A Discontinuous Differential Equation

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Abstract

This paper explores a system of discontinuous differential equations involving floor functions[1]. Such equations don't make sense in terms of general theorem concerning existence and uniqueness of solutions of ordinary differential equations. The focus of this paper is on possible methods to regularize this system of "odd" equations. Two types of regularizations are discussed and the behavior of corresponding solutions is investigated.

1 Introduction

This paper concerns a set of discontinuous differential equations which describe a system of coupled maps. These equations involve floor functions and have the following form:

$$\frac{dc_j(t)}{dt} = floor(c_{j-1}(t)) - 2floor(c_j(t)) + floor(c_{j+1}(t)) \tag{1}$$

Here j is site index with $1 \leq j \leq N$ and $floor(z)$ represents the floor function of number z.

A general first order differential equation with some initial condition takes the form $y'(t) = f(t, y)$ with $y(t_0) = y_0$. General theorem concerning the existence and uniqueness of its solution requires that $f(t, y)$ and its partial derivative with respect to y be continuous in a certain neighborhood region. However, the floor function $floor(z)$ is not continuous itself, not mentioning any derivative. Thus, equation (1) does not make sense in its original form.

[1]Floor function is a function that rounds the specified number down and returns the largest number that is less than or equal to the specified number.

Our goal in this paper is to find possible methods to regularize equation (1) such that the equations we have after regularizations do make sense.

The paper is organized as follows. Section 2 shortly reviews the origin of this problem. Section 3 presents the first type of regularization in which we replace the derivative with forward difference. The behavior of the solution is discussed both numerically and analytically. In section 4 we move on to present the second type of regularization in which we use a series of functions to approximate floor function. The behavior of the solution is discussed numerically in this section, and finally in section 5 we present some further questions for this problem.

2 Origin of Problem

The following system of equations is used to describe the dynamics of sliding charge-density waves (CDW's):

$$x_j(\tau+1) = x_j(\tau) + floor[k \sum_{i(nn)} (x_i(\tau) - x_j(\tau)) + A(\tau)]$$

Here i and j are site indices, τ is time index and the sum is over nearest neighbors. This system of equations describes the evolution of the positions x_j of N particles in deep periodic potential wells, with nearest neighbor particles connected by springs of spring constant $k \ll 1$, in the presence of force impulse $A(\tau)$. The floor functions arise because after each pulse every particle falls into the nearest potential minimum [1].

Let $\tilde{c_j}(\tau) = k \sum_{i(nn)} (x_i(\tau) - x_j(\tau))$ and we obtain a new system of equations in the following form:

$$\tilde{c_j}(\tau+1) - \tilde{c_j}(\tau) = k \sum_{i(nn)} [floor(\tilde{c_i}(\tau) + A(\tau)) - floor(\tilde{c_j}(\tau) + A(\tau))]$$

Consider a one-dimensional case with force impulse $A(\tau) = 0$ and we have:

$$\tilde{c_j}(\tau+1) - \tilde{c_j}(\tau) = k[floor(\tilde{c}_{j-1}(\tau)) - 2floor(\tilde{c_j}(\tau)) + floor(\tilde{c}_{j+1}(\tau))]$$

Furthermore, if we take $t = \tau * k$ to be rescaled time index and $\tilde{c}(\tau) = c(\tau * k) = c(t)$ and we have:

$$c_j(t+k) - c_j(k) = k[floor(c_{j-1}(t)) - 2floor(c_j(t)) + floor(c_{j+1}(t))]$$

The limit $k \to 0$ seems to lead us to our system of equation (1) but before we can do that, one question needs to be answered: does the solution of the previous equation converge in the limit $k \to 0$? This provides us an idea of regularizing the system of equations (1).

3 Discretization

In this section, we replace the derivative in the left-hand side with forward difference formula and investigate the behavior of the solution both numerically and analytically. Our goal is to show the strong convergence of c_j as well as the weak convergence of $floor(c_j)$ in the limit $k \to 0$.

