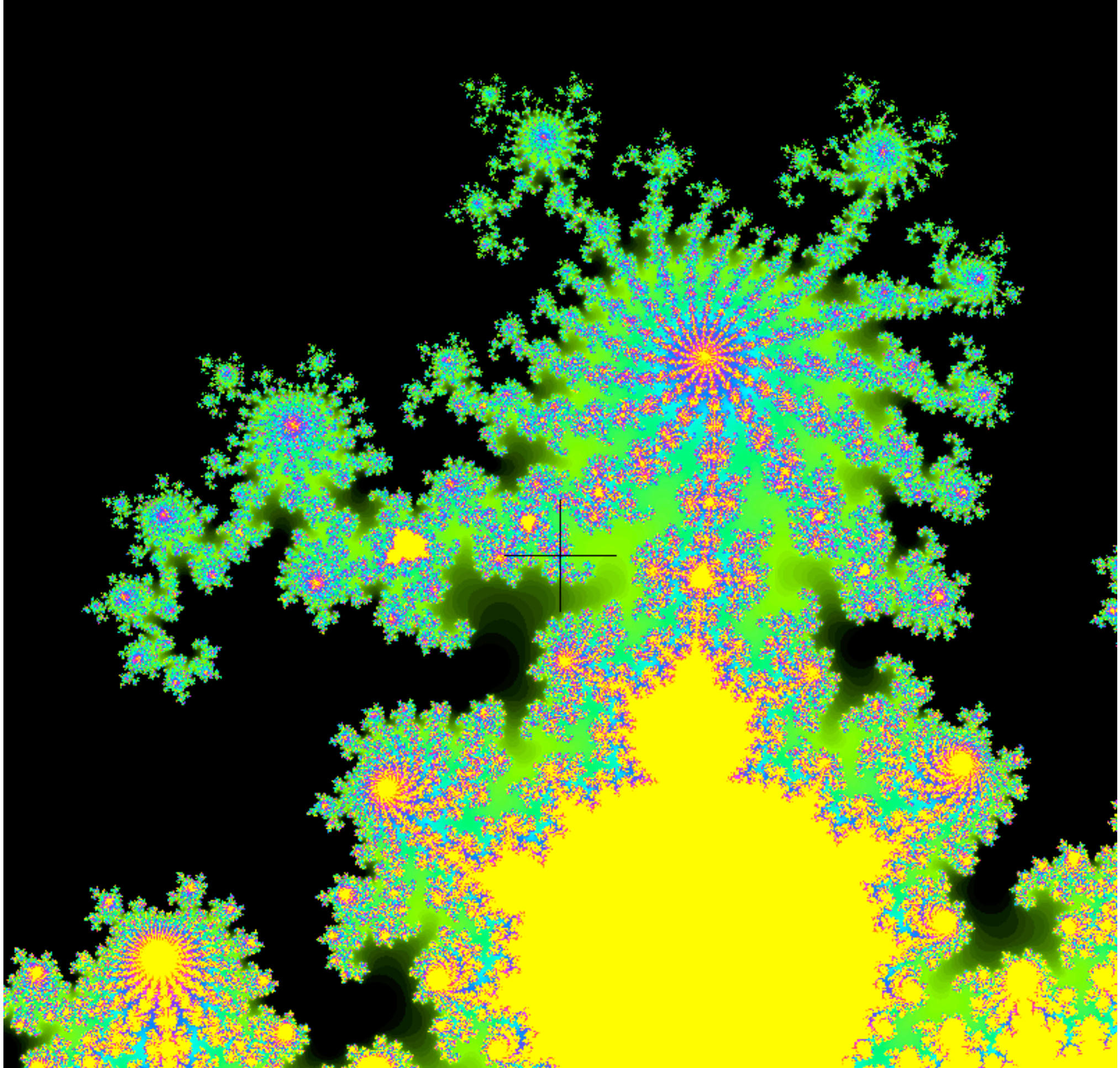
Color the point c black if the orbit does not escape to infinity.

