MATH 583B

due April 23, 2019

Principles and Methods of Applied Mathematics

Section 001, Spring 2019 (instructor: Misha Stepanov)

Homework 10

1. Find the curve $y(x) > 0$, with $y(x_1) = y_1$ and $y(x_2) = y_2$, that minimizes the functional

$$\int_{x_1}^{x_2} \mathrm{d}x \, \frac{\sqrt{1 + y_x^2(x)}}{y(x)}$$

You may think about the problem as finding the path that minimizes travel time, when the speed limit is $v(x, y) = y$. Another interpretation is: the shortest paths in the upper half-plane with the metric $(\mathrm{d}s)^2 = \big((\mathrm{d}x)^2 + (\mathrm{d}y)^2\big) \,/\, y^2$.

2. Find the shape of a chain that has the length $\sqrt{2} < L < 2$ with its ends at the points $(0, 0)$ and $(1, 1)$ in xy-plane. The line $y = 0$ is the ground level, and the chain can't go below it. (You may assume that you can solve any system of equations involving transcendental functions or an integral representing the length of a curve, *etc.*)

3. Minimize the functional $\displaystyle\int_{-1}^{1} \mathrm{d}x \, \Big(K\big(y_x - 1\big)^2 + y^2\Big)$, where $K > 0$, while $y(-1)$ and $y(1)$ are arbitrary. Sketch the solution $y(x)$ for small K and for large K.

4. Consider the system $\ddot{x} = -U'(x)$ with

$$U(x) = \begin{cases} x, & x > 0; \\ +\infty, & x < 0. \end{cases}$$

Draw the phase portrait (show the qualitative features accurately).

5. Consider the system $\ddot{x} = -U'(x)$ with

$$U(x) = \frac{x^2}{2} - \frac{2x^3}{3} + \frac{x^4}{4}$$

Draw the phase portrait (show the qualitative features accurately).

6. Consider the propagation of surface waves, with g being the acceleration of gravity, ρ being the volume density of the fluid, and σ being the surface tension. Let $k = 2\pi/\lambda$ be the wave number (λ is the wavelength). Form (from g, ρ, σ, and k) combinations with the dimension of time. Deduce (up to a constant factor) the dispersion laws (*i.e.*, how the frequency ω depends on k) for gravitational waves (small k limit, or σ is not important) and capillary waves (large k limit, or g is not important).