Now the problem is expressed as follows:

$$c_j(t+k) - c_j(k) = k[floor(c_{j-1}(t)) - 2floor(c_j(t)) + floor(c_{j+1}(t))] \quad (2)$$

with fixed boundary conditions:

$$c_0 = a, c_{N+1} = b \text{ where } a, b \in \aleph \quad (3)$$

and random initial conditions:

$$c_j(t=0) = \text{random number}, \; 1 \le j \le N \quad (4)$$

Due to the special structure of equation (2), we can reduce the boundary conditions (3) to a simpler form. First, let $l = b - a$ and we can set $c_0 = 0, c_{N+1} = l$ instead of $c_0 = a, c_{N+1} = b$. This is because the following transform $c_i \to c_i - a$, $0 \le i \le N+1$ does not change the dynamics of our system[2]. Second, we can furthermore require that $0 \le l \le N$. The proof is similar: if $l = p * (N+1) + q$ with $0 \le q < N+1$, the following transform $c_i \to c_i - i * p, 0 \le i \le N+1$ does not change the dynamics of the system. To sum up, we can reduce the boundary conditions to the following form:

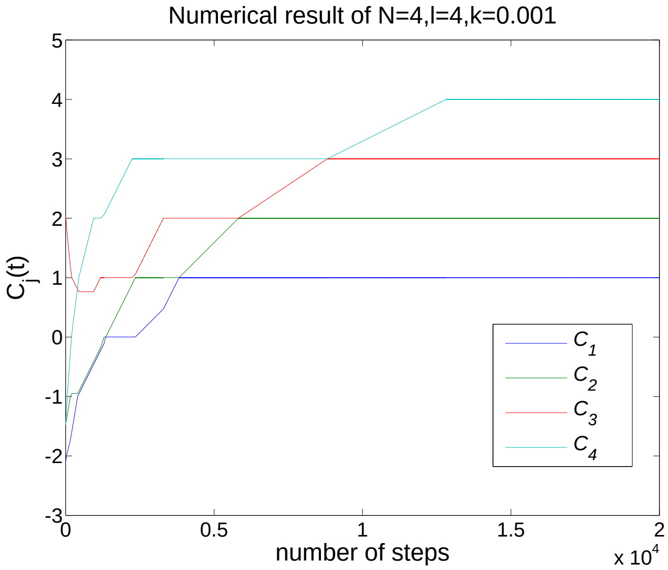
$$c_0 = 0, c_{N+1} = l, \text{ with } 0 \le l \le N \quad (5)$$

We first focus on long-term behavior of solution of equations (2) with initial conditions (4) and boundary conditions (5). We conducted numerical simulations using different combinations of N and l with certain values of k and observed interesting results.

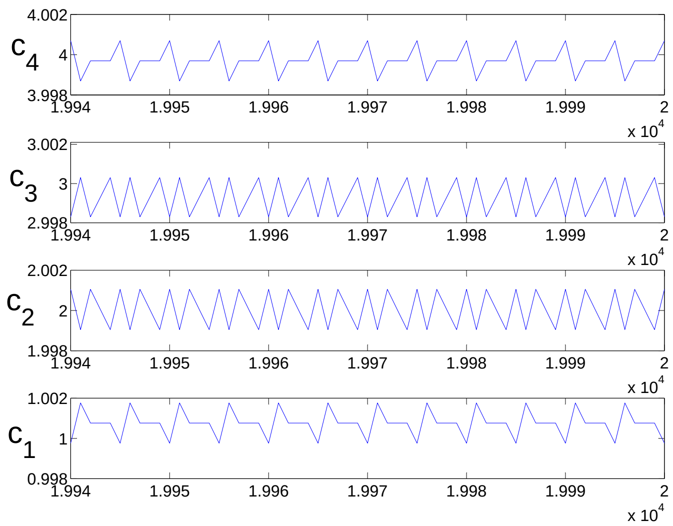
As shown in figure 1(a) where $N = 4, l = 4$, the system seems to reach a stable final state after approximately 12500 steps[3](the horizontal part in the figure) where c_j takes the value of an integer (specifically the integer j for $1 \le j \le 4$ in this case). This is not precise if we take a closer look at the last 60 steps of figure 1(a). Figure 1(b) shows that each site c_j is oscillating

[2]The solution depends on initial conditions. After such transform, each set of initial conditions also needs to be transformed.

[3]One step means one operation on the system of equations (2). Namely, the operation of calculating $c_j(t+k)$ from $c_j(t)$ for a given step size k. The number of steps m can be seen as a measure of time $t = m * k$.