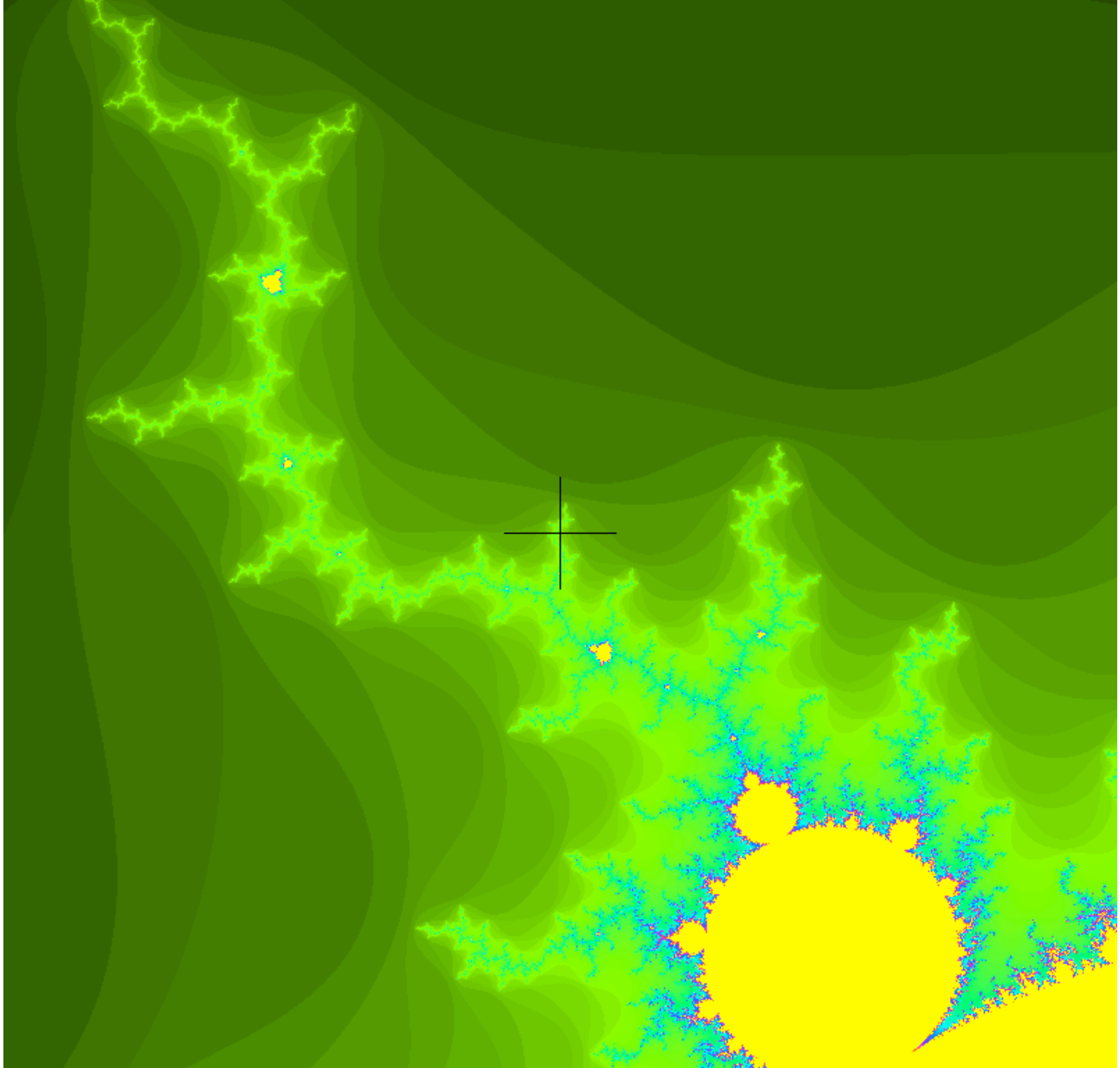
In practice, take a large distance and look at the number of steps it takes for the orbit to get outside that distance. Color accordingly.

