

ON LOCAL REPRESENTATION DENSITIES OF HERMITIAN FORMS AND SPECIAL CYCLES II

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ABSTRACT. In this paper, we prove that there are certain relations among representation densities and provide an efficient way to compute representation densities by using these relations. As an application, we compute some arithmetic intersection numbers of special cycles on unitary Shimura varieties and propose a conjecture on these.

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1. INTRODUCTION

In [KR11] and [KR14], Kudla and Rapoport made a precise conjectural formula for the arithmetic intersection numbers of special cycles on unitary Shimura varieties with hyperspecial level structure. This is called the Kudla-Rapoport conjecture. This is proved by Li and Zhang in 2019 [LZ22]. In [Cho22a], we made a variant of the Kudla-Rapoport conjecture in the case that the unitary Shimura variety has minuscule parahoric level structures. This conjectural formula predicts a relation between the arithmetic intersection numbers of special cycles on unitary Rapoport-Zink spaces and the derivative of some representation densities of hermitian forms. To prove this conjecture, one may hope to compute the arithmetic intersection numbers of special cycles directly and compare these with the analytic side. However, there is currently no way to compute the geometric side directly except some low dimensional cases. For this reason, we hope to get an idea from the analytic side of the conjecture since it is at least possible to compute.

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The analytic side of the Kudla-Rapoport conjecture is related to representation densities of hermitian forms. Therefore, we would like to test our geometric guesses by computing some representation densities. Let p be an odd prime and let F be an unramified extension of \mathbb{Q}_p with ring of integers O_F and residue field \mathbb{F}_q . Let E be a quadratic unramified extension of F with ring of integers O_E . For $n \times n$ hermitian matrices A and B , we define the representation density $\alpha(A, B)$ by

$$\alpha(A, B) = \lim_{d \rightarrow \infty} (q^{-d})^{n(2m-n)} |\{X \in M_{m,n}(O_E/\pi^d O_E) | A[X] \equiv B \pmod{\pi^d}\}|.$$

We have an explicit formula for these representation densities in [Hir98], however, the actual computation is very complicated. For example, for $A = A_{40} := \begin{pmatrix} p^4 & 0 \\ 0 & 1 \end{pmatrix}$ and $B = A_{42} := \begin{pmatrix} p^4 & 0 \\ 0 & p^2 \end{pmatrix}$, we used several pages to compute that $\alpha(A_{40}, A_{42}) = q^6(1 + \frac{1}{q})^2$ with the above formula. Therefore, we need a more efficient way to compute representation densities.

In the present paper, we prove that there are certain relations among representation densities and by using this, we give an efficient way to compute representation densities. We also apply this method to compute some arithmetic intersection numbers of special cycles and then give a conjecture on these.

We now describe our results in more detail. We define the set of hermitian matrices $X_n(E) = \{X \in GL_n(E) | {}^t X^* = X\}$, and $X_n(O_E) = X_n(E) \cap M_{n,n}(O_E)$. Let Λ_n^+ be the set of $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$ such that $\lambda_1 \geq \dots \geq \lambda_n \geq 0$. For $\lambda \in \Lambda_n^+$, we write A_λ for $\text{diag}(p^{\lambda_1}, \dots, p^{\lambda_n})$. Then, the complete set of representatives of $GL_n(O_E)$ -equivalence classes in $X_n(O_E)$ is given by $\{\lambda \in \Lambda_n^+ | A_\lambda\}$.

Let $\Lambda_{n,s}^+$ be the subset of Λ_n^+ such that

$$\Lambda_{n,s}^+ = \{\xi = (\xi_1, \dots, \xi_n) \in \Lambda_n^+ \mid \xi_{n-s} \geq 1, \xi_{n-s+1} = \dots = \xi_n = 0\}.$$

For $\xi \in \Lambda_{n,s}^+$, we define $\xi_{a,b}^+$ as

$$\xi_{a,b}^+ := (\xi_1, \dots, \xi_{n-s}, \overbrace{2, \dots, 2}^a, \overbrace{1, \dots, 1}^b, \overbrace{0, \dots, 0}^{s-a-b})$$

For integers $1 \leq s \leq n$, $0 \leq i \leq s$, we define $d_{n,s,i}$ by the coefficients of the polynomial

$$\begin{aligned} & \sum_{i=0}^s d_{n,s,i} X^i \\ &= (1 - (-q)^{-n} X)(1 - (-q)^{-n+1} X) \dots (1 - (-q)^{-n+s-1} X) \end{aligned}$$

Then, our main theorem is as follows.

Theorem 1.1. *(Theorem 2.5) Let s and n be integers such that $1 \leq s \leq n$, and let $\xi \in \Lambda_{n,s}^+$. Then, for any $B \in X_n(O_E)$ such that $\pi^{-1}B$ is also*

integral (i.e., the representative of B is A_λ for some $\lambda \in \Lambda_n^+$ such that $\lambda \geq (1, \dots, 1)$), we have the following equality.

$$\sum_{i=0}^s d_{n,s,i} \alpha(A_{\xi_{0,i}^+}, B) - \frac{(-1)^s}{(-q)^{ns}} \sum_{i=0}^s d_{n,s,i} \alpha(A_{\xi_{i,s-i}^+}, B) = 0.$$

For example, when $n = 2$, the theorem implies that for any B such that $\pi^{-1}B$ is integral and $\lambda \geq 1$, we have

$$\alpha(A_{\lambda 0}, B) = \frac{1}{q^4} \alpha(A_{\lambda 2}, B)$$

Therefore, we can simply compute that

$$\alpha(A_{40}, A_{42}) = \frac{1}{q^4} \alpha(A_{42}, A_{42}) = q^4 \alpha(A_{20}, A_{20}) = q^4 \left(1 + \frac{1}{q}\right) \alpha(A_2, A_2) = q^6 \left(1 + \frac{1}{q}\right)^2.$$

Our theorem is true for arbitrary n . Therefore, this can be applied to compute any representation densities.

When we apply the theorem to compute representation densities, it is better to use normalized representation densities and a normalized version of the theorem. Also, the Cho-Yamauchi's formula for the analytic part of the Kudla-Rapoport conjecture ([LZ22, Theorem 3.5.1], [Cho22b, Theorem 4.7]) is formulated in terms of normalized representation densities.

For $A, B \in X_n(O_E)$, we define the normalized representation density $A(B)$ as

$$A(B) := \frac{\alpha(A, B)}{\alpha(A, A)}.$$

Then this has the following reductions.

Proposition 1.2. (Proposition 2.14) For $\xi, \lambda \in \Lambda_{n-1}^+$, we consider $(\xi, 0), (\lambda, 0)$ in Λ_n^+ . Then, we have

$$A_{\xi,0}(A_{\lambda,0}) = A_\xi(A_\lambda).$$

Proposition 1.3. (Proposition 2.15) For ξ, λ in Λ_n^+ , such that $\xi, \lambda \geq (1, 1, \dots, 1)$, we consider

$$\begin{aligned} \xi - 1 &:= (\xi_1 - 1, \dots, \xi_n - 1) \in \Lambda_n^+ \\ \lambda - 1 &:= (\lambda_1 - 1, \dots, \lambda_n - 1) \in \Lambda_n^+. \end{aligned}$$

Then, we have

$$A_\xi(A_\lambda) = A_{\xi-1}(A_{\lambda-1}).$$

Also, Theorem 1.1 gives the following corollaries when $n = 2$ and $n = 3$.

Corollary 1.4 ($n = 2$). Assume that $\lambda \geq 3$. For any $B \in X_2(O_E)$ such that $\pi^{-1}B$ is integral, the following equalities hold.

- (1) $A_{00}(B) = (q+1)A_{11}(B) - qA_{22}(B)$.
- (2) $A_{10}(B) = A_{21}(B)$.
- (3) $A_{20}(B) = q(q-1)A_{22}(B)$.
- (4) $A_{\lambda 0}(B) = q^2 A_{\lambda 2}(B)$.

Corollary 1.5 ($n = 3$). *(Corollary 2.20) Assume that $\lambda, \kappa \geq 3$. For any $B \in X_3(O_E)$ such that $\pi^{-1}B$ is integral, the following equalities hold.*

- (1) $A_{000}(B) = (q+1)A_{211}(B) - q^3A_{222}(B)$.
- (2) $A_{100}(B) = (q^3+1)A_{111}(B) - qA_{221}(B)$.
- (3) $A_{110}(B) = A_{211}(B)$.
- (4) $A_{\lambda\kappa 0}(B) = q^4A_{\lambda\kappa 2}(B)$.
- (5) $A_{\lambda 20}(B) = q^3(q-1)A_{\lambda 22}(B)$.
- (6) $A_{220}(B) = q^2(q^2-q+1)A_{222}(B)$.
- (7) $A_{\lambda 00}(B) = q^2(q+1)A_{\lambda 11}(B) - q^5A_{\lambda 22}(B)$.
- (8) $A_{200}(B) = q^2(q+1)A_{211}(B) - q^3(q^2-q+1)A_{222}(B)$.
- (9) $A_{210}(B) = q(q-1)A_{221}(B)$.
- (10) $A_{\lambda 10}(B) = q^2A_{\lambda 21}(B)$.

In Corollary 2.23, we give a similar corollary of Theorem 1.1 in case where $n = 4$.

These corollaries can be used to compute normalized representation densities. The most important advantage of this method is that we can reduce the computations of representation densities for B to the ones for $\pi^{-1}B$. For example, consider $A_{300}(A_{832})$. By Corollary 1.4, Corollary 1.5, Proposition 1.2, and Proposition 1.3, we can compute that

$$\begin{aligned}
A_{300}(A_{832}) &= q^2(q+1)A_{311}(A_{832}) - q^5A_{322}(A_{832}) \\
&= q^2(q+1)A_{200}(A_{721}) - q^5A_{100}(A_{610}) \\
&= q^4(q+1)^2A_{211}(A_{721}) - q^5A_{10}(A_{61}) \\
&= q^4(q+1)^2A_{100}(A_{610}) - q^5A_{21}(A_{61}) \\
&= q^4(q+1)^2A_{10}(A_{61}) - q^5A_{10}(A_{50}) \\
&= \{q^4(q+1)^2 - q^5\}A_1(A_5) \\
&= q^6 + q^5 + q^4.
\end{aligned}$$

Another important advantage of this method is that we do not need to compute all representation densities separately: when we compute a representation density, we can use other representation densities. We refer to Example 3.4 for this.

In Section 3, we apply Theorem 1.1 to compute some arithmetic intersection numbers of special cycles on unitary Rapoport-Zink spaces. In fact, the arithmetic intersection numbers of special cycles on the Drinfeld upper half plane are the main motivation of our Theorem 1.1. We refer the reader to Example 3.3 for this.

Our computational method allows us to test some geometric guesses on the arithmetic intersection numbers of special cycles. In Example 3.4, we test our geometric guess by using some computations of representation densities. Using this example, we make Conjecture 3.5. This is modelled on the analytic part of [LZ22] (in particular [LZ22, Theorem 4.2.1, Theorem 5.4.1]). We believe that we can use our computations to make more precise conjectures on the arithmetic intersection numbers of special cycles. We refer the reader to Remark 3.6 for this.

2. RELATIONS AMONG REPRESENTATION DENSITIES

In this section, we prove some relations among representation densities.

2.1. Representation densities. In this subsection, we will recall the definition of representation densities and their formulas.

We fix a prime $p > 2$. Let F be a finite extension of \mathbb{Q}_p with ring of integers O_F and residue field \mathbb{F}_q . We fix a uniformizer π of O_F . Let E be a quadratic unramified extension of F with ring of integers O_E . Denote by $*$ the nontrivial Galois automorphism of E over F .

We define the set of hermitian matrices

$$X_n(E) = \{X \in GL_n(E) \mid {}^t X^* = X\}.$$

We define $X_n(O_E) = X_n(E) \cap M_{n,n}(O_E)$ and $K_n = GL_n(O_E)$.

For $g \in GL_n(E)$, and $X \in X_n(E)$, we define the group action of $GL_n(E)$ on $X_n(E)$ by $g \cdot X = gX^t g^*$.

For $X \in M_{m,n}(E)$ and $A \in X_m(E)$, we denote by $A[X] = {}^t X^* A X$.

Let $\Lambda_n^+ := \{\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{Z}^n \mid \lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n \geq 0\}$. For $\lambda \in \Lambda_n^+$, we define $A_\lambda := \text{diag}(\pi^{\lambda_1}, \pi^{\lambda_2}, \dots, \pi^{\lambda_n})$.

Note that the complete set of representatives of $K_n \backslash X_n(O_E)$ is given by the set of diagonal matrices A_λ for $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n) \in \Lambda_n^+$.

Definition 2.1. For $A \in X_m(O_E)$ and $B \in X_n(O_E)$, we define $\alpha(A, B)$ by

$$\alpha(A, B) = \lim_{d \rightarrow \infty} (q^{-d})^{n(2m-n)} |\{X \in M_{m,n}(O_E/\pi^d O_E) \mid A[X] \equiv B \pmod{\pi^d}\}|.$$

Now, we recall an explicit formula for the representation densities from [Hir98].

For each $\mu \in (\mu_1, \dots, \mu_n) \in \Lambda_n^+$, we denote by

$$\tilde{\mu} := (\mu_1 + 1, \dots, \mu_n + 1);$$

$$|\mu| := \sum_{i=1}^n \mu_i;$$

$$n(\mu) := \sum_{i=1}^n (i-1)\mu_i;$$

$$\mu'_i := |\{j \mid \mu_j \geq i\}|, \text{ for } i \geq 1.$$

Also, for $\mu \in \Lambda_n^+$, $\xi \in \Lambda_m^+$, we write

$$\langle \xi', \mu' \rangle := \sum_{i \geq 1} \xi'_i \mu'_i.$$

For μ and λ in Λ_n^+ , we denote by $\mu \leq \lambda$ if $\mu_i \leq \lambda_i$ for all $1 \leq i \leq n$.