(a)



(b)

Figure 1: (a) Plot of c_j versus number of steps for equations (2) with boundary conditions (5) and random initial conditions (4). System parameters are $k = 0.001, N = 4, l = 4$. (b) Plot of c_j versus number of steps for the last 60 steps of (a).

around an integer (specifically the integer j for $1 \leq j \leq 4$ in this case). It means that for each j, $floor(c_j)$ is also oscillating by taking the values of both j and $j-1$ within each cycle of oscillation. The period of the cycle of oscillation[4] is 5 for every site.

$N = 4, l = 4$ is one of the two typical kinds of combination in the sense that l and $N+1$ are coprime in this case. Figure 2(a) and 2(b) are numerical results with system parameters $N = 3, l = 2$. Note that now l and $N+1$ are not coprime any more. As shown in figure 2(a),in terms of the long-term behavior of solution we still have stable structure after certain number of steps. However, figure 2(b) shows some difference. While c_1 and c_3 are still oscillating around an integer, the value of c_2 is always larger than 1. It means that $floor(c_2)$ remains unchanged instead of oscillating. The period of cycle is 2 for every site.

Figure 1(a) and 2(a) enable us to calculate the average value of $floor(c_j)$ in detail. Let u_j denote the average value of $floor(c_j)$ over a complete cycle. When $N = 4, l = 4$, we have: $u_1 = \frac{4}{5}$, $u_2 = \frac{8}{5}$, $u_3 = \frac{12}{5}$, $u_4 = \frac{16}{5}$. When $N = 3, l = 2$, we have: $u_1 = \frac{1}{2}$, $u_2 = 1$, $u_3 = \frac{3}{2}$. Another point worth mentioning is that the amplitude of oscillation in both figure 1(a) and figure 2(a) is 0.002, given that $k = 0.001$.

Based on these two cases and other numerical simulations we have conducted, we propose the following conjecture concerning the long-term behavior of solutions of equations (2):

Conjecture:

1. Periodic behavior, specifically oscillation[5], must exist.

2. The period of cycle: suppose $\frac{l}{N+1}$ can be fully reduced to $\frac{q}{p}$, then the period of cycle is p.

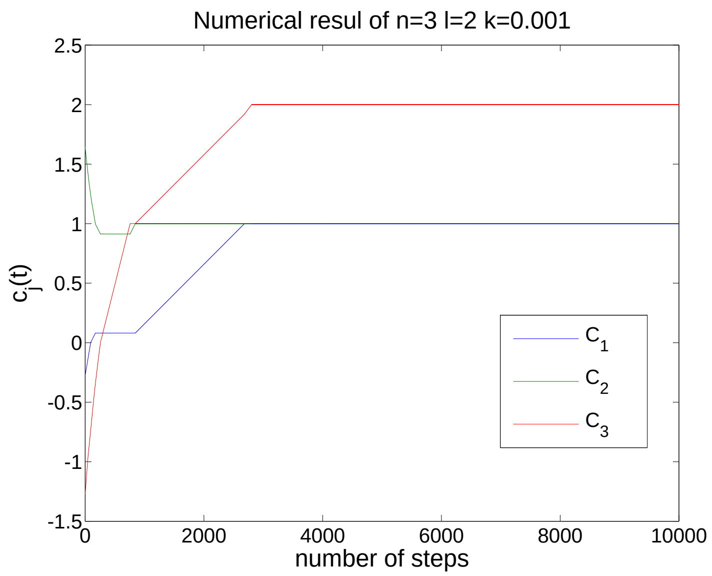
3. Average value of $floor(c_j)$ over a complete cycle: $\frac{j*l}{N+1}$ for $1 \leq j \leq N$.

4. Pattern of oscillation: (1) If $\frac{j*l}{N+1}$ is not an integer, then c_j oscillates around the integer $floor(\frac{j*l}{N+1}) + 1$ and $floor(c_j)$ takes the value of both $floor(\frac{j*l}{N+1})$ and $floor(\frac{j*l}{N+1}) + 1$. Otherwise, c_j oscillates within the interval $(\frac{j*l}{N+1}, \frac{j*l}{N+1} + 1)$ and $floor(c_j)$ can only take the value of $\frac{j*l}{N+1}$. (2) The amplitude of oscillation is proportional to $O(k)$ for any site.