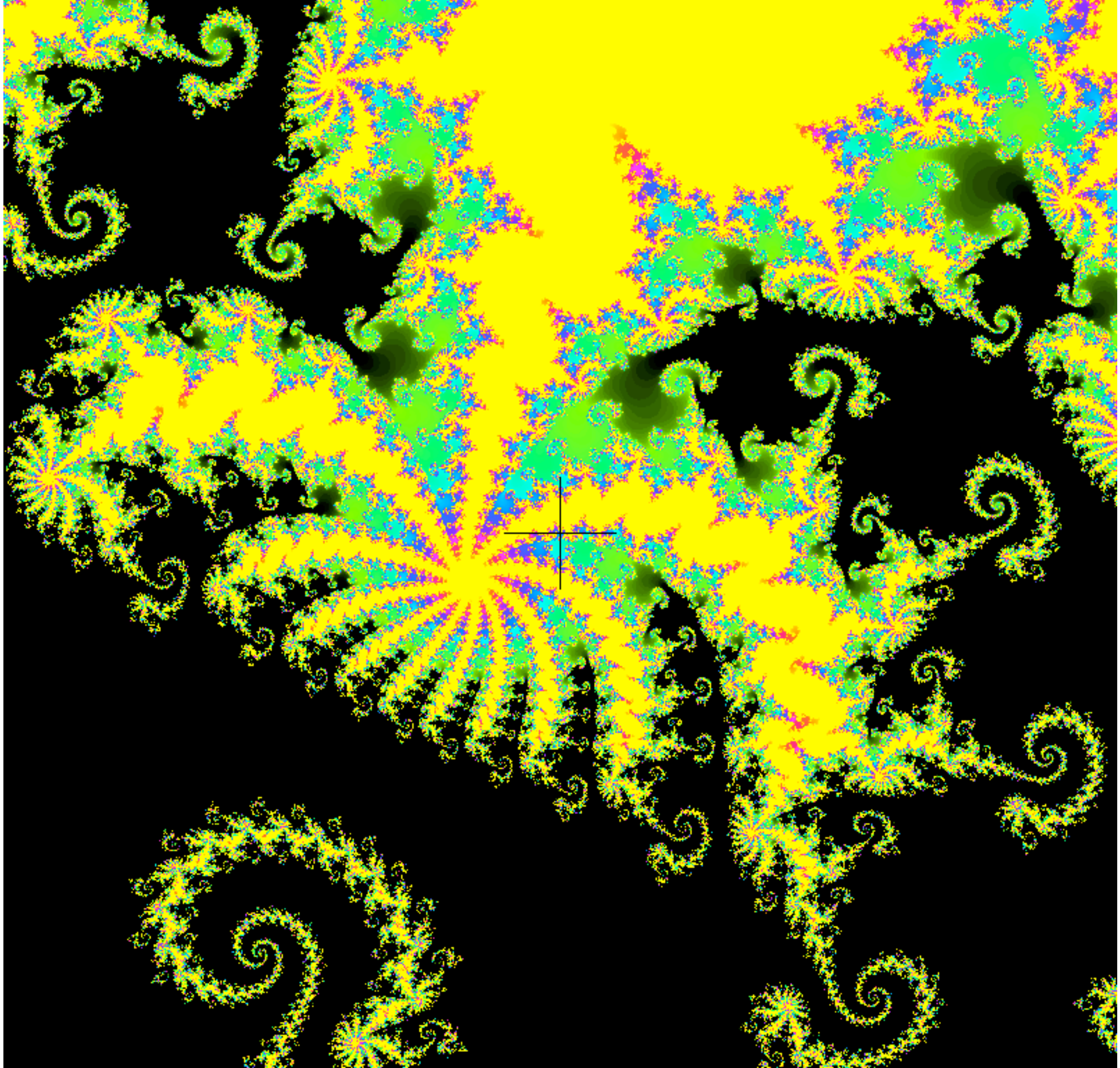


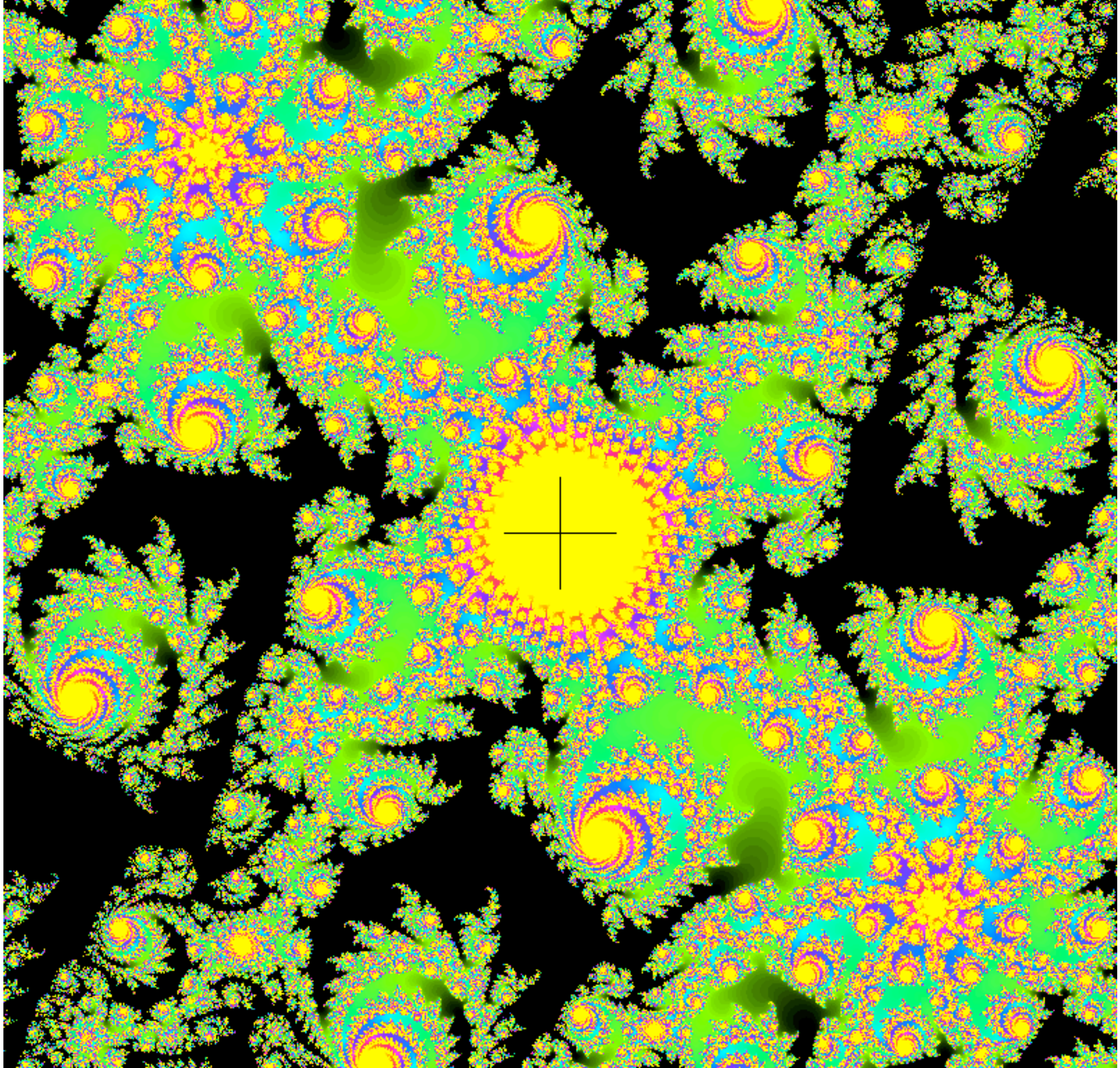












The Journey through the Mandelbrot set

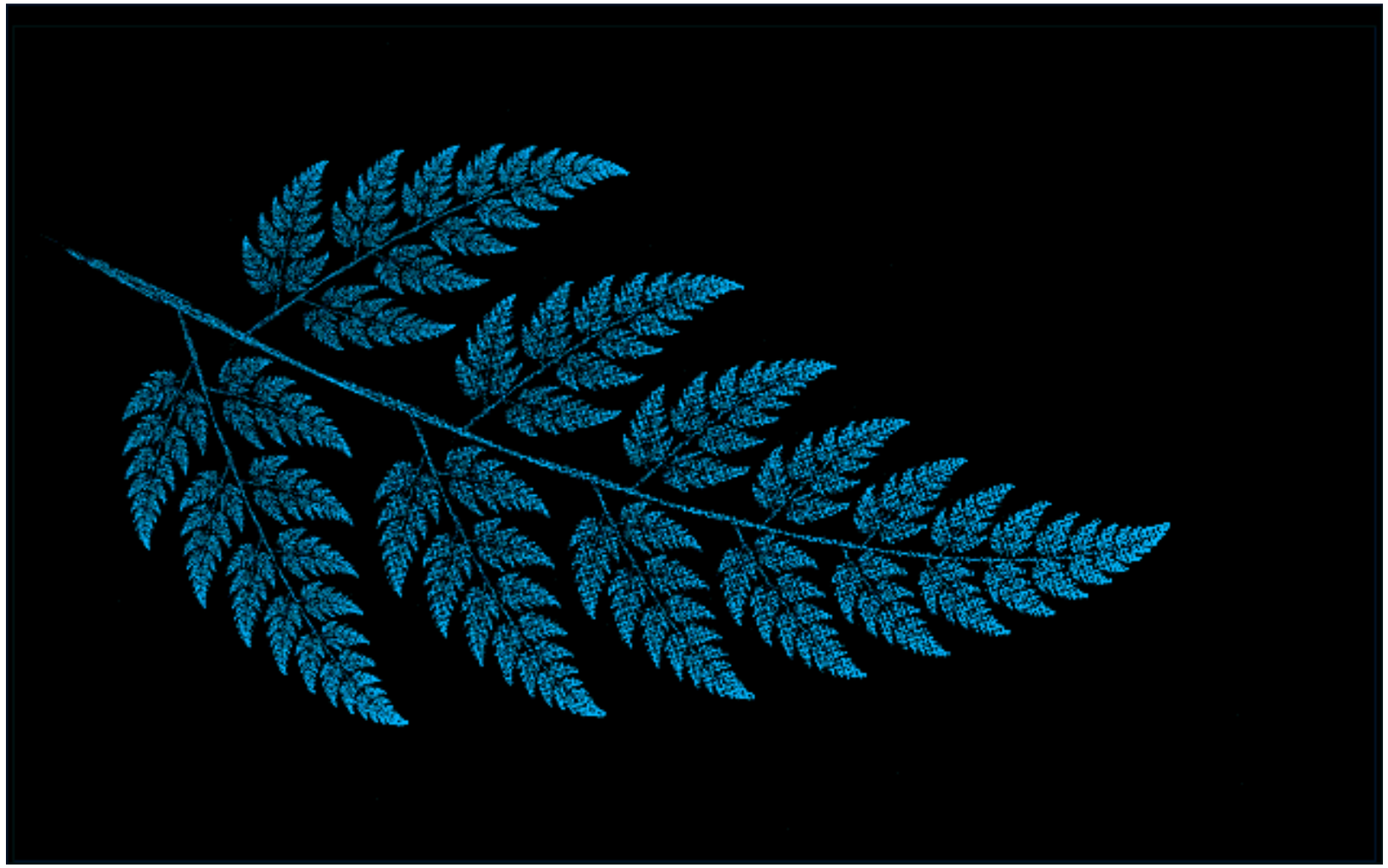
- Structure on all scales.
- There are different kinds of structures at any given scale.
- The structures on different scales are not exactly the same.
- No exact self-similarity.

Statistical self-similarity.

If one is shown a picture of a part of the Mandelbrot set, one can't look at it and find the scale (the level of magnification).

Fractals Everywhere

- Art and Architecture.
- Image processing and storage.
- The Geometry of nature.
- Description of dynamical systems.
- Strange Attractors.
- Many applied sciences including Oil exploration, Ore extraction, etc.
- How does a cookie crumble?