For nonnegative integers $u \geq v \geq 0$, we define

$$\begin{bmatrix} u \\ v \end{bmatrix} := \frac{\prod_{i=1}^u (1 - (-q)^{-i})}{\prod_{i=1}^v (1 - (-q)^{-i}) \prod_{i=1}^{u-v} (1 - (-q)^{-i})}$$

Finally, for $\mu, \lambda \in \Lambda_n^+$, we define

$$I_j(\mu, \lambda) := \sum_{i=\mu'_{j+1}}^{\min((\tilde{\lambda})'_{j+1}, \mu'_j)} (-q)^{i(2(\tilde{\lambda})'_{j+1}+1-i)/2} \begin{bmatrix} (\tilde{\lambda})'_{j+1} - \mu'_{j+1} \\ (\tilde{\lambda})'_{j+1} - i \end{bmatrix} \begin{bmatrix} (\tilde{\lambda})'_j - i \\ (\tilde{\lambda})'_j - \mu'_j \end{bmatrix}.$$

Now, we can state the following formula for representation densities.

Proposition 2.2. ([Hir98, Theorem II]) *For $\lambda \in \Lambda_n^+$ and $\xi \in \Lambda_m^+$ with $m \geq n$, we have*

$$\alpha(A_\xi, A_\lambda) = \sum_{\substack{\mu \in \Lambda_n^+ \\ \mu \leq \lambda}} (-1)^{|\mu|} (-q)^{-n(\mu) + (n-m-1)|\mu| + \langle \xi', \mu' \rangle} \prod_{j \geq 1} I_j(\mu, \lambda).$$

2.2. Relations among representation densities. In this subsection, we will prove some relations among representation densities.

Definition 2.3. We fix an integer $n \geq 1$. For integers $1 \leq s \leq n$, $0 \leq i \leq s$ we define $d_{n,s,i}$ by the coefficients of the polynomial

$$\begin{aligned} & \sum_{i=0}^s d_{n,s,i} X^i \\ &= (1 - (-q)^{-n} X)(1 - (-q)^{-n+1} X) \dots (1 - (-q)^{-n+s-1} X) \end{aligned}$$

For example, $d_{n,k,0} = 1$ for any k , $d_{n,1,1} = -\frac{1}{(-q)^n}$, $d_{n,2,1} = -\frac{(-q)+1}{(-q)^n}$,
 $d_{n,2,2} = \frac{1}{(-q)^{2n-1}}$.

Definition 2.4. Let s and n be integers such that $1 \leq s \leq n$.

(1) We define the subset $\Lambda_{n,s}^+$ of Λ_n^+ as

$$\Lambda_{n,s}^+ = \{\xi = (\xi_1, \dots, \xi_n) \in \Lambda_n^+ \mid \xi_{n-s} \geq 1, \xi_{n-s+1} = \dots = \xi_n = 0\}.$$

(2) For $\xi \in \Lambda_{n,s}^+$ and $a, b \geq 0$ such that $a + b \leq s$, we write $\xi_{a,b}^+$ for the element in Λ_n^+ representing

$$\xi_{a,b}^+ := (\xi_1, \dots, \xi_{n-s}, \overbrace{2, \dots, 2}^a, \overbrace{1, \dots, 1}^b, \overbrace{0, \dots, 0}^{s-a-b})$$

In other words, we replace a 0s by a 2s, b 0s by b 1s.

For example, for $\xi = (4, 3, 0, 0) \in \Lambda_{4,2}^+$, we have that

$$\begin{aligned} \xi_{0,1}^+ &= (4, 3, 1, 0) \\ \xi_{1,1}^+ &= (4, 3, 2, 1) \\ \xi_{2,0}^+ &= (4, 3, 2, 2), \end{aligned}$$

and for $\xi = (4, 1, 0, 0) \in \Lambda_{4,2}^+$, we have,

$$\begin{aligned} \xi_{0,1}^+ &= (4, 1, 1, 0) \\ \xi_{1,1}^+ &= (4, 2, 1, 1) \\ \xi_{2,0}^+ &= (4, 2, 2, 1). \end{aligned}$$

Now, we can state our main theorem as follows.

Theorem 2.5. *Let s and n be integers such that $1 \leq s \leq n$, and let $\xi \in \Lambda_{n,s}^+$. Then, for any $B \in X_n(O_E)$ such that $\pi^{-1}B$ is also integral (i.e., the representative of B in $K_n \setminus X_n(O_E)$ is A_λ for some $\lambda \in \Lambda_n^+$ such that $\lambda \geq (1, \dots, 1)$), we have the following equality.*

$$\sum_{i=0}^s d_{n,s,i} \alpha(A_{\xi_{0,i}^+}, B) - \frac{(-1)^s}{(-q)^{ns}} \sum_{i=0}^s d_{n,s,i} \alpha(A_{\xi_{i,s-i}^+}, B) = 0.$$

Example 2.6. (1) For any B in $X_2(O_E)$ such that $\pi^{-1}B$ is integral, we have that

$$\begin{aligned} & \left\{ \alpha(A_{10}, B) - \frac{1}{(-q)^2} \alpha(A_{11}, B) \right\} \\ & + \frac{1}{(-q)^2} \left\{ \alpha(A_{11}, B) - \frac{1}{(-q)^2} \alpha(A_{21}, B) \right\} = 0. \end{aligned}$$

$$\text{Therefore, } \alpha(A_{10}, B) = \frac{1}{(-q)^4} \alpha(A_{21}, B).$$

(2) For any B in $X_2(O_E)$ such that $\pi^{-1}B$ is integral,

$$\begin{aligned} & \alpha(A_{00}, B) - \frac{(-q)+1}{(-q)^2} \alpha(A_{10}, B) + \frac{1}{(-q)^3} \alpha(A_{11}, B) \\ & = \frac{1}{(-q)^4} \left\{ \alpha(A_{11}, B) - \frac{(-q)+1}{(-q)^2} \alpha(A_{21}, B) + \frac{1}{(-q)^3} \alpha(A_{22}, B) \right\} \end{aligned}$$

(3) For $s = 1$ and $\xi = (\bar{\xi}, 0) \in \Lambda_{n,1}^+$ and for any B such that $\pi^{-1}B$ is integral, we have that

$$\begin{aligned} & \left\{ \alpha(A_{\bar{\xi},0}, B) - \frac{1}{(-q)^n} \alpha(A_{\bar{\xi},1}, B) \right\} \\ & + \frac{1}{(-q)^n} \left\{ \alpha(A_{\bar{\xi},1}, B) - \frac{1}{(-q)^n} \alpha(A_{\bar{\xi},2}, B) \right\} = 0 \end{aligned}$$

$$\text{Therefore, } \alpha(A_{\bar{\xi},0}, B) = \frac{1}{(-q)^{2n}} \alpha(A_{\bar{\xi},2}, B)$$

To prove Theorem 2.5, we need to use Proposition 2.2. Let us introduce the following definition.

Definition 2.7. Let l and n be integers such that $0 \leq l \leq n$.

(1) We define the subset Λ_{n-l}^{2+} of Λ_{n-l}^+ as

$$\Lambda_{n-l}^{2+} = \{ \bar{\mu} = (\bar{\mu}_1, \dots, \bar{\mu}_{n-l}) \in \Lambda_{n-l}^+ \mid \bar{\mu}_i \geq 2 \}.$$

- (2) For integers $0 \leq k \leq l$ and an element $\bar{\mu}_l = (\bar{\mu}_1, \dots, \bar{\mu}_{n-l}) \in \Lambda_{n-l}^{2+}$ (i.e. $\mu_{l,i} \geq 2$), we define

$$\mu_{l,k} := (\bar{\mu}_l, \overbrace{1, \dots, 1}^k, \overbrace{0, \dots, 0}^{l-k}) \in \Lambda_n^+.$$

By Proposition 2.2, we have that

$$\begin{aligned} \alpha(A_{\xi_{a,b}^+}, B) &= \alpha(A_{\xi_{a,b}^+}, A_\lambda) \\ &= \sum_{\substack{\mu \in \Lambda_n^+ \\ \mu \leq \tilde{\lambda}}} (-1)^{|\mu|} (-q)^{-n(\mu) - |\mu| + \langle (\xi_{a,b}^+)', \mu' \rangle} \prod_{j \geq 1} I_j(\mu, \lambda). \end{aligned}$$

Let us further decompose this as follows.

(2.2.1)

$$\begin{aligned} \alpha(A_{\xi_{a,b}^+}, A_\lambda) &= \sum_{\substack{\mu \in \Lambda_n^+ \\ \mu \leq \tilde{\lambda}}} (-1)^{|\mu|} (-q)^{-n(\mu) - |\mu| + \langle (\xi_{a,b}^+)', \mu' \rangle} \prod_{j \geq 1} I_j(\mu, \lambda) \\ &= \sum_{l=0}^n \sum_{\substack{\bar{\mu}_l \in \Lambda_{n-l}^{2+} \\ \bar{\mu}_l \leq \tilde{\lambda}}} \sum_{k=0}^l (-1)^{|\mu_{k,l}|} (-q)^{-n(\mu_{k,l}) - |\mu_{k,l}| + \langle (\xi_{a,b}^+)', \mu'_{k,l} \rangle} \prod_{j \geq 1} I_j(\mu_{k,l}, \lambda). \end{aligned}$$

Here, $\bar{\mu}_l \leq \tilde{\lambda}$ means that $\bar{\mu}_1 \leq \tilde{\lambda}_1 = \lambda_1 + 1, \dots, \bar{\mu}_{n-l} \leq \tilde{\lambda}_{n-l} = \lambda_{n-l} + 1$. The condition that $\mu_{k,l} \leq \tilde{\lambda}$ automatically follows from this since $\lambda_i + 1$ is greater than 1 for all i .

Now, let us introduce the following definition.

Definition 2.8. For $\bar{\mu}_l = (\bar{\mu}_1, \dots, \bar{\mu}_{n-l}) \in \Lambda_{n-l}^{2+}$, we define

$$\begin{aligned} \alpha(A_{\xi_{a,b}^+}, A_\lambda; \bar{\mu}_l) \\ := \sum_{k=0}^l (-1)^{|\mu_{k,l}|} (-q)^{-n(\mu_{k,l}) - |\mu_{k,l}| + \langle (\xi_{a,b}^+)', \mu'_{k,l} \rangle} \prod_{j \geq 1} I_j(\mu_{k,l}, \lambda). \end{aligned}$$

Lemma 2.9. For $\bar{\mu}_l = (\bar{\mu}_1, \dots, \bar{\mu}_{n-l}) \in \Lambda_{n-l}^{2+}$, we have that

$$\sum_{i=0}^s d_{n,s,i} \alpha(A_{\xi_{0,i}^+}, A_\lambda; \bar{\mu}_l) - \frac{(-1)^s}{(-q)^{ns}} \sum_{i=0}^s d_{n,s,i} \alpha(A_{\xi_{i,s-i}^+}, A_\lambda; \bar{\mu}_l) = 0.$$

Proof. Note that

$$\begin{aligned} |\mu_{k,l}| &= |\bar{\mu}_l| + k, \\ n(\mu_{k,l}) &= n(\bar{\mu}_l) + (n-l)k + \frac{k(k-1)}{2}. \end{aligned}$$

Also, note that for $i \geq 2$, we have

$$(\mu_{0,l})'_i = (\mu_{1,l})'_i = \cdots = (\mu_{l,l})'_i.$$

Recall that

$$\begin{aligned} & I_j(\mu_{k,l}, \lambda) \\ &= \sum_{i=(\mu_{k,l})'_{j+1}}^{\min((\tilde{\lambda})'_{j+1}, (\mu_{k,l})'_j)} (-q)^{i(2(\tilde{\lambda})'_{j+1}+1-i)/2} \begin{bmatrix} (\tilde{\lambda})'_{j+1} - (\mu_{k,l})'_{j+1} \\ (\tilde{\lambda})'_{j+1} - i \end{bmatrix} \begin{bmatrix} (\tilde{\lambda})'_j - i \\ (\tilde{\lambda})'_j - (\mu_{k,l})'_j \end{bmatrix}. \end{aligned}$$

This depends only on $(\mu_{k,l})'_j$ and $(\mu_{k,l})'_{j+1}$. Therefore, for $j \geq 2$, $I_j(\mu_{k,l}, \lambda)$ are the same for all k .