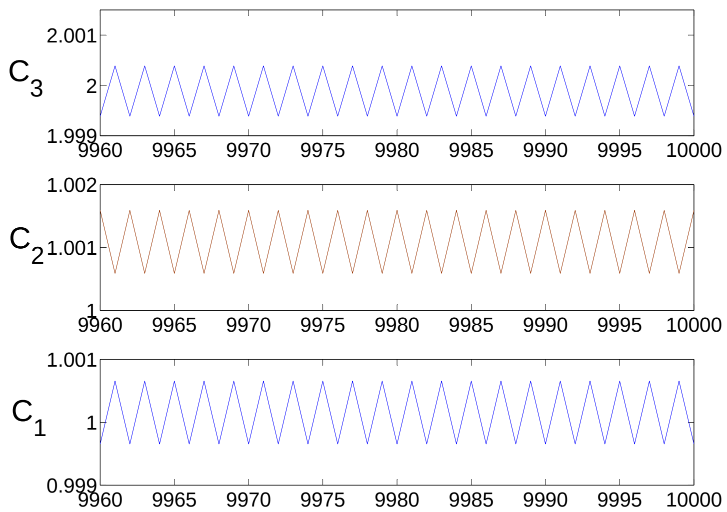
It is beneficial to add some remarks. (1) The conjecture is consistent with all the results we have so far. (2) A special case concerning the cycle period is when $l = 0$. In this case the period is actually 1 meaning that c_j stays

[4]in the sense of step number

[5]The special case of a period of 1 also counts as oscillation in this paper.



(a)



(b)

Figure 2: (a) Plot of c_j versus number of steps for equations (2) with boundary conditions (5) and random initial conditions (4). System parameters are $k = 0.001, N = 3, l = 2$. (b) Plot of c_j versus number of steps for the last 40 steps of (a).

constant for each j. We consider this case as a special kind of oscillation in this paper. (3) When $\frac{j*l}{N+1}$ is an integer, c_j can stay anywhere within the interval $(\frac{j*l}{N+1}, \frac{j*l}{N+1}+1)$, oscillating or being constant. As an example, consider the simple case with $N=1, l=0$ and $c_j(0)=c$ where $c \in (0,1)$. (4) Most importantly, the conjecture is independent of the exact value of step size k.

We now move on to an incomplete proof of the conjecture.

Proof of part 1. Part 1 is the theoretical foundation of the whole conjecture. The basic idea of proving part 1 is that the system only has finite number of states[6]. Suppose $c_{j_0} = \max\{c_j, 1 \le j \le N\}$ and $floor(c_{j_0}) \ge l$, then c_{j_0} is non-increasing due to the special structure of equations (2). Similarly, if $c_{j_1} = \max\{c_j, 1 \le j \le N\}$ and $floor(c_{j_1}) \le 0$, then c_{j_1} is non-decreasing. Therefore, the whole system is bounded by the initial conditions plus the boundary conditions (5). For every j, moreover, c_j can only change its value by a multiple of k due to equations (2). As a result, the system can only have a finite number of states and will inevitably return to an old state and start to behave in a periodical way.

Proof of part 3. Suppose the cycle period is p meaning that $c_j(t+p*k) = c_j(t)$ for each j. u_j, the average value of c_j, can be expressed as:

$$u_j = \frac{1}{p}\sum_{i=1}^{p} floor(c_j(t+i*k)) \quad 1 \le j \le N \tag{6}$$

Sum the equations (2) over a complete cycle and we have:

$$\begin{aligned} u_2 + u_0 - 2u_1 &= 0 \\ u_3 + u_1 - 2u_2 &= 0 \\ &\cdots \\ u_{N+1} + u_{N-1} - 2u_N &= 0 \end{aligned}$$

Note that $u_0 = 0, u_{N+1} = l$ and the solution of this system of equations is:

$$u_j = \frac{j*l}{N+1} \quad 1 \le j \le N$$

Due to the definition (6), it is natural to require that $p*u_1 = p*\frac{l}{N+1} \in \aleph$. Therefore if $\frac{l}{N+1}$ can be fully reduced to $\frac{u}{v}$, p must be a multiple of v. A special case is when $l=0$ which does not give any restriction on p. In this case, we can expect a cycle of period 1. Actually, this is also the only case that we may have a cycle of period 1. So far in this paragraph is an

[6]A state of the system means a set of numbers $\{c_j, 1 \le j \le N\}$.

incomplete proof of part 2. We will discuss the proof of the rest part of our conjecture in the final section.