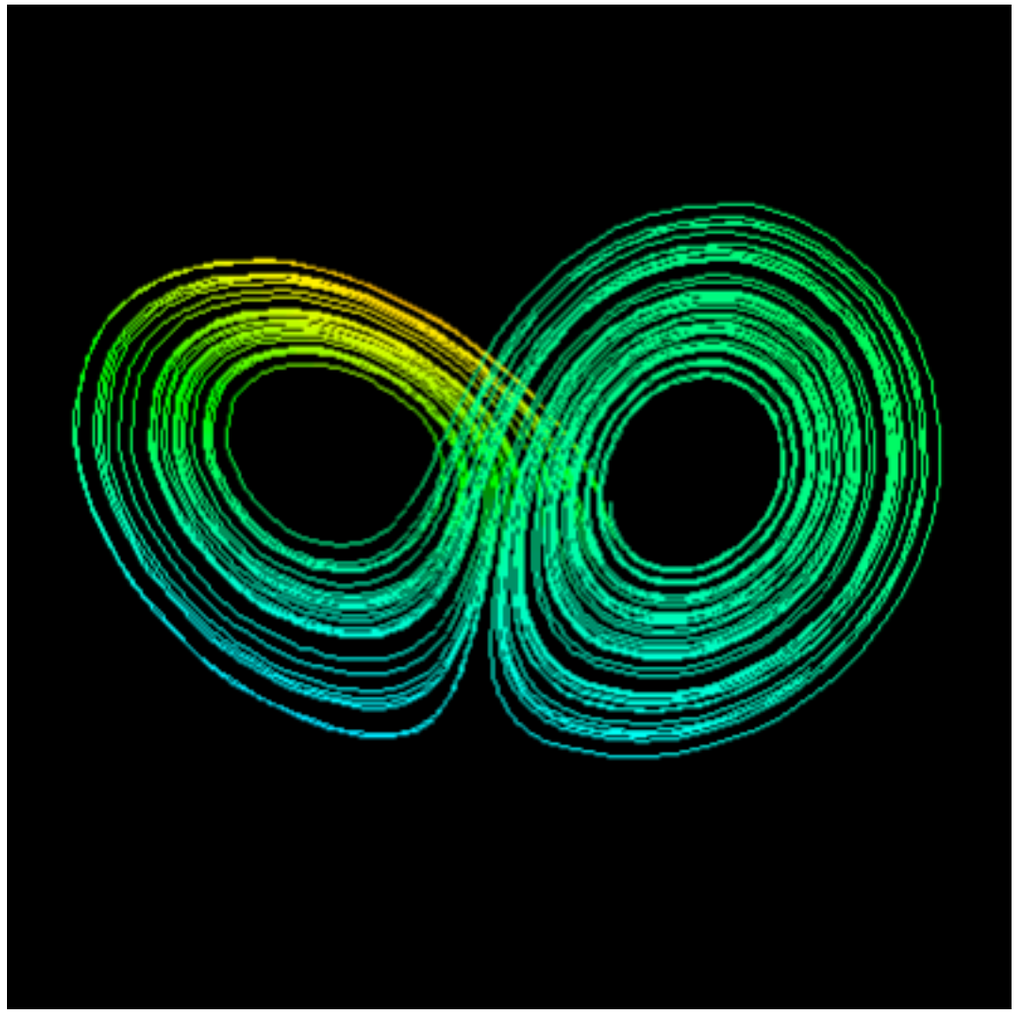


A fractal fern

Frosty the “snowman”. Image courtesy Frank Roussel fractal page.

A tree. Image courtesy E. Stephen Mack.

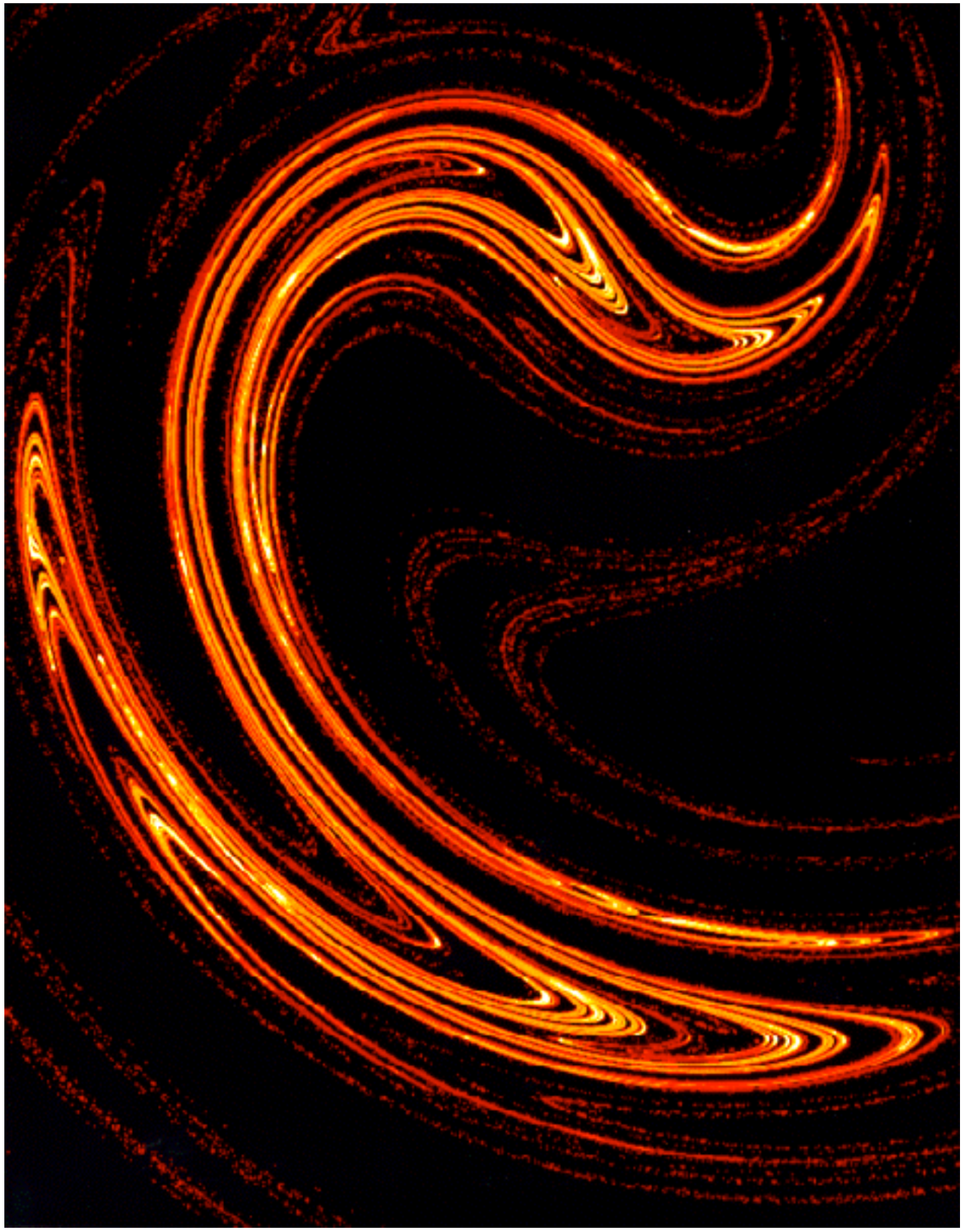
A computer generated Martian landscape. Image courtesy J. C. Sprott.