We can therefore write that

$$\begin{aligned} & \alpha(A_{\xi_{a,b}^+}, A\lambda; \overline{\mu_l}) \\ &= \sum_{k=0}^l (-1)^{|\mu_{k,l}|} (-q)^{-n(\mu_{k,l}) - |\mu_{k,l}| + \langle (\xi_{a,b}^+)', \mu'_{k,l} \rangle} \prod_{j \geq 1} I_j(\mu_{k,l}, \lambda) \\ &= (-1)^{|\overline{\mu_l}|} (-q)^{-n(\overline{\mu_l}) - |\overline{\mu_l}|} \prod_{j \geq 2} I_j(\mu_{k,l}, \lambda) \\ & \quad \times \sum_{k=0}^l (-1)^k (-q)^{-(n-l)k - k(k+1)/2 + \langle (\xi_{a,b}^+)', \mu'_{k,l} \rangle} I_1(\mu_{k,l}, \lambda) \end{aligned}$$

Therefore, it suffices to prove that

$$\begin{aligned} (2.2.2) \quad & \sum_{i=0}^s d_{n,s,i} \sum_{k=0}^l (-1)^k (-q)^{-(n-l)k - k(k+1)/2 + \langle (\xi_{0,i}^+)', \mu'_{k,l} \rangle} I_1(\mu_{k,l}, \lambda) \\ & - \frac{(-1)^s}{(-q)^{ns}} \sum_{i=0}^s d_{n,s,i} \sum_{k=0}^l (-1)^k (-q)^{-(n-l)k - k(k+1)/2 + \langle (\xi_{i,s-i}^+)', \mu'_{k,l} \rangle} I_1(\mu_{k,l}, \lambda) \\ &= 0. \end{aligned}$$

Note that

$$\begin{aligned} (\xi_{a,b}^+)'_1 &= \xi'_1 + a + b = n - s + a + b, \\ (\xi_{a,b}^+)'_2 &= \xi'_2 + a, \\ (\xi_{a,b}^+)'_j &= \xi'_j, \quad \forall j \geq 3, \end{aligned}$$

and

$$\begin{aligned} (\mu_{k,l})'_1 &= n - l + k, \\ (\mu_{k,l})'_2 &= n - l. \end{aligned}$$

Also, note that for $j \geq 2$, $(\mu_{k,l})'_j$ are the same for all k

Therefore, we have that

$$\begin{aligned} \sum_{i=0}^s d_{n,s,i}(-q)^{\langle(\xi_{0,i}^+)'\rangle, \mu'_{k,l}} &= \sum_{i=0}^s d_{n,s,i}(-q)^{(n-s+i)(n-l+k) + (\xi'_2)(n-l) + \sum_{j=3}^{\infty} \xi'_j(\mu_{k,l})'_j} \\ &= (-q)^{(n-s)(n-l+k) + (\xi'_2)(n-l) + \sum_{j=3}^{\infty} \xi'_j(\mu_{k,l})'_j} \\ &\quad \times \prod_{i=0}^{s-1} (1 - (-q)^{-l+k+i}). \end{aligned}$$

Also,

$$\begin{aligned} \sum_{i=0}^s d_{n,s,i}(-q)^{\langle(\xi_{i,s-i}^+)'\rangle, \mu'_{k,l}} &= \sum_{i=0}^s d_{n,s,i}(-q)^{n(n-l+k) + (\xi'_2+i)(n-l) + \sum_{j=3}^{\infty} \xi'_j(\mu_{k,l})'_j} \\ &= (-q)^{n(n-l+k) + (\xi'_2)(n-l) + \sum_{j=3}^{\infty} \xi'_j(\mu_{k,l})'_j} \\ &\quad \times \prod_{i=0}^{s-1} (1 - (-q)^{-l+i}) \end{aligned}$$

These imply that the equation (2.2.2) is equivalent to

$$\begin{aligned} (2.2.3) \quad &\sum_{k=0}^l (-1)^k (-q)^{(n-s)k - (n-l)k - k(k+1)/2} \prod_{i=0}^{s-1} (1 - (-q)^{-l+k+i}) I_1(\mu_{k,l}, \lambda) \\ &- \frac{(-1)^s}{(-q)^{ns}} \sum_{k=0}^l (-1)^k (-q)^{s(n-l+k) + (n-s)k - (n-l)k - k(k+1)/2} \prod_{i=0}^{s-1} (1 - (-q)^{-l+i}) I_1(\mu_{k,l}, \lambda) \\ &= 0. \end{aligned}$$

\iff

$$\begin{aligned} &\sum_{k=0}^l (-1)^k (-q)^{(l-s)k - k(k+1)/2} I_1(\mu_{k,l}, \lambda) \\ &\times \left\{ \prod_{i=0}^{s-1} (1 - (-q)^{-l+k+i}) - (-1)^s (-q)^{s(-l+k)} \prod_{i=0}^{s-1} (1 - (-q)^{-l+i}) \right\} = 0. \end{aligned}$$

We have that

$$\begin{aligned} &I_1(\mu_{k,l}, \lambda) \\ &= \sum_{i=(\mu_{k,l})'_2}^{\min((\tilde{\lambda})'_2, (\mu_{k,l})'_1)} (-q)^{i(2(\tilde{\lambda})'_2 + 1 - i)/2} \begin{bmatrix} (\tilde{\lambda})'_2 - (\mu_{k,l})'_2 \\ (\tilde{\lambda})'_2 - i \end{bmatrix} \begin{bmatrix} (\tilde{\lambda})'_1 - i \\ (\tilde{\lambda})'_1 - (\mu_{k,l})'_1 \end{bmatrix}. \end{aligned}$$

Since we assume that $\lambda \geq (1, 1, \dots, 1)$, we have that

$$(\tilde{\lambda})'_2 = (\tilde{\lambda})'_1 = n.$$

Therefore,

$$\begin{aligned}
I_1(\mu_{k,l}, \lambda) &= \sum_{i=n-l}^{n-l+k} (-q)^{i(2n+1-i)/2} \begin{bmatrix} l \\ n-i \end{bmatrix} \begin{bmatrix} n-i \\ l-k \end{bmatrix} \\
&= \sum_{j=0}^k (-q)^{(j+n-l)(n+l+1-j)/2} \begin{bmatrix} l \\ l-j \end{bmatrix} \begin{bmatrix} l-j \\ l-k \end{bmatrix} \\
&= (-q)^{(n-l)(n+l+1)/2} \sum_{j=0}^k (-q)^{(2l+1-j)j/2} \begin{bmatrix} l \\ l-j \end{bmatrix} \begin{bmatrix} l-j \\ l-k \end{bmatrix}
\end{aligned}$$

This implies that the equation (2.2.3) is equivalent to

$$\begin{aligned}
(2.2.4) \quad & \sum_{k=0}^l (-1)^k (-q)^{(l-s)k-k(k+1)/2} \sum_{j=0}^k (-q)^{(2l+1-j)j/2} \begin{bmatrix} l \\ l-j \end{bmatrix} \begin{bmatrix} l-j \\ l-k \end{bmatrix} \\
& \times \left\{ \prod_{i=0}^{s-1} (1 - (-q)^{-l+k+i}) - (-1)^s (-q)^{s(-l+k)} \prod_{i=0}^{s-1} (1 - (-q)^{-l+i}) \right\} = 0.
\end{aligned}$$

This equality will be proved separately in Lemma 2.11. \square

Lemma 2.10.

$$\sum_{j=0}^k (-q)^{(2l+1-j)j/2} \begin{bmatrix} l \\ l-j \end{bmatrix} \begin{bmatrix} l-j \\ l-k \end{bmatrix} = \begin{bmatrix} l \\ k \end{bmatrix} \prod_{j=0}^{k-1} (1 + (-q)^{l-j}).$$

Proof. First, note that

$$\begin{aligned}
\begin{bmatrix} l \\ l-j \end{bmatrix} \begin{bmatrix} l-j \\ l-k \end{bmatrix} &= \frac{\prod_{i=1}^l (1 - (-q)^{-i})}{\prod_{i=1}^{l-j} (1 - (-q)^{-i}) \prod_{i=1}^j (1 - (-q)^{-i})} \\
&\times \frac{\prod_{i=1}^{l-j} (1 - (-q)^{-i})}{\prod_{i=1}^{k-j} (1 - (-q)^{-i}) \prod_{i=1}^{l-k} (1 - (-q)^{-i})} \\
&= \frac{\prod_{i=1}^l (1 - (-q)^{-i})}{\prod_{i=1}^{l-k} (1 - (-q)^{-i}) \prod_{i=1}^j (1 - (-q)^{-i}) \prod_{i=1}^{k-j} (1 - (-q)^{-i})} \\
&= \frac{\prod_{i=1}^l (1 - (-q)^{-i})}{\prod_{i=1}^k (1 - (-q)^{-i}) \prod_{i=1}^{l-k} (1 - (-q)^{-i})} \\
&\times \frac{\prod_{i=1}^k (1 - (-q)^{-i})}{\prod_{i=1}^j (1 - (-q)^{-i}) \prod_{i=1}^{k-j} (1 - (-q)^{-i})} \\
&= \begin{bmatrix} l \\ k \end{bmatrix} \begin{bmatrix} k \\ j \end{bmatrix}
\end{aligned}$$

Therefore, it suffices to show that

$$(2.2.5) \quad \sum_{j=0}^k (-q)^{(2l+1-j)j/2} \begin{bmatrix} k \\ j \end{bmatrix} = \prod_{j=0}^{k-1} (1 + (-q)^{l-j}).$$

We prove (2.2.5) by induction on k . When $k = 1$, we have that

$$\sum_{j=0}^1 (-q)^{(2l+1-j)j/2} \begin{bmatrix} 1 \\ j \end{bmatrix} = 1 + (-q)^l.$$

Therefore, (2.2.5) is true for $k = 1$. Now, assume that (2.2.5) is true for k .

Note that

$$\begin{aligned} \begin{bmatrix} k+1 \\ j \end{bmatrix} &= \frac{\prod_{i=1}^{k+1} (1 - (-q)^{-i})}{\prod_{i=1}^j (1 - (-q)^{-i}) \prod_{i=1}^{k+1-j} (1 - (-q)^{-i})} \\ &= \frac{(1 - (-q)^{-k-1})}{(1 - (-q)^{-k-1+j})} \begin{bmatrix} k \\ j \end{bmatrix} \end{aligned}$$

Therefore,

$$\begin{aligned} \begin{bmatrix} k+1 \\ j \end{bmatrix} - \begin{bmatrix} k \\ j \end{bmatrix} &= \frac{((-q)^{-k-1+j} - (-q)^{-k-1})}{(1 - (-q)^{-k-1+j})} \begin{bmatrix} k \\ j \end{bmatrix} \\ &= (-q)^{-k-1+j} \frac{(1 - (-q)^{-j})}{(1 - (-q)^{-k-1+j})} \frac{(1 - (-q)^{-k+j-1})}{(1 - (-q)^{-j})} \begin{bmatrix} k \\ j-1 \end{bmatrix} \\ &= (-q)^{-k-1+j} \begin{bmatrix} k \\ j-1 \end{bmatrix} \end{aligned}$$

This implies that

$$\begin{aligned} &\sum_{j=0}^{k+1} (-q)^{(2l+1-j)j/2} \begin{bmatrix} k+1 \\ j \end{bmatrix} - \sum_{j=0}^k (-q)^{(2l+1-j)j/2} \begin{bmatrix} k \\ j \end{bmatrix} \\ &= \sum_{j=0}^k (-q)^{(2l+1-j)j/2} \left\{ \begin{bmatrix} k+1 \\ j \end{bmatrix} - \begin{bmatrix} k \\ j \end{bmatrix} \right\} + (-q)^{(2l-k)(k+1)/2}. \end{aligned}$$

Since $\begin{bmatrix} k+1 \\ 0 \end{bmatrix} = \begin{bmatrix} k \\ 0 \end{bmatrix} = 1$, we have that

$$\begin{aligned}
& \sum_{j=0}^k (-q)^{(2l+1-j)j/2} \left\{ \begin{bmatrix} k+1 \\ j \end{bmatrix} - \begin{bmatrix} k \\ j \end{bmatrix} \right\} + (-q)^{(2l-k)(k+1)/2} \\
&= \sum_{j=1}^k (-q)^{(2l+1-j)j/2} \left\{ \begin{bmatrix} k+1 \\ j \end{bmatrix} - \begin{bmatrix} k \\ j \end{bmatrix} \right\} + (-q)^{(2l-k)(k+1)/2} \\
&= \sum_{j=1}^k (-q)^{(2l+1-j)j/2} (-q)^{-k-1+j} \begin{bmatrix} k \\ j-1 \end{bmatrix} + (-q)^{(2l-k)(k+1)/2} \\
&= \sum_{j=0}^{k-1} (-q)^{(2l-j)(j+1)/2} (-q)^{-k+j} \begin{bmatrix} k \\ j \end{bmatrix} + (-q)^{(2l-k)(k+1)/2} \\
&= \sum_{j=0}^k (-q)^{(2l-j)(j+1)/2} (-q)^{-k+j} \begin{bmatrix} k \\ j \end{bmatrix} \\
&= (-q)^{l-k} \sum_{j=0}^k (-q)^{(2l+1-j)j/2} \begin{bmatrix} k \\ j \end{bmatrix}
\end{aligned}$$

Therefore, we have that

$$\sum_{j=0}^{k+1} (-q)^{(2l+1-j)j/2} \begin{bmatrix} k+1 \\ j \end{bmatrix} = (1 + (-q)^{l-k}) \sum_{j=0}^k (-q)^{(2l+1-j)j/2} \begin{bmatrix} k \\ j \end{bmatrix}.$$

By our inductive hypothesis, we have

$$\sum_{j=0}^{k+1} (-q)^{(2l+1-j)j/2} \begin{bmatrix} k+1 \\ j \end{bmatrix} = (1 + (-q)^{l-k}) \prod_{j=0}^{k-1} (1 + (-q)^{l-j}).$$

This finishes the proof of (2.2.5) and hence the proof of the lemma. \square

Lemma 2.11. *We have that*

$$\begin{aligned}
& \sum_{k=0}^l (-1)^k (-q)^{(l-s)k-k(k+1)/2} \sum_{j=0}^k (-q)^{(2l+1-j)j/2} \begin{bmatrix} l \\ l-j \end{bmatrix} \begin{bmatrix} l-j \\ l-k \end{bmatrix} \\
& \times \left\{ \prod_{i=0}^{s-1} (1 - (-q)^{-l+k+i}) - (-1)^s (-q)^{s(-l+k)} \prod_{i=0}^{s-1} (1 - (-q)^{-l+i}) \right\} = 0.
\end{aligned}$$