The conjecture implies both the strong convergence of c_j and the weak convergence of $floor(c_j)$ in the limit $k \to 0$ in the long term. We consider the simple case when l and $N+1$ are coprime. As the value of k decreases, the amplitude of oscillation also decreases. Moreover, the average value of $floor(c_j)$ is restricted to be $\frac{j*l}{N+1}$. Therefore, c_j must oscillate around the integer $floor(\frac{j*l}{N+1})+1$ which guarantees the strong convergence of c_j. However, the strong convergence of c_j should be understood in the sense of the weak convergence of $floor(c_j)$. Independent of the strong convergence of c_j, $floor(c_j)$ always oscillates between two values: $floor(\frac{j*l}{N+1})$ and $floor(\frac{j*l}{N+1})+1$. The average value of $floor(c_j)$ over a complete cycle is a constant which implies the weak convergence of $floor(c_j)$.

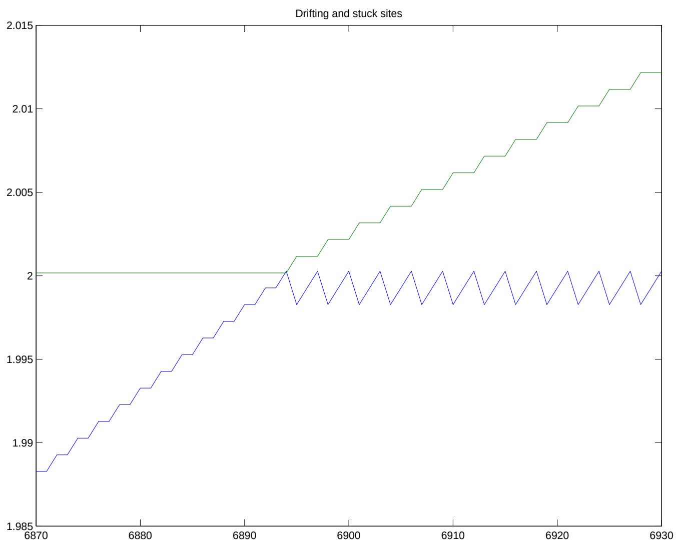


Figure 3: Plot of a transition. It is part of a numerical simulation with system parameters $N=4, l=4, k=0.001$. As shown in the figure, c_2 is drifting at the beginning and c_3 is oscillating. When c_2 reaches the integer 2, the transition occurs. After the transition, c_2 starts to oscillate while c_3 starts to drift.

We need more careful discussion for the normal case when l and $N+1$ may not be coprime. In such case, we argue that c_j of those sites such that $\frac{j*l}{N+1} \in \aleph$ is determined by the initial conditions of the system[7] but it needs theoretical proof. However, the weak convergence of $floor(c_j)$ is still guaranteed.

We now apply the idea of oscillation and time averaging procedure in the proof of conjecture part 3 to the time evolving process of the system.