The Lorenz Attractor

Generated by the Program "Chaotic Flows"

Thanks to John Lindner, Bryan Prusha and Josh Bozeday, University of Wooster



The Ikeda attractor

Image courtesy the University of Maryland Chaos Group

Characterization of Fractals

Box Counting Dimension or Capacity Dimension - D_0.

Draw the set on a graph paper with squares of size ϵ.

Count the number of boxes needed to cover the set $N(\epsilon)$.

Make the size of the grid on the graph paper smaller and count the number of boxes again.

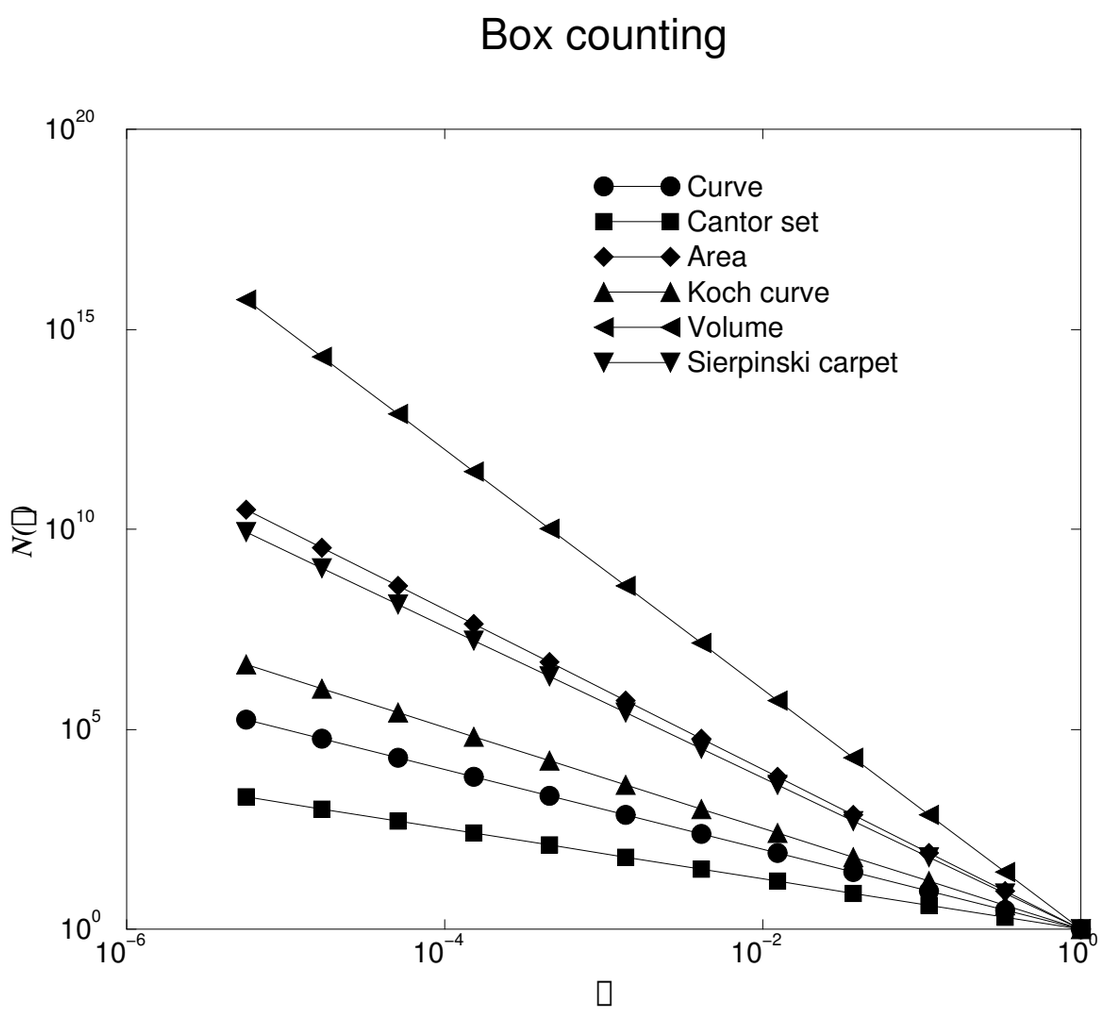
Plot $N(\epsilon)$ as a function of ϵ on a log-log plot.