Proof. By Lemma 2.10, we need to show that

$$\begin{aligned}
& \sum_{k=0}^l (-1)^k (-q)^{(l-s)k-k(k+1)/2} \begin{bmatrix} l \\ k \end{bmatrix} \prod_{j=0}^{k-1} (1 + (-q)^{l-j}) \\
& \times \left\{ \prod_{i=0}^{s-1} (1 - (-q)^{-l+k+i}) - (-1)^s (-q)^{s(-l+k)} \prod_{i=0}^{s-1} (1 - (-q)^{-l+i}) \right\} = 0.
\end{aligned}$$

Let

$$\begin{aligned} \alpha_{sk} &:= (-1)^k (-q)^{(l-s)k-k(k+1)/2} \begin{bmatrix} l \\ k \end{bmatrix} \prod_{j=0}^{k-1} (1 + (-q)^{l-j}) \\ &\times \left\{ \prod_{i=0}^{s-1} (1 - (-q)^{-l+k+i}) - (-1)^s (-q)^{s(-l+k)} \prod_{i=0}^{s-1} (1 - (-q)^{-l+i}) \right\} \end{aligned}$$

Since $\begin{bmatrix} l \\ k \end{bmatrix} = \frac{\prod_{i=l-k+1}^l (1 - (-q)^{-i})}{\prod_{i=1}^k (1 - (-q)^{-i})}$ and $(1 + (-q)^{l-j}) = (-q)^{l-j} (1 + (-q)^{-l+j})$ we have

$$\begin{aligned} \alpha_{sk} &= \frac{\prod_{i=l-k+1}^l (1 - (-q)^{-2i})}{\prod_{i=1}^k (1 - (-q)^{-i})} \\ &\times \left\{ (-1)^k (-q)^{2lk-k^2-sk} \prod_{i=0}^{s-1} (1 - (-q)^{-l+k+i}) \right. \\ &\quad \left. - (-1)^{s+k} (-q)^{2lk-k^2-sl} \prod_{i=0}^{s-1} (1 - (-q)^{-l+i}) \right\}. \end{aligned}$$

Now, we want to prove that $\sum_{k=0}^l \alpha_{sk} = 0$.

Assume that $l < s$. Then, α_{sk} is zero for all k , and hence $\sum_{k=0}^l \alpha_{sk} = 0$.

Now, assume that $l \geq s$.

For $0 \leq t \leq l$, we write $B_{t,1}$ and $B_{t,2}$ for

$$\begin{aligned} B_{t,1} &:= (-1)^t (-q)^{2lt-t^2-st} \prod_{i=0}^{s-1} (1 - (-q)^{-l+t+i}); \\ B_{t,2} &:= -(-1)^{s+t} (-q)^{2lt-t^2-sl} \prod_{i=0}^{s-1} (1 - (-q)^{-l+i}). \end{aligned}$$

Also, for $0 \leq t \leq l$, $0 \leq k \leq s-1$, we write $L_{t,k}$ and $R_{t,k}$ for

$$\begin{aligned} L_{t,k} &:= (-1)^{t-k} (-q)^{l(2t-k)-t(t+s-k)} \prod_{i=1}^k (1 - (-q)^{-l+s-i}) \\ &\quad \times \prod_{i=1}^{s-k-1} (1 - (-q)^{-l+t+i}), \end{aligned}$$

$$\begin{aligned} R_{t,k} &:= (-1)^{t-k} (-q)^{l(2t-k-1)-t(t+s-k-1)} \prod_{i=1}^k (1 - (-q)^{-l+s-i}) \\ &\quad \times \prod_{i=0}^{s-k-1} (1 - (-q)^{-l+t+i}). \end{aligned}$$

Here, we choose the following convention. The term $\prod_{i=1}^k (1 - (-q)^{-l+s-i})$ is 1 if $k = 0$, and the term $\prod_{i=1}^{s-k-1} (1 - (-q)^{-l+t+i})$ is 1 if $k = s-1$.

Claim 2.12. *For $0 \leq t \leq l$, and $0 \leq k \leq s-1$ we have the followings.*

$$(1) \alpha_{s0} = (1 - (-q)^{-2l}) \sum_{k=0}^{s-1} L_{0,k}.$$

$$(2) L_{t,s-1} (1 - (-q)^{-t-1}) + B_{t+1,2} = R_{t+1,s-1}.$$

(3) For $2 \leq r \leq s$, we have

$$L_{t,s-r} (1 - (-q)^{-t-1}) + R_{t+1,s-r+1} = L_{t+1,s-r+1} (1 - (-q)^{-2l+2t+2}) + R_{t+1,s-r}.$$

$$(4) \quad R_{t+1,0} + B_{t+1,1} = L_{t+1,0}(1 - (-q)^{-2l+2t+2}).$$

$$(5) \quad \sum_{k=0}^{s-1} L_{t,k}(1 - (-q)^{-t-1}) + B_{t+1,1} + B_{t+1,2} = \sum_{k=0}^{s-1} L_{t+1,k}(1 - (-q)^{-2l+2t+2})$$

Proof of Claim 2.12. (1) Note that

$$\begin{aligned} \alpha_{s0} &= (1 - (-1)^s(-q)^{-sl}) \prod_{i=0}^{s-1} (1 - (-q)^{-l+i}) \\ &= (1 + (-q)^{-l}) \sum_{k=0}^{s-1} (-1)^k (-q)^{-kl} \prod_{i=0}^{s-1} (1 - (-q)^{-l+i}) \\ &= (1 - (-q)^{-2l}) \sum_{k=0}^{s-1} L_{0,k}. \end{aligned}$$

This finishes the proof of (1).

(2) We have that

$$\begin{aligned} &L_{t,s-1}(1 - (-q)^{-t-1}) + B_{t+1,2} \\ &= (-1)^{t-s+1} (-q)^{l(2t-s+1)-t(t+1)} \prod_{i=1}^{s-1} (1 - (-q)^{-l+s-i})(1 - (-q)^{-t-1}) \\ &\quad - (-1)^{s+t+1} (-q)^{l(2t-s+2)-(t+1)^2} \prod_{i=0}^{s-1} (1 - (-q)^{-l+i}). \\ &= (-1)^{t-s+2} (-q)^{l(2t-s+2)-(t+1)^2} \prod_{i=1}^{s-1} (1 - (-q)^{-l+s-i}) \\ &\quad \times \{1 - (-q)^l - (-q)^{-l+t+1}(1 - (-q)^{-t-1})\} \\ &= (-1)^{t-s+2} (-q)^{l(2t-s+2)-(t+1)^2} \prod_{i=1}^{s-1} (1 - (-q)^{-l+s-i})(1 - (-q)^{-l+t+1}) \\ &= R_{t+1,s-1}. \end{aligned}$$

This finishes the proof of (2).

(3) We have that

$$\begin{aligned} &L_{t,s-r}(1 - (-q)^{-t-1}) + R_{t+1,s-r+1} \\ &= (-1)^{t-s+r} (-q)^{l(2t-s+r)-t(t+r)} \prod_{i=1}^{s-r} (1 - (-q)^{-l+s-i}) \\ &\quad \prod_{i=1}^{r-1} (1 - (-q)^{-l+t+i})(1 - (-q)^{-t-1}) \\ &\quad + (-1)^{t-s+r} (-q)^{l(2t-s+r)-(t+1)(t+r-1)} \prod_{i=1}^{s-r+1} (1 - (-q)^{-l+s-i}) \\ &\quad \prod_{i=0}^{r-2} (1 - (-q)^{-l+t+1+i}) \\ &= (-1)^{t-s+r} \prod_{i=1}^{s-r} (1 - (-q)^{-l+s-i}) \prod_{i=1}^{r-1} (1 - (-q)^{-l+t+i}) \\ &\quad (-q)^{l(2t-s+r)-(t+1)(t+r-1)} \{(-q)^{r-1}(1 - (-q)^{-t-1}) + (1 - (-q)^{-l+r-1})\} \\ &= -(-q)^{-t+r-2} (1 + (-q)^{-l+t+1}) (-1)^{t-s+r} (-q)^{l(2t-s+r)-(t+1)(t+r-1)} \\ &\quad (-q)^{l-r+1} (-q)^{-l+r-1} \prod_{i=1}^{s-r} (1 - (-q)^{-l+s-i}) \prod_{i=1}^{r-1} (1 - (-q)^{-l+t+i}) \\ &\quad + ((-q)^{r-1} + 1) (-1)^{t-s+r} (-q)^{l(2t-s+r)-(t+1)(t+r-1)} \\ &\quad \prod_{i=1}^{s-r} (1 - (-q)^{-l+s-i}) \prod_{i=1}^{r-1} (1 - (-q)^{-l+t+i}) \end{aligned}$$

$$\begin{aligned}
&= (1 - (-q)^{-2l+2t+2})(-1)^{t-s+r}(-q)^{l(2t-s+r+1)-(t+1)(t+r)} \\
&\quad \prod_{i=1}^{s-r+1}(1 - (-q)^{-l+s-i}) \prod_{i=2}^{r-1}(1 - (-q)^{-l+t+i}) \\
&\quad - (1 - (-q)^{-2l+2t+2})(-1)^{t-s+r}(-q)^{l(2t-s+r+1)-(t+1)(t+r)} \\
&\quad \prod_{i=1}^{s-r}(1 - (-q)^{-l+s-i}) \prod_{i=2}^{r-1}(1 - (-q)^{-l+t+i}) \\
&\quad + ((-q)^{r-1} + 1)(-1)^{t-s+r}(-q)^{l(2t-s+r)-(t+1)(t+r-1)} \\
&\quad \prod_{i=1}^{s-r}(1 - (-q)^{-l+s-i}) \prod_{i=1}^{r-1}(1 - (-q)^{-l+t+i}) \\
&= L_{t+1,s-r+1}(1 - (-q)^{-2l+2t+2}) \\
&\quad + (-1)^{t-s+r+1}(-q)^{l(2t-s+r+1)-(t+1)(t+r)} \\
&\quad \prod_{i=1}^{s-r}(1 - (-q)^{-l+s-i}) \prod_{i=1}^{r-1}(1 - (-q)^{-l+t+i}) \\
&\quad \quad \times \{1 + (-q)^{-l+t+1} - ((-q)^{r-1} + 1)(-q)^{-l+t+1}\} \\
&= L_{t+1,s-r+1}(1 - (-q)^{-2l+2t+2}) + R_{t+1,s-r}.
\end{aligned}$$

This finishes the proof of (3).

(4) We have that

$$\begin{aligned}
&R_{t+1,0} + B_{t+1,1} \\
&= (-1)^{t+1}(-q)^{l(2t+1)-(t+1)(t+s)} \prod_{i=0}^{s-1}(1 - (-q)^{-l+t+1+i}) \\
&\quad + (-1)^{t+1}(-q)^{l(2t+2)-(t+1)(t+s+1)} \prod_{i=0}^{s-1}(1 - (-q)^{-l+t+1+i}) \\
&= (-1)^{t+1}(-q)^{l(2t+2)-(t+1)(t+s+1)} \prod_{i=0}^{s-1}(1 - (-q)^{-l+t+1+i})(1 + (-q)^{-l+t+1}) \\
&= L_{t+1,0}(1 - (-q)^{-2l+2t+2}).
\end{aligned}$$

This finishes the proof of (4).

(5) This is obvious from (1), (2), (3), and (4). □

Now, by Claim 2.12 (1), we have that $\alpha_{s0} = (1 - (-q)^{-2l}) \sum_{k=0}^{s-1} L_{0,k}$. Also, by definition, we have that

$$\alpha_{st} = \frac{\prod_{i=0}^{t-1}(1 - (-q)^{-2l+2i})}{\prod_{i=1}^t(1 - (-q)^{-i})} (B_{t,1} + B_{t+2}).$$

Therefore, by Claim 2.12 (5) we have

$$\begin{aligned}
\alpha_{s0} + \alpha_{s1} &= \frac{(1 - (-q)^{-2l})}{(1 - (-q)^{-1})} \{ \sum_{k=0}^{s-1} L_{0,k}(1 - (-q)^{-1}) + B_{1,1} + B_{1,2} \} \\
&= \frac{(1 - (-q)^{-2l})(1 - (-q)^{-2l+2})}{(1 - (-q)^{-1})} \{ \sum_{k=0}^{s-1} L_{1,k} \}
\end{aligned}$$

Now, we claim that for any $t \leq l$,

$$(2.2.6) \quad \sum_{i=0}^t \alpha_{si} = \frac{\prod_{i=0}^t (1 - (-q)^{-2l+2i})}{\prod_{i=1}^t (1 - (-q)^{-i})} \sum_{k=0}^{s-1} L_{t,k}.$$

We prove this by induction on t . Assume that (2.2.6) is true for t , then we have

$$\sum_{i=0}^{t+1} \alpha_{si} = \frac{\prod_{i=0}^t (1 - (-q)^{-2l+2i})}{\prod_{i=1}^{t+1} (1 - (-q)^{-i})} \left\{ \sum_{k=0}^{s-1} L_{t,k} (1 - (-q)^{-t-1}) + B_{t+1,1} + B_{t+1,2} \right\}.$$

By Claim 2.12 (5), we have

$$\sum_{i=0}^{t+1} \alpha_{si} = \frac{\prod_{i=0}^{t+1} (1 - (-q)^{-2l+2i})}{\prod_{i=1}^{t+1} (1 - (-q)^{-i})} \left\{ \sum_{k=0}^{s-1} L_{t+1,k} \right\}.$$

This finishes the proof of (2.2.6).