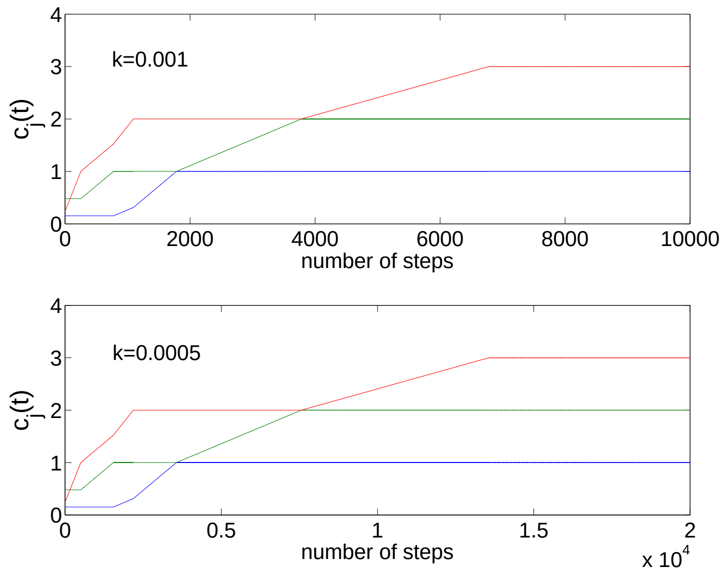


Figure 4: Plots of c_j versus number of steps with same system parameters $N = 3, l = 3$ and same initial conditions. The upper plot is the result of $k = 0.001$ and the lower one is using $k = 0.0005$. If we use time index $t = m * k$ where m is the step number, the two plots look the same in this scale.

As shown in figure 1(a) and 2(a), there are some transient horizontal regions before the systems stop evolving[8]. We argue that there are two types of sites: "drifting" sites and "stuck" sites [2]. Namely, a drifting site is in transit between two differnent numbers (may not be two integers) and a stuck site oscillates periodically around a number (may not be an integer). When a site changes its pattern or oscillation of drifting, or changes from drifting to oscillating or in the opposite direction, we call it a transition.

[7]Recall that c_j can stay anywhere within the interval $(\frac{j*l}{N+1}, \frac{j*l}{N+1} + 1)$ in this case.

[8]The expression here may not be so accurate. We consider the appearence of periodic behavior in the long times as the end of evolving

A transition occurs when a drifting site reaches the value of an integer (as shown in figure (3)). The whole evolving process can be considered as a sequence of drifting or oscillating processes and transitions.

When k is sufficiently small, every drifting site c_j provides a stationary environment for its neighbors in the sense that $floor(c_j)$ does not change during the drifting process. Suppose c_{j_0} and c_{j_1} are a pair of drifting sites within a certain period of time. They serve as a set of fixed boundary conditions for the sites between them[9]. Therefore, we can apply the time averaging procedure as before and expect strong convergence of c_j and weak convergence of $floor(c_j)$ for $j_0 < j < j_1$.

We have argued that the amplitude of oscillation is proportional to $O(k)$. Moreover, it is shown numerically that the number of steps a transition needs remains finite in the limit $k \to 0$. Thus the time a transition lasts for is propotional to $O(k)$. As shown in figure (4), the dynamics of the systems are almost the same for $k = 0.001$ and $k = 0.0005$. In fact the slight difference is proportional to $O(k)$ and is invisible in this scale.

So far, it is reasonable to conclude that we can define the $k \to 0$ limit of the system of equations (2). Therefore, the regularization of discretization is well-defined and reasonable.

4 Approximating Floor Functions

The basic idea of this section is to construct a sequence of continuously differentiable functions $q_\epsilon(x)$ to approximate floor function $floor(x)$ and investigate the behavior of solutions in the limit $\epsilon \to 0$.

Let $frac(x)$ denote the fractional part[10] of real number x. We utilize hyperbolic tangent function $\tanh(x)$ and define $q_\epsilon(x)$ as follows:

$$q_\epsilon(x) = \begin{cases} floor(x) + \frac{1}{2}\left\{ \tanh\left(\frac{\frac{(1-\epsilon)+1}{2} - frac(x)}{2[frac(x)-(1-\epsilon)][frac(x)-1]}\right) + 1 \right\} & frac(x) \in (1-\epsilon, 1) \\ floor(x) & frac(x) \in [0, 1-\epsilon] \end{cases}$$

Figure (5) demonstrates how this sequence of functions $q_\epsilon(x)$ approximate $floor(x)$ in the limit $\epsilon \to 0$.

[9]It also explains why the cycle is less than $N+1$ when l and $N+1$ are not coprime. When $\frac{j*l}{N+1}$ is an integer, $floor(\frac{j*l}{N+1})$ stays constant and the j^{th} site cuts the whole chain into pieces. Each piece has fixed boundary conditions and can oscillate separately with a period of $p < N+1$.

[10]Therefore $frac(x) = x - floor(x)$.