One obtains a straight line, *i.e.*, it is a scaling law.

The slope of the line gives the negative of the dimension, so that

$$N(\epsilon) \sim \epsilon^{-D_0}$$

Capacity dimension for various sets



This figure shows the capacity dimension of various sets.

Fractals

A fractal is a set with a box counting dimension that is not an integer.

First defined by Benoit Mandelbrot.

- For the Cantor set, $D_0 = 0.6309$.
- For the Koch curve, $D_0 = 1.2619$.
- For the Sierpiński carpet, $D_0 = 1.8928$.
- For the Menger sponge, $D_0 = 2.7712$.

For $0 \leq D_0 \leq 1$, Number of points is infinite but length is zero, for $1 \leq D_0 \leq 2$, length is infinite but area is zero and for $2 \leq D_0 \leq 3$, area is infinite but volume is zero.

Scaling Range

- A non integer fractal dimension D_0 between 1 and 2 measures the roughness of the curve.
- A curve occurring in nature cannot be rough on every scale all the way down to zero. There is a cutoff at Atomic scales.
- On the log-log plot, there is a scaling range in which one obtains a straight line with a slope equal to $-D_0$.
- There is a cutoff both at small scales and at large scales where the behavior deviates from this straight line.

- The object looks like a fractal only on scales in the scaling range.
- On scales much bigger than the object, it looks like a point $D_0 = 0$ and on very small scales, it looks smooth and has a finite volume $D_0 = 3$.
- We can have objects which have no scaling range, so that the fractal dimension depends on the scale we are looking at.
- We can have objects whose fractal dimension changes from one point to another. Such objects have a spectrum of Pointwise dimensions.
- Multifractals - Dimension Spectrum.

Summary

- We are usually interested in describing a system at many scales.
- A system that is the same at different scales is self-similar.
- Systems that are exactly self-similar lack a preferred scale.
- Systems without a preferred scale obey scaling laws.
- Empirically, it is observed that scaling laws describe a wide variety of phenomena.
- A geometrical object that is self similar is called a fractal. A fractal has structure on arbitrarily small scales.

Summary

- Iterations of nonlinear dynamical systems lead to structures on arbitrarily small scales and hence to fractals.
- We can characterize a fractal by its Box-counting dimension.
- Fractals are usually not exactly self-similar. They are self-similar in a statistical sense.
- The Box-counting yields the fractal dimension over a scaling range.
- Often, the box counting dimension does not uniquely characterize a fractal. Usually, the fractal object has a spectrum of dimensions.