Therefore, we see that

$$\sum_{i=0}^l \alpha_{si} = \frac{\prod_{i=0}^l (1 - (-q)^{-2l+2i})}{\prod_{i=1}^l (1 - (-q)^{-i})} \left\{ \sum_{k=0}^{s-1} L_{l,k} \right\} = 0,$$

and this completes the proof of the lemma. \square

Proof of Theorem 2.5. This follows from (2.2.1) and Lemma 2.9. \square

2.3. Normalized representation densities. When we apply Theorem 2.5 to compute representation densities, it is better to use a normalized version of the theorem. In this section, we consider a normalized version of Theorem 2.5 and compute some representation densities.

Definition 2.13. For $A, B \in X_n(O_E)$, we define $A(B)$ by

$$A(B) := \frac{\alpha(A, B)}{\alpha(A, A)}.$$

These normalized representation densities have the following reduction properties.

Proposition 2.14. For $\xi, \lambda \in \Lambda_{n-1}^+$, we consider $(\xi, 0), (\lambda, 0)$ in Λ_n^+ . Then, we have

$$A_{\xi,0}(A_{\lambda,0}) = A_{\xi}(A_{\lambda}).$$

Proof. By [KR11, Corollary 9.12], we have that

$$\begin{aligned}\alpha(A_{\xi,0}, A_{\lambda,0}) &= \alpha(A_{\xi,0}, A_0)\alpha(A_{\xi}, A_{\lambda}), \\ \alpha(A_{\xi,0}, A_{\xi,0}) &= \alpha(A_{\xi,0}, A_0)\alpha(A_{\xi}, A_{\xi}).\end{aligned}$$

Therefore, we have

$$A_{\xi,0}(A_{\lambda,0}) = A_{\xi}(A_{\lambda}).$$

□

Proposition 2.15. *For ξ, λ in Λ_n^+ , such that $\xi, \lambda \geq (1, 1, \dots, 1)$, we consider*

$$\begin{aligned}\xi - 1 &:= (\xi_1 - 1, \dots, \xi_n - 1) \in \Lambda_n^+ \\ \lambda - 1 &:= (\lambda_1 - 1, \dots, \lambda_n - 1) \in \Lambda_n^+.\end{aligned}$$

Then, we have

$$A_{\xi}(A_{\lambda}) = A_{\xi-1}(A_{\lambda-1}).$$

Proof. This follows directly from the fact that

$$\begin{aligned}\alpha(A_{\xi}, A_{\xi}) &= q^{n^2}\alpha(A_{\xi-1}, A_{\xi-1}), \\ \alpha(A_{\xi}, A_{\lambda}) &= q^{n^2}\alpha(A_{\xi-1}, A_{\lambda-1})\end{aligned}$$

□

Example 2.16 ($n = 1$). By Theorem 2.5, for $B \in X_1(O_E)$ such that $\pi^{-1}B$ is integral, we have that

$$\left\{\alpha(A_0, B) + \frac{1}{q}\alpha(A_1, B)\right\} + \frac{1}{(-q)}\left\{\alpha(A_1, B) + \frac{1}{q}\alpha(A_2, B)\right\} = 0.$$

Therefore, $\alpha(A_0, B) = \frac{1}{q^2}\alpha(A_2, B)$.

Since $\alpha(A_0, A_0) = (1 + \frac{1}{q})$ and $\alpha(A_2, A_2) = q^2(1 + \frac{1}{q})$, we have

$$A_0(B) = A_2(B).$$

For example, for $B = A_6 = (\pi^6)$, we have

$$A_0(A_6) = A_2(A_6) = A_0(A_4) = A_2(A_4) = A_0(A_2) = A_2(A_2) = 1.$$

In general, $A_0(B)$ is 1 for any B such that $\text{val det } B \equiv 0 \pmod{2}$.

Corollary 2.17 ($n = 2$). *Assume that $\lambda \geq 3$. For any $B \in X_2(O_E)$ such that $\pi^{-1}B$ is integral, the following equalities hold.*

$$(1) \quad A_{00}(B) = (q+1)A_{11}(B) - qA_{22}(B).$$

$$(2) \quad A_{10}(B) = A_{21}(B).$$

$$(3) \quad A_{20}(B) = q(q-1)A_{22}(B).$$

$$(4) \quad A_{\lambda 0}(B) = q^2 A_{\lambda 2}(B).$$

Proof. By Theorem 2.5 for $n = 2$ and $s = 2$, we have that

$$(2.3.1) \quad \begin{aligned} & \{\alpha(A_{00}, B) + (\frac{1}{q} - \frac{1}{q^2})\alpha(A_{10}, B) - \frac{1}{q^3}\alpha(A_{11}, B)\} \\ & - \frac{1}{q^4}\{\alpha(A_{11}, B) + (\frac{1}{q} - \frac{1}{q^2})\alpha(A_{21}, B) - \frac{1}{q^3}\alpha(A_{22}, B)\} = 0 \end{aligned}$$

Also, by [Cho22b, Lemma 4.6], we can compute that

$$\begin{aligned} \alpha(A_{00}, A_{00}) &= (1 + \frac{1}{q})(1 - \frac{1}{q^2}), \\ \alpha(A_{10}, A_{10}) &= q(1 + \frac{1}{q})^2, \\ \alpha(A_{11}, A_{11}) &= q^4(1 + \frac{1}{q})(1 - \frac{1}{q^2}), \\ \alpha(A_{21}, A_{21}) &= q^5(1 + \frac{1}{q})^2, \\ \alpha(A_{22}, A_{22}) &= q^8(1 + \frac{1}{q})(1 - \frac{1}{q^2}), \end{aligned}$$

Therefore, (2.3.1) is

$$A_{00}(B) + A_{10}(B) - (q + 1)A_{11}(B) - A_{21}(B) + qA_{22}(B) = 0.$$

Note that $A(B) = 0$ if $\text{val det } A \not\equiv \text{val det } B \pmod{2}$. Therefore, we have

$$A_{00}(B) = (q + 1)A_{11}(B) - qA_{22}(B),$$

and

$$A_{10}(B) = A_{21}(B).$$

This finishes the proof of (1) and (2).

Now, for $\lambda \geq 2$, by Theorem 2.5 for $n = 2$, $s = 1$, we have

$$\{\alpha(A_{\lambda 0}, B) - \frac{1}{q^2}\alpha(A_{\lambda 1}, B)\} + \frac{1}{q^2}\{\alpha(A_{\lambda 1}, B) - \frac{1}{q^2}\alpha(A_{\lambda 2}, B)\} = 0.$$

Therefore,

$$(2.3.2) \quad \alpha(A_{\lambda 0}, B) = \frac{1}{q^4}\alpha(A_{\lambda 2}, B).$$

Now, for $\lambda \geq 3$, we have

$$\begin{aligned} \alpha(A_{\lambda 0}, A_{\lambda, 0}) &= (1 + \frac{1}{q})\alpha(A_{\lambda}, A_{\lambda}) = (1 + \frac{1}{q})q^2\alpha(A_{\lambda-2}, A_{\lambda-2}), \\ \alpha(A_{\lambda 2}, A_{\lambda, 2}) &= q^8(1 + \frac{1}{q})\alpha(A_{\lambda-2}, A_{\lambda-2}). \end{aligned}$$

Also, for $\lambda = 2$, we have

$$\alpha(A_{20}, A_{20}) = q^2 \left(1 + \frac{1}{q}\right)^2,$$

$$\alpha(A_{22}, A_{22}) = q^8 \left(1 + \frac{1}{q}\right) \left(1 - \frac{1}{q^2}\right).$$

These imply that (2.3.2) is

$$\begin{aligned} A_{\lambda 0}(B) &= q^2 A_{\lambda 2}(B), \text{ if } \lambda \geq 3; \\ A_{20}(B) &= q(q-1)A_{22}(B). \end{aligned}$$

This finishes the proof of (3) and (4). \square

Remark 2.18. When $n = 2$, Corollary 2.17 gives an efficient way to compute representation densities. For example, consider $A_{60}(A_{86})$. If we use Proposition 2.2, we need several pages to compute this. But, by Corollary 2.17, we can compute that

$$\begin{aligned} A_{60}(A_{86}) &= q^2 A_{62}(A_{86}) = q^2 A_{40}(A_{64}) = q^4 A_{42}(A_{64}) = q^4 A_{20}(A_{42}) \\ &= q^5 (q-1) A_{22}(A_{42}) = q^5 (q-1) A_{00}(A_{20}) = q^5 (q-1) A_0(A_2) = q^5 (q-1). \end{aligned}$$

Lemma 2.19 ($n = 3$). *Assume that $\lambda, \kappa \geq 3$. Then, we have the following equalities.*

- (1) $\alpha(A_{\lambda\kappa 0}, A_{\lambda\kappa 0}) = q^8 \left(1 + \frac{1}{q}\right) \alpha(A_{\lambda-2, \kappa-2}, A_{\lambda-2, \kappa-2})$.
- (2) $\alpha(A_{\lambda\kappa 1}, A_{\lambda\kappa 1}) = q^{13} \left(1 + \frac{1}{q}\right) \alpha(A_{\lambda-2, \kappa-2}, A_{\lambda-2, \kappa-2})$.
- (3) $\alpha(A_{\lambda\kappa 2}, A_{\lambda\kappa 2}) = q^{18} \left(1 + \frac{1}{q}\right) \alpha(A_{\lambda-2, \kappa-2}, A_{\lambda-2, \kappa-2})$.
- (4) $\alpha(A_{\lambda 20}, A_{\lambda 20}) = q^8 \left(1 + \frac{1}{q}\right)^2 \alpha(A_{\lambda-2}, A_{\lambda-2})$.
- (5) $\alpha(A_{\lambda 21}, A_{\lambda 21}) = q^{13} \left(1 + \frac{1}{q}\right)^2 \alpha(A_{\lambda-2}, A_{\lambda-2})$.
- (6) $\alpha(A_{\lambda 22}, A_{\lambda 22}) = q^{18} \left(1 + \frac{1}{q}\right) \left(1 - \frac{1}{q^2}\right) \alpha(A_{\lambda-2}, A_{\lambda-2})$.
- (7) $\alpha(A_{220}, A_{220}) = q^8 \left(1 + \frac{1}{q}\right)^2 \left(1 - \frac{1}{q^2}\right)$.
- (8) $\alpha(A_{221}, A_{221}) = q^{13} \left(1 + \frac{1}{q}\right)^2 \left(1 - \frac{1}{q^2}\right)$.
- (9) $\alpha(A_{222}, A_{222}) = q^{18} \left(1 + \frac{1}{q}\right) \left(1 - \frac{1}{q^2}\right) \left(1 + \frac{1}{q^3}\right)$.
- (10) $\alpha(A_{\lambda 00}, A_{\lambda 00}) = q^2 \left(1 + \frac{1}{q}\right) \left(1 - \frac{1}{q^2}\right) \alpha(A_{\lambda-2}, A_{\lambda-2})$.

- (11) $\alpha(A_{\lambda 10}, A_{\lambda 10}) = q^5(1 + \frac{1}{q})^2 \alpha(A_{\lambda-2}, A_{\lambda-2})$.
- (12) $\alpha(A_{\lambda 11}, A_{\lambda 11}) = q^{10}(1 + \frac{1}{q})(1 - \frac{1}{q^2}) \alpha(A_{\lambda-2}, A_{\lambda-2})$.
- (13) $\alpha(A_{200}, A_{200}) = q^2(1 + \frac{1}{q})^2(1 - \frac{1}{q^2})$.
- (14) $\alpha(A_{210}, A_{210}) = q^5(1 + \frac{1}{q})^3$.
- (15) $\alpha(A_{211}, A_{211}) = q^{10}(1 + \frac{1}{q})^2(1 - \frac{1}{q^2})$.
- (16) $\alpha(A_{000}, A_{000}) = (1 + \frac{1}{q})(1 - \frac{1}{q^2})(1 + \frac{1}{q^3})$.
- (17) $\alpha(A_{100}, A_{100}) = q^1(1 + \frac{1}{q})^2(1 - \frac{1}{q^2})$.
- (18) $\alpha(A_{110}, A_{110}) = q^4(1 + \frac{1}{q})^2(1 - \frac{1}{q^2})$.
- (19) $\alpha(A_{111}, A_{111}) = q^9(1 + \frac{1}{q})(1 - \frac{1}{q^2})(1 + \frac{1}{q^3})$.

Proof. This follows directly from [Cho22b, Lemma 4.6]. \square

Corollary 2.20 ($n = 3$). *Assume that $\lambda, \kappa \geq 3$. For any $B \in X_3(O_E)$ such that $\pi^{-1}B$ is integral, the following equalities hold.*

- (1) $A_{000}(B) = qA_{110}(B) + A_{211}(B) - q^3A_{222}(B)$
 $= (q + 1)A_{211}(B) - q^3A_{222}(B)$.
- (2) $A_{100}(B) = (q^3 + 1)A_{111}(B) - qA_{221}(B)$.
- (3) $A_{110}(B) = A_{211}(B)$.
- (4) $A_{\lambda\kappa 0}(B) = q^4A_{\lambda\kappa 2}(B)$.
- (5) $A_{\lambda 20}(B) = q^3(q - 1)A_{\lambda 22}(B)$.
- (6) $A_{220}(B) = q^2(q^2 - q + 1)A_{222}(B)$.
- (7) $A_{\lambda 00}(B) = q^2(q + 1)A_{\lambda 11}(B) - q^5A_{\lambda 22}(B)$.
- (8) $A_{200}(B) = q^2(q + 1)A_{211}(B) - q^3(q^2 - q + 1)A_{222}(B)$.
- (9) $A_{210}(B) = q(q - 1)A_{221}(B)$.
- (10) $A_{\lambda 10}(B) = q^2A_{\lambda 21}(B)$.