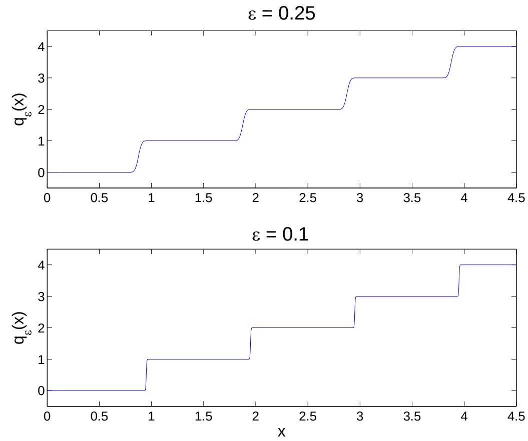
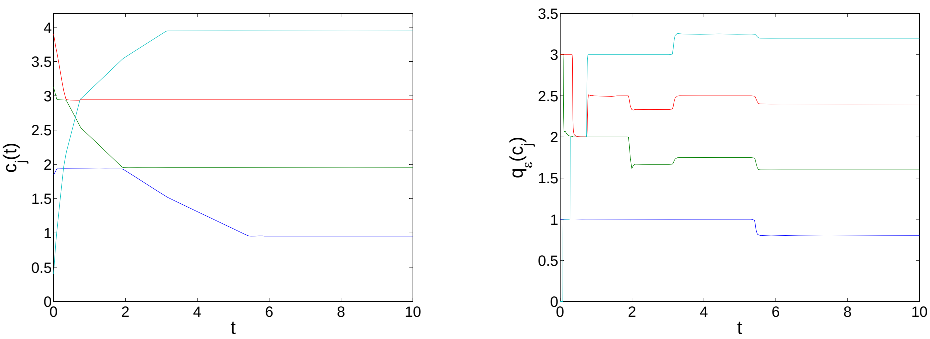


Figure 5: Plots of $q_\epsilon(x)$ with $\epsilon = 0.25$ (b) $\epsilon = 0.1$.



(a) (b)

Figure 6: Numerical results of (a) c_j vs. t and (b) $q_\epsilon(c_j(t))$ vs. t with system parameters $N = 4, l = 4, \epsilon = 0.1$.

Due to numerical results, we can expect strong convergence of both c_j and $q_\epsilon(j)$. We present this result using a simple example with system parameters $N = 4, l = 4$. Figure 6(a) shows that we can still treat the integer j as the limit of c_j, which is consistent with the results we obtain using regularization of discretization[11]. Figure 6(b) shows that we even have strong convergence of $q_\epsilon(c_j)$ and the limit of convergence is $\frac{j*l}{N+1}$. Recall that we only have weak convergence of $floor(c_j)$ when using regularization of discretization. However, the value of u_j, which represents the average value of $floor(c_j)$, is exactly $\frac{j*l}{N+1}$.

The example with $N = 4, l = 4, \epsilon = 0.1$ itself provides insight into the behavior of the system, especially the possible limits of c_j and $q_\epsilon(c_j)$ in the limit $\epsilon \to 0$. The strong convergence can be demonstrated directly by calculating numerically the difference between real values of c_j and $q_\epsilon(c_j)$ and their expected limits. As a result, we can argue that the regularization of approximating floor functions is reasonable.

5 Conclusion and Further Questions

In this paper we have shown that the two types of regularization, namely discretization of the derivative and approximation of floor functions, are well-defined and reasonable. They both imply strong convergence of c_j in a certain limit and moreover, they share the same limit of convergence. However, the first type of regularization features a weak convergence of $floor(c_j)$ while the second type features a strong convergence of $q_\epsilon(c_j)$ in contrast. They also share the same limit of convergence.

Some further questions concern the theoretical proof of the results presented in section 3 and 4. One idea to prove the conjecture in section 3 is to show that there can not be any drifting sites when the system is oscillating. Because u_j, the average value of $floor(c_j)$, is a constant, $floor(c_j)$ can only take two values if u_j is not an integer or one value if u_j is an integer. Again, since there are no drifting sites, the amplitude of oscillation can only be $O(k)$. Therefore, the pattern of the oscillation is fixed. We can move on to prove part 2 of the conjecture concerning cycle period based on the pattern of the oscillation.

I would like to thank Professor Shankar Venkataramani here for his help with my research project.

[11] Recall that in this case, c_j oscillates around the integer j with an oscillation amplitude proportional to $O(k)$.

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