Proof. This follows directly from Lemma 2.19 and Theorem 2.5. \square

Example 2.21. This Corollary 2.20 gives an efficient way to compute representation densities for $n = 3$. For example, let us compute $A_{200}(A_{332})$. By Corollary 2.20 we have the following formula.

$$\begin{aligned} A_{200}(A_{332}) &= q^2(q+1)A_{211}(A_{332}) - q^3(q^2 - q + 1)A_{222}(A_{332}) \\ &= q^2(q+1)A_{100}(A_{221}) - q^3(q^2 - q + 1)A_{000}(A_{110}) \end{aligned}$$

Also,

$$A_{000}(A_{110}) = A_{00}(A_{11}) = (q+1)A_{11}(A_{11}) = (q+1),$$

and

$$\begin{aligned} A_{100}(A_{221}) &= (q^3 + 1)A_{111}(A_{221}) - qA_{221}(A_{221}) \\ &= (q^3 + 1)A_{000}(A_{110}) - q \\ &= (q^3 + 1)A_{00}(A_{11}) - q \\ &= (q^3 + 1)(q+1) - q \\ &= q^4 + q^3 + 1. \end{aligned}$$

Therefore,

$$\begin{aligned} A_{200}(A_{332}) &= q^2(q+1)(q^4 + q^3 + 1) - q^3(q^2 - q + 1)(q+1) \\ &= (q+1)(q^6 + q^4 - q^3 + q^2). \end{aligned}$$

Lemma 2.22 ($n = 4$). *Assume that $\lambda, \kappa, \mu \geq 3$. Then, we have the following equalities.*

- (1) $\alpha(A_{\lambda\kappa\mu 0}, A_{\lambda\kappa\mu 0}) = q^{18}(1 + \frac{1}{q})\alpha(A_{\lambda-2, \kappa-2, \mu-2}, A_{\lambda-2, \kappa-2, \mu-2})$.
- (2) $\alpha(A_{\lambda\kappa\mu 1}, A_{\lambda\kappa\mu 1}) = q^{25}(1 + \frac{1}{q})\alpha(A_{\lambda-2, \kappa-2, \mu-2}, A_{\lambda-2, \kappa-2, \mu-2})$.
- (3) $\alpha(A_{\lambda\kappa\mu 2}, A_{\lambda\kappa\mu 2}) = q^{32}(1 + \frac{1}{q})\alpha(A_{\lambda-2, \kappa-2, \mu-2}, A_{\lambda-2, \kappa-2, \mu-2})$.
- (4) $\alpha(A_{\lambda\kappa 20}, A_{\lambda\kappa 20}) = q^{18}(1 + \frac{1}{q})^2\alpha(A_{\lambda-2, \kappa-2}, A_{\lambda-2, \kappa-2})$.
- (5) $\alpha(A_{\lambda\kappa 21}, A_{\lambda\kappa 21}) = q^{25}(1 + \frac{1}{q})^2\alpha(A_{\lambda-2, \kappa-2}, A_{\lambda-2, \kappa-2})$.
- (6) $\alpha(A_{\lambda\kappa 22}, A_{\lambda\kappa 22}) = q^{32}(1 + \frac{1}{q})(1 - \frac{1}{q^2})\alpha(A_{\lambda-2, \kappa-2}, A_{\lambda-2, \kappa-2})$.
- (7) $\alpha(A_{\lambda 220}, A_{\lambda 220}) = q^{18}(1 + \frac{1}{q})^2(1 - \frac{1}{q^2})\alpha(A_{\lambda-2}, A_{\lambda-2})$.
- (8) $\alpha(A_{\lambda 221}, A_{\lambda 221}) = q^{25}(1 + \frac{1}{q})^2(1 - \frac{1}{q^2})\alpha(A_{\lambda-2}, A_{\lambda-2})$.
- (9) $\alpha(A_{\lambda 222}, A_{\lambda 222}) = q^{32}(1 + \frac{1}{q})(1 - \frac{1}{q^2})(1 + \frac{1}{q^3})\alpha(A_{\lambda-2}, A_{\lambda-2})$.
- (10) $\alpha(A_{2220}, A_{2220}) = q^{18}(1 + \frac{1}{q})^2(1 - \frac{1}{q^2})(1 + \frac{1}{q^3})$.
- (11) $\alpha(A_{2221}, A_{2221}) = q^{25}(1 + \frac{1}{q})^2(1 - \frac{1}{q^2})(1 + \frac{1}{q^3})$.

- (12) $\alpha(A_{2222}, A_{2222}) = q^{32}(1 + \frac{1}{q})(1 - \frac{1}{q^2})(1 + \frac{1}{q^3})(1 - \frac{1}{q^4})$.
- (13) $\alpha(A_{\lambda\kappa 00}, A_{\lambda\kappa 00}) = q^8(1 + \frac{1}{q})(1 - \frac{1}{q^2})\alpha(A_{\lambda-2, \kappa-2}, A_{\lambda-2, \kappa-2})$.
- (14) $\alpha(A_{\lambda\kappa 10}, A_{\lambda\kappa 10}) = q^{13}(1 + \frac{1}{q})^2\alpha(A_{\lambda-2, \kappa-2}, A_{\lambda-2, \kappa-2})$.
- (15) $\alpha(A_{\lambda\kappa 11}, A_{\lambda\kappa 11}) = q^{20}(1 + \frac{1}{q})(1 - \frac{1}{q^2})\alpha(A_{\lambda-2, \kappa-2}, A_{\lambda-2, \kappa-2})$.
- (16) $\alpha(A_{\lambda 200}, A_{\lambda 200}) = q^8(1 + \frac{1}{q})^2(1 - \frac{1}{q^2})\alpha(A_{\lambda-2}, A_{\lambda-2})$.
- (17) $\alpha(A_{\lambda 210}, A_{\lambda 210}) = q^{13}(1 + \frac{1}{q})^3\alpha(A_{\lambda-2}, A_{\lambda-2})$.
- (18) $\alpha(A_{\lambda 211}, A_{\lambda 211}) = q^{20}(1 + \frac{1}{q})^2(1 - \frac{1}{q^2})\alpha(A_{\lambda-2}, A_{\lambda-2})$.
- (19) $\alpha(A_{2200}, A_{2200}) = q^8(1 + \frac{1}{q})^2(1 - \frac{1}{q^2})^2$.
- (20) $\alpha(A_{2210}, A_{2210}) = q^{13}(1 + \frac{1}{q})^3(1 - \frac{1}{q^2})$.
- (21) $\alpha(A_{2211}, A_{2211}) = q^{20}(1 + \frac{1}{q})^2(1 - \frac{1}{q^2})^2$.
- (22) $\alpha(A_{\lambda 000}, A_{\lambda 000}) = q^2(1 + \frac{1}{q})(1 - \frac{1}{q^2})(1 + \frac{1}{q^3})\alpha(A_{\lambda-2}, A_{\lambda-2})$.
- (23) $\alpha(A_{\lambda 100}, A_{\lambda 100}) = q^5(1 + \frac{1}{q})^2(1 - \frac{1}{q^2})\alpha(A_{\lambda-2}, A_{\lambda-2})$.
- (24) $\alpha(A_{\lambda 110}, A_{\lambda 110}) = q^{10}(1 + \frac{1}{q})^2(1 - \frac{1}{q^2})\alpha(A_{\lambda-2}, A_{\lambda-2})$.
- (25) $\alpha(A_{\lambda 111}, A_{\lambda 111}) = q^{17}(1 + \frac{1}{q})(1 - \frac{1}{q^2})(1 + \frac{1}{q^3})\alpha(A_{\lambda-2}, A_{\lambda-2})$.
- (26) $\alpha(A_{2000}, A_{2000}) = q^2(1 + \frac{1}{q})^2(1 - \frac{1}{q^2})(1 + \frac{1}{q^3})$.
- (27) $\alpha(A_{2100}, A_{2100}) = q^5(1 + \frac{1}{q})^3(1 - \frac{1}{q^2})$.
- (28) $\alpha(A_{2110}, A_{2110}) = q^{10}(1 + \frac{1}{q})^3(1 - \frac{1}{q^2})$.
- (29) $\alpha(A_{2111}, A_{2111}) = q^{17}(1 + \frac{1}{q})^2(1 - \frac{1}{q^2})(1 + \frac{1}{q^3})$.
- (30) $\alpha(A_{0000}, A_{0000}) = (1 + \frac{1}{q})(1 - \frac{1}{q^2})(1 + \frac{1}{q^3})(1 - \frac{1}{q^4})$.
- (31) $\alpha(A_{1000}, A_{1000}) = q(1 + \frac{1}{q})^2(1 - \frac{1}{q^2})(1 + \frac{1}{q^3})$.

$$(32) \quad \alpha(A_{1100}, A_{1100}) = q^4 \left(1 + \frac{1}{q}\right)^2 \left(1 - \frac{1}{q^2}\right)^2.$$

$$(33) \quad \alpha(A_{1110}, A_{1110}) = q^9 \left(1 + \frac{1}{q}\right)^2 \left(1 - \frac{1}{q^2}\right) \left(1 + \frac{1}{q^3}\right).$$

$$(34) \quad \alpha(A_{1111}, A_{1111}) = q^{16} \left(1 + \frac{1}{q}\right) \left(1 - \frac{1}{q^2}\right) \left(1 + \frac{1}{q^3}\right) \left(1 - \frac{1}{q^4}\right).$$

Proof. This follows directly from [Cho22b, Lemma 4.6]. \square

Corollary 2.23 ($n = 4$). *Assume that $\lambda, \kappa, \mu \geq 3$. For any $B \in X_4(O_E)$ such that $\pi^{-1}B$ is integral, the following equalities hold.*

$$(1) \quad A_{\lambda\kappa\mu 0}(B) = q^6 A_{\lambda\kappa\mu 2}(B).$$

$$(2) \quad A_{\lambda\kappa 20}(B) = q^5 (q - 1) A_{\lambda\kappa 22}(B).$$

$$(3) \quad A_{\lambda\kappa 10}(B) = q^4 A_{\lambda\kappa 21}(B).$$

$$(4) \quad A_{\lambda 220}(B) = q^4 (q^2 - q + 1) A_{\lambda 222}(B).$$

$$(5) \quad A_{\lambda 210}(B) = q^3 (q - 1) A_{\lambda 221}(B).$$

$$(6) \quad A_{\lambda 110}(B) = q^2 A_{\lambda 211}(B).$$

$$(7) \quad A_{2220}(B) = q^3 (q^3 - q^2 + q - 1) A_{2222}(B).$$

$$(8) \quad A_{2210}(B) = q^2 (q^2 - q + 1) A_{2221}(B).$$

$$(9) \quad A_{2110}(B) = q (q - 1) A_{2211}(B).$$

$$(10) \quad A_{1110}(B) = A_{2111}(B).$$

$$(11) \quad A_{\lambda\kappa 00}(B) = q^4 (q + 1) A_{\lambda\kappa 11}(B) - q^9 A_{\lambda\kappa 22}(B).$$

$$(12) \quad A_{\lambda 200}(B) = q^4 (q + 1) A_{\lambda 211}(B) - q^7 (q^2 - q + 1) A_{\lambda 222}(B).$$

$$(13) \quad A_{\lambda 100}(B) = q^2 (q^3 + 1) A_{\lambda 111}(B) - q^5 A_{\lambda 221}(B).$$

$$(14) \quad A_{2200}(B) = q^4 (q + 1) A_{2211}(B) - q^5 (q^2 - q + 1) (q^2 + 1) A_{2222}(B).$$

$$(15) \quad A_{2100}(B) = q^2 (q^3 + 1) A_{2111}(B) - q^3 (q^2 - q + 1) A_{2221}(B).$$

$$(16) \quad A_{1100}(B) = (q^3 + 1) (q^2 + 1) A_{1111}(B) - q A_{2211}(B).$$

$$(17) \quad A_{\lambda 000}(B) = q^3 A_{\lambda 110}(B) + q^4 A_{\lambda 211}(B) - q^9 A_{\lambda 222}(B) \\ = q^4(q+1)A_{\lambda 211}(B) - q^9 A_{\lambda 222}(B).$$

$$(18) \quad A_{2000}(B) = q^3 A_{2110}(B) + q^3(q-1)A_{2211}(B) \\ - q^6(q^3 - q^2 + q - 1)A_{2222}(B) \\ = q^3(q^2 - 1)A_{2211}(B) - q^6(q^3 - q^2 + q - 1)A_{2222}(B).$$

$$(19) \quad A_{1000}(B) = q(q^2 - q + 1)A_{1110}(B) + (q^2 - q + 1)A_{2111}(B) - q^3 A_{2221}(B) \\ = (q^3 + 1)A_{2111}(B) - q^3 A_{2221}(B).$$

$$(20) \quad A_{0000}(B) = qA_{1100}(B) - (q^6 - 1)A_{1111}(B) - qA_{2211}(B) + q^6 A_{2222}(B) \\ = (q+1)(q^3 + 1)A_{1111}(B) - q(q+1)A_{2211}(B) + q^6 A_{2222}(B).$$

Proof. This follows directly from Lemma 2.22 and Theorem 2.5. \square

Remark 2.24. The most important advantage of this Corollary 2.23 is that we can reduce the computation of representation densities for B to the ones for $\pi^{-1}B$. For example, to compute $A_{1000}(B)$, we only need to compute $(q^3 + 1)A_{1000}(\pi^{-1}B) - q^3 A_{1110}(\pi^{-1}B)$. In Section 3, we will provide several examples of computations of representation densities by using this Corollary.

3. SPECIAL CYCLES AND REPRESENTATION DENSITIES

In this section, we apply Theorem 2.5 to understand the arithmetic intersection theory of special cycles on unitary Shimura varieties.

3.1. Special cycles on unitary Rapoport-Zink spaces. In this subsection, we recall some facts and definitions of unitary Rapoport-Zink spaces $\mathcal{N}_{E/F}^h(1, n-1)$ and special cycles $\mathcal{Z}(x)$ from [Cho18] and [Cho22a]. In this section, we assume that F is an unramified extension of \mathbb{Q}_p . We fix integers n and $0 \leq h \leq n$. Let \check{E} be the completion of a maximal unramified extension of E and let $O_{\check{E}}$ be its ring of integers. We denote by k the residue field of $O_{\check{E}}$.

Let $(\mathbb{X}, i_{\mathbb{X}}, \lambda_{\mathbb{X}})$ be the following triple.

- (1) \mathbb{X} is a supersingular strict formal O_F -module of F -height $2n$ over \mathbb{F}_{q^2} .
- (2) $i_{\mathbb{X}} : O_E \rightarrow \text{End } \mathbb{X}$ is an O_E -action on \mathbb{X} that extends the O_F -action on \mathbb{X} . This satisfies the following signature condition of $(1, n-1)$: for all $a \in O_E$, we have

$$\text{Charpol}(i_{\mathbb{X}}(a), \text{Lie } \mathbb{X}) = (T - a)(T - a^*)^{n-1}.$$

- (3) $\lambda_{\mathbb{X}} : \mathbb{X} \rightarrow \mathbb{X}^{\vee}$ is a polarization such that the corresponding Rosati involution induces $*$ on O_E . We assume that $\ker \lambda_{\mathbb{X}} \subset \mathbb{X}[\pi]$ and its order is q^{2h} .

This triple is unique up to quasi-isogeny.

Let (Nilp) be the category of $O_{\tilde{E}}$ -schemes S such that π is locally nilpotent on S . We define $\mathcal{N}_{E/F}^h(1, n-1)$ as a functor sending S to the set of isomorphism classes of tuples $(X, i_X, \lambda_X, \rho_X)$.

Here, X is a supersingular strict formal O_F -module of F -height $2n$ over S and i_X is an O_E -action on X satisfying the signature condition of $(1, n-1)$. $\lambda_X : X \rightarrow X^\vee$ is a polarization such that its Rosati involution induces $*$ on E . $\rho_X : X_{\overline{S}} \rightarrow \mathbb{X} \times_{\mathbb{F}_{q^2}} \overline{S}$ is an O_E -linear quasi-isogeny of height 0, where $\overline{S} = S \times_{O_E} \mathbb{F}_{q^2}$. We also assume that the following diagram commutes up to a constant in O_F^\times :

$$\begin{array}{ccc} X_{\overline{S}} & \xrightarrow{\lambda_{X_{\overline{S}}}} & X_{\overline{S}}^\vee \\ \downarrow \rho_X & & \rho_X^\vee \uparrow \\ \mathbb{X}_{\overline{S}} & \xrightarrow{\lambda_{\mathbb{X}_{\overline{S}}}} & \mathbb{X}_{\overline{S}}^\vee. \end{array}$$

Then $\mathcal{N}_{E/F}^h(1, n-1)$ is representable by a formal scheme over $\text{Spf } O_{\tilde{E}}$ which is locally formally of finite type and regular ([Cho18, Proposition 3.33]).

To define special cycles on $\mathcal{N} := \mathcal{N}_{E/F}^h(1, n-1)$, we need to fix a triple $(\overline{Y}, i_{\overline{Y}}, \lambda_{\overline{Y}})$: Let \overline{Y} be a strict formal O_F -module of F -height 2 over k , and let $i_{\overline{Y}}$ be an O_E -action satisfying the signature condition of signature $(0, 1)$. Also, $\lambda_{\overline{Y}}$ is a principal polarization. Then, we can consider the corresponding moduli space $\mathcal{N}^0(0, 1)$.

We define $\mathbb{V} = \text{Hom}_{O_E}(\overline{Y}, \mathbb{X}) \otimes_{\mathbb{Z}} \mathbb{Q}$ to be the space of special homomorphisms. For each $x, y \in \mathbb{V}$, we define the hermitian form h on \mathbb{V} as

$$h(x, y) = \lambda_{\overline{Y}}^{-1} \circ y^\vee \circ \lambda_{\mathbb{X}} \circ x \in \text{End}_{O_E}(\overline{Y}) \otimes \mathbb{Q} \stackrel{i_{\overline{Y}}^{-1}}{\simeq} E.$$

For $x \in \mathbb{V}$, we define the special cycles $\mathcal{Z}(x)$ to be the closed formal scheme of $\mathcal{N}^0 \times \mathcal{N}$ as follows: For each S in (Nilp) , $\mathcal{Z}(x)(S)$ is the set of all points $(\overline{Y}, i_{\overline{Y}}, \lambda_{\overline{Y}}, \rho_{\overline{Y}}, X, i_X, \lambda_X, \rho_X)$ in $\mathcal{N}^0 \times \mathcal{N}(S)$ such that the quasi-homomorphism

$$\rho_X^{-1} \circ x \circ \rho_{\overline{Y}} : \overline{Y} \times_S \overline{S} \rightarrow X \times_S \overline{S}$$

extends to a homomorphism from \overline{Y} to X .

For a basis $\{x_1, x_2, \dots, x_n\}$ of \mathbb{V} , we denote by $\langle \mathcal{Z}(x_1), \mathcal{Z}(x_2), \dots, \mathcal{Z}(x_n) \rangle$ the arithmetic intersection number in $\mathcal{N} = \mathcal{N}_{E/F}^h(1, n-1)$:

$$\langle \mathcal{Z}(x_1), \mathcal{Z}(x_2), \dots, \mathcal{Z}(x_n) \rangle := \chi(O_{\mathcal{Z}(x_1)} \otimes_{\mathbb{L}}^{\mathbb{L}} \cdots \otimes_{\mathbb{L}}^{\mathbb{L}} O_{\mathcal{Z}(x_n)}).$$

Here χ is the Euler-Poincare characteristic and $\otimes_{\mathbb{L}}$ is the derived tensor product of $O_{\mathcal{N}}$ -modules.

Now, we can state the following conjecture from [Cho22a, Conjecture 3.17, Conjecture 3.25] and [Cho22b, Conjecture 4.9].

Conjecture 3.1. ([Cho22a, Conjecture 3.17, Conjecture 3.25], [Cho22b, Conjecture 4.9]) *For a basis $\{x_1, \dots, x_{2n}\}$ of \mathbb{V} , and special cycles $\mathcal{Z}(x_1), \dots, \mathcal{Z}(x_{2n})$ in $\mathcal{N}_{E/F}^n(1, 2n-1)$, we have*

$$\langle \mathcal{Z}(x_1), \dots, \mathcal{Z}(x_{2n}) \rangle = \sum_{\lambda \in \Lambda_{2n}^+} D_\lambda A_\lambda(B) - \sum_{0 \leq i \leq n-1} \mathfrak{b}_i^0 A_{(1^i, 0^{2n-i})}(B).$$

Here B is the $2n \times 2n$ hermitian matrix $(h(x_i, x_j))$ in $X_{2n}(O_E)$ and $D_\lambda, \mathfrak{b}_i^0$ are some constants (we refer to [Cho22b] for the precise definitions of these constants).

Remark 3.2. Conjecture 3.1 gives conjectural formulas for the arithmetic intersection numbers of special cycles in $\mathcal{N}_{E/F}^h(1, n-1)$ for any h after some reductions. Indeed, [Cho22a, Conjecture 3.17] and [Cho22a, Conjecture 3.25] are not only about special cycles \mathcal{Z} , these are also about special cycles \mathcal{Y} (even though we will not discuss \mathcal{Y} -cycles here). In the case that $h = 0$, this is the Kudla-Rapoport conjecture [KR11, Conjecture 1.3] and is proved in [LZ22]. The case when $h = 1$ is also proved in [LZ22].

3.2. Some applications. In this subsection, we apply Theorem 2.5 to compute some arithmetic intersection numbers of special cycles and suggest a way to understand these.

Example 3.3. [$\mathcal{N}_{E/F}^1(1, 1)$] In this example, we consider the arithmetic intersection numbers of special cycles in $\mathcal{N}_{E/F}^1(1, 1)$ and answer to the question raised in [Cho22b, Remark 4.11]. In fact, this is our main motivation for Theorem 2.5. In [San17], Sankaran proved that for special homomorphisms $x, y \in \mathbb{V}$, the arithmetic intersection numbers of special cycles $\langle \mathcal{Z}(x), \mathcal{Z}(y) \rangle$ in $\mathcal{N}_{E/F}^1(1, 1)$ is

$$\frac{q}{(q+1)^2} \left\{ \alpha'(A_{10}, B) - \frac{q^2}{q^2-1} \alpha(A_{00}, B) \right\},$$

where $B = \begin{pmatrix} h(x, x) & h(x, y) \\ h(y, x) & h(y, y) \end{pmatrix}$. Let us write $A_{\alpha\beta}(x, y)$ for $A_{\alpha\beta}(B)$. In terms of weighted lattice counting (i.e a version of the Cho-Yamauchi's formula [CY20]) in [Cho22b], this can be written as

$$\begin{aligned} \langle \mathcal{Z}(x), \mathcal{Z}(y) \rangle &= -(q-1)A_{11}(x, y) - (q^2-1) \sum_{\lambda, \kappa \geq 2} A_{\lambda\kappa}(x, y) \\ &\quad + \sum_{\lambda \geq 2} A_{\lambda 1}(x, y) + \sum_{\lambda \geq 2} A_{\lambda 0}(x, y). \end{aligned}$$

Therefore, we have that

$$\begin{aligned}
(3.2.1) \quad & \langle \mathcal{Z}(x), \mathcal{Z}(y) \rangle - \langle \mathcal{Z}\left(\frac{x}{\pi}\right), \mathcal{Z}\left(\frac{y}{\pi}\right) \rangle \\
&= -(q-1)A_{11}(x, y) - (q^2-1) \sum_{\lambda, \kappa \geq 2} A_{\lambda\kappa}(x, y) + \sum_{\lambda \geq 2} A_{\lambda 1}(x, y) + \sum_{\lambda \geq 2} A_{\lambda 0}(x, y) \\
&+ (q-1)A_{11}\left(\frac{x}{\pi}, \frac{y}{\pi}\right) + (q^2-1) \sum_{\lambda, \kappa \geq 2} A_{\lambda\kappa}\left(\frac{x}{\pi}, \frac{y}{\pi}\right) - \sum_{\lambda \geq 2} A_{\lambda 1}\left(\frac{x}{\pi}, \frac{y}{\pi}\right) - \sum_{\lambda \geq 2} A_{\lambda 0}\left(\frac{x}{\pi}, \frac{y}{\pi}\right) \\
&= -(q-1)A_{11}(x, y) - (q^2-1) \sum_{\lambda, \kappa \geq 2} A_{\lambda\kappa}(x, y) + \sum_{\lambda \geq 2} A_{\lambda 1}(x, y) + \sum_{\lambda \geq 2} A_{\lambda 0}(x, y) \\
&+ (q-1)A_{33}(x, y) + (q^2-1) \sum_{\lambda, \kappa \geq 4} A_{\lambda\kappa}(x, y) - \sum_{\lambda \geq 4} A_{\lambda 3}(x, y) - \sum_{\lambda \geq 4} A_{\lambda 2}(x, y) \\
&= -(q-1)A_{11}(x, y) - (q^2-1)A_{22}(x, y) - (q^2-q)A_{33}(x, y) + A_{20}(x, y) + A_{31}(x, y) \\
&+ \sum_{\lambda \geq 4} A_{\lambda 0}(x, y) - q^2 \sum_{\lambda \geq 4} A_{\lambda 2}(x, y) + \sum_{\lambda \geq 4} A_{\lambda 1}(x, y) - q^2 \sum_{\lambda \geq 4} A_{\lambda 3}(x, y)
\end{aligned}$$

On the other hand, by [San13, Theorem 3.14], we know that for $y \in \mathbb{V}$ with $\text{val}(h(y, y)) \geq 2$, we have

$$\mathcal{Z}(y) - \mathcal{Z}\left(\frac{y}{\pi}\right) = \sum_{\substack{y/\pi \in \Lambda \\ \Lambda \simeq A_{-1, -1}}} \mathbb{P}_{\Lambda}^1 + \sum_{\substack{y/\pi \in \Lambda \\ \Lambda \simeq A_{00}}} \mathbb{P}_{\Lambda}^1$$

Here $\Lambda \simeq A_{\alpha\beta}$ means that the lattice Λ has the hermitian matrix $A_{\alpha\beta}$. Also, by [San17, Lemma 2.11], we have that

$$\langle \mathcal{Z}(x), \mathbb{P}_{\Lambda}^1 \rangle = -q1_{\Lambda}\left(\frac{x}{\pi}\right),$$

if $\Lambda \simeq A_{-1, -1}$, and

$$\langle \mathcal{Z}(x), \mathbb{P}_{\Lambda}^1 \rangle = 1_{\Lambda}(x),$$

if $\Lambda \simeq A_{00}$.

Therefore, we have that

$$\begin{aligned}
\langle \mathcal{Z}(x), \mathcal{Z}(y) - \mathcal{Z}\left(\frac{y}{\pi}\right) \rangle &= -qA_{-1, -1}\left(\frac{x}{\pi}, \frac{y}{\pi}\right) + A_{00}\left(x, \frac{y}{\pi}\right) \\
&= -qA_{11}(x, y) + A_{00}\left(x, \frac{y}{\pi}\right).
\end{aligned}$$

Also, we have that

$$\begin{aligned}
& \langle \mathcal{Z}(x), \mathcal{Z}(y) \rangle - \langle \mathcal{Z}\left(\frac{x}{\pi}\right), \mathcal{Z}\left(\frac{y}{\pi}\right) \rangle \\
&= \langle \mathcal{Z}(x), \mathcal{Z}(y) - \mathcal{Z}\left(\frac{y}{\pi}\right) \rangle + \langle \mathcal{Z}(x) - \mathcal{Z}\left(\frac{x}{\pi}\right), \mathcal{Z}\left(\frac{y}{\pi}\right) \rangle \\
&= -qA_{11}(x, y) + A_{00}\left(x, \frac{y}{\pi}\right) - qA_{11}\left(x, \frac{y}{\pi}\right) + A_{00}\left(\frac{x}{\pi}, \frac{y}{\pi}\right) \\
&= -qA_{11}(x, y) + A_{00}\left(x, \frac{y}{\pi}\right) - qA_{11}\left(x, \frac{y}{\pi}\right) + A_{22}(x, y).
\end{aligned}$$

Note that this equation looks very different from (3.2.1).

Combining this and (3.2.1), we have that

$$\begin{aligned}
(3.2.2) \quad & -qA_{11}(x, y) + A_{00}\left(x, \frac{y}{\pi}\right) - qA_{11}\left(x, \frac{y}{\pi}\right) + A_{22}(x, y) \\
&= -(q-1)A_{11}(x, y) - (q^2-1)A_{22}(x, y) - (q^2-q)A_{33}(x, y) + A_{20}(x, y) + A_{31}(x, y) \\
&+ \sum_{\lambda \geq 4} A_{\lambda 0}(x, y) - q^2 \sum_{\lambda \geq 4} A_{\lambda 2}(x, y) + \sum_{\lambda \geq 4} A_{\lambda 1}(x, y) - q^2 \sum_{\lambda \geq 4} A_{\lambda 3}(x, y)
\end{aligned}$$

Why do we have this equality? This question is the starting point for our Theorem 2.5.

Note that $A_{00}(x, y) - qA_{11}(x, y) = 1$ for any x, y such that $\begin{pmatrix} h(x, x) & h(x, y) \\ h(y, x) & h(y, y) \end{pmatrix}$ is integral. Therefore, we have that

$$A_{00}(x, y) - qA_{11}(x, y) = A_{00}\left(x, \frac{y}{\pi}\right) - qA_{11}\left(x, \frac{y}{\pi}\right).$$

This implies that (3.2.2) is equivalent to

$$\begin{aligned}
0 &= A_{00}(x, y) - (q+1)A_{11}(x, y) + q^2A_{22}(x, y) - A_{20}(x, y) \\
&+ (q^2-q)A_{33}(x, y) - A_{31}(x, y) \\
&- \sum_{\lambda \geq 4} \{A_{\lambda 0}(x, y) - q^2A_{\lambda 2}(x, y)\} - \sum_{\lambda \geq 4} \{A_{\lambda 1}(x, y) - q^2A_{\lambda 3}(x, y)\} \\
&= \{A_{00}(x, y) - (q+1)A_{11}(x, y) + qA_{22}(x, y)\} \\
&- \{A_{20} - q(q-1)A_{22}(x, y)\} - \{A_{31}(x, y) - q(q-1)A_{33}(x, y)\} \\
&- \sum_{\lambda \geq 4} \{A_{\lambda 0}(x, y) - q^2A_{\lambda 2}(x, y)\} - \sum_{\lambda \geq 4} \{A_{\lambda 1}(x, y) - q^2A_{\lambda 3}(x, y)\}.
\end{aligned}$$

Now, Corollary 2.17 gives the reason why this equality holds for any x, y such that $\frac{1}{\pi^2} \begin{pmatrix} h(x, x) & h(x, y) \\ h(y, x) & h(y, y) \end{pmatrix}$ is integral.

Example 3.4. $[\mathcal{N}_{E/F}^0(1, 3)]$ Assume that $\alpha > \beta \geq \gamma \geq \delta \geq 0$ and α is even. Let $L_{\beta\gamma\delta}$ be a lattice of rank 3 in \mathbb{V} with the hermitian matrix $A_{\beta\gamma\delta}$ and let

x be a special homomorphism in \mathbb{V} with $\text{val}(h(x, x)) = \alpha$. Assume that x is orthogonal to $L_{\beta\gamma\delta}$. By the Kudla-Rapoport conjecture [KR11, Conjecture 1.3] and its proof [LZ22], we have that the arithmetic intersection number of special cycles $\langle \mathcal{Z}(L_{\beta\gamma\delta}), \mathcal{Z}(x) \rangle$ is $A'_{0000}(A_{\alpha\beta\gamma\delta})$. We will show that

$$(3.2.3) \quad \begin{aligned} A'_{0000}(A_{\alpha\beta\gamma\delta}) &= A_{100}(A_{\beta\gamma\delta})A'_{0000}(A_{\alpha 100}) \\ &+ \sum_{\lambda \geq 3} A_{\lambda 00}(A_{\beta\gamma\delta})\{A'_{0000}(A_{\alpha\lambda 00}) - A'_{0000}(A_{\alpha(\lambda-2)00})\} \\ &+ (1 - q^2) \sum_{\lambda \geq \kappa \geq 1} A_{\lambda\kappa 0}(A_{\beta\gamma\delta})A_{1110}(A_{\alpha\lambda\kappa 0}) \\ &+ (1 + q)(1 - q^2) \sum_{\lambda \geq \kappa \geq \mu \geq 1} A_{\lambda\kappa\mu}(A_{\beta\gamma\delta})A_{1110}(A_{\alpha\lambda\kappa\mu}). \end{aligned}$$

in some cases. This is modelled on the analytic part of [LZ22] (in particular, [LZ22, Theorem 4.2.1, Theorem 5.4.1]). Note that this formula relates $A'_{0000}(A_{\alpha\beta\gamma\delta})$ to the counting of vertex lattices of type 3 (i.e., its hermitian matrix is A_{1110}) plus additional parts (these correspond to the horizontal parts of $\mathcal{Z}(L_{\beta\gamma\delta})$ by [LZ22, Theorem 4.2.1]).

By the Cho-Yamauchi's formula in unitary case [LZ22, Corollary 3.5.3], we have

$$(3.2.4) \quad \begin{aligned} A'_{0000}(A_{\alpha\beta\gamma\delta}) &= \sum_{\lambda \geq 1} A_{\lambda 000}(A_{\alpha\beta\gamma\delta}) + (1 + q) \sum_{\lambda \geq \kappa \geq 1} A_{\lambda\kappa 00}(A_{\alpha\beta\gamma\delta}) \\ &+ (1 + q)(1 - q^2) \sum_{\lambda \geq \kappa \geq \mu \geq 1} A_{\lambda\kappa\mu 0}(A_{\alpha\beta\gamma\delta}) \\ &+ (1 + q)(1 - q^2)(1 + q^3) \sum_{\lambda \geq \kappa \geq \mu \geq \eta \geq 1} A_{\lambda\kappa\mu\eta}(A_{\alpha\beta\gamma\delta}). \end{aligned}$$

Now, let us show that (3.2.3) holds in the following cases.

- (1) ($L_{\beta\gamma\delta} = L_{100}$ case) Indeed, in $L_{\beta\gamma\delta} = L_{\nu 00}$ case, (3.2.3) holds since all $A_{\lambda 00}(A_{\nu 00})$ are 1 for all $\nu \geq \lambda$. But, we will compute $A'_{0000}(A_{\alpha 100})$ for later use. By (3.2.4) we have that

$$\begin{aligned} A'_{0000}(A_{\alpha 100}) &= A_{1000}(A_{\alpha 100}) + \cdots + A_{(\alpha-1)000}(A_{\alpha 100}) \\ &+ (1 + q)\{A_{2100}(A_{\alpha 100}) + \cdots + A_{\alpha 100}(A_{\alpha 100})\}. \end{aligned}$$

Also, by Corollary 2.23, we have

$$\left\{ \begin{array}{l} A_{1000}(A_{\alpha 100}) = A_{10}(A_{\alpha 1}) = 1, \\ A_{3000}(A_{\alpha 100}) = A_{30}(A_{\alpha 1}) = q^2 A_{32}(A_{\alpha 1}) = 0, \\ \vdots \\ A_{(\alpha-1)000}(A_{\alpha 100}) = A_{(\alpha-1)0}(A_{\alpha 1}) = q^2 A_{(\alpha-1)2}(A_{\alpha 1}) = 0, \\ \\ A_{2100}(A_{\alpha 100}) = A_{21}(A_{\alpha 1}) = A_{10}(A_{(\alpha-1)0}) = 1, \\ \vdots \\ A_{\alpha 100}(A_{\alpha 100}) = 1. \end{array} \right.$$

This implies that

$$A_{0000}(A_{\alpha 100})' = (q+1)\frac{\alpha}{2} + 1.$$

Also (3.2.3) holds since $A_{100}(A_{100}) = 1$.

(2) ($L_{\beta\gamma\delta} = L_{111}$ case) By (3.2.4) we have that

$$\begin{aligned} A'_{0000}(A_{\alpha 111}) &= A_{1000}(A_{\alpha 111}) + \cdots + A_{(\alpha-1)000}(A_{\alpha 111}) \\ &\quad + (1+q)\{A_{2100}(A_{\alpha 111}) + \cdots + A_{\alpha 111}(A_{\alpha 111})\} \\ &\quad + (1+q)(1-q^2)\{A_{1110}(A_{\alpha 111}) + \cdots + A_{(\alpha-1)110}(A_{\alpha 111})\} \\ &\quad + (1+q)(1-q^2)(1+q^3)\{A_{2111}(A_{\alpha 111}) + \cdots + A_{\alpha 111}(A_{\alpha 111})\}. \end{aligned}$$

Now, let us compute the following representation densities by using Corollary 2.23.

$$\begin{cases} A_{2111}(A_{\alpha 111}) = A_2(A_{\alpha}) = 1, \\ \vdots \\ A_{\alpha 111}(A_{\alpha 111}) = 1, \end{cases}$$

$$\begin{cases} A_{1110}(A_{\alpha 111}) = A_{2111}(A_{\alpha 111}) = 1, \\ A_{3110}(A_{\alpha 111}) = A_{3211}(A_{\alpha 111}) = 0, \\ \vdots \\ A_{(\alpha-1)110}(A_{\alpha 111}) = q^2 A_{(\alpha-1)211}(A_{\alpha 111}) = 0, \end{cases}$$

$$\begin{cases} A_{2100}(A_{\alpha 111}) = q^2(q^3+1)A_{2111}(A_{\alpha 111}) = q^2(q^3+1), \\ \vdots \\ A_{\alpha 100}(A_{\alpha 111}) = q^2(q^3+1)A_{\alpha 111}(A_{\alpha 111}) = q^2(q^3+1), \end{cases}$$

$$\begin{cases} A_{1000}(A_{\alpha 111}) = (q^3+1)A_{2111}(A_{\alpha 111}) = (q^3+1), \\ A_{3000}(A_{\alpha 111}) = q^4(q+1)A_{3211}(A_{\alpha 111}) = 0, \\ \vdots \\ A_{(\alpha-1)000}(A_{\alpha 111}) = q^4(q+1)A_{(\alpha-1)211}(A_{\alpha 111}) = 0. \end{cases}$$

Therefore, we have that

$$\begin{aligned} A'_{0000}(A_{\alpha 111}) &= \frac{\alpha}{2}(1+q)(1-q^2)(1+q^3) + (1+q)(1-q^2) + \frac{\alpha}{2}q^2(q^3+1) + (q^3+1) \\ &= \frac{\alpha}{2}(q+1)(q^3+1) + 2 + q - q^2. \end{aligned}$$

Now, note that $A_{100}(A_{111}) = (q^3 + 1)A_{111}(A_{111}) = q^3 + 1$, and $A_{1110}(A_{\alpha 111}) = 1$. This implies that

$$\begin{aligned} & A_{100}(A_{111})A'_{0000}(A_{\alpha 100}) + (1+q)(1-q^2)A_{1110}(A_{\alpha 111}) \\ &= (q^3 + 1)\left\{(q+1)\frac{\alpha}{2} + 1\right\} + (1+q)(1-q^2) \\ &= (q^3 + 1)(q+1)\frac{\alpha}{2} + 2 + q - q^2. \end{aligned}$$

Therefore, (3.2.3) holds in this case.

- (3) ($L_{\beta\gamma\delta} = L_{300}$ case) Indeed, in $L_{\beta\gamma\delta} = L_{\nu 00}$ case, (3.2.3) holds since all $A_{\lambda 00}(A_{\nu 00})$ are 1 for all $\nu \geq \lambda$. But, we will compute $A'_{0000}(A_{\alpha 300})$ and $A'_{0000}(A_{\alpha 300}) - A'_{0000}(A_{\alpha 100})$ for later use. By (3.2.4) we have that

$$\begin{aligned} A'_{0000}(A_{\alpha 300}) &= A_{1000}(A_{\alpha 300}) + \cdots + A_{(\alpha-1)000}(A_{\alpha 300}) \\ &\quad + (1+q)\{A_{2100}(A_{\alpha 300}) + \cdots + A_{\alpha 100}(A_{\alpha 300}) \\ &\quad + A_{3200}(A_{\alpha 300}) + A_{5200}(A_{\alpha 300}) + \cdots + A_{(\alpha-1)200}(A_{\alpha 300}) \\ &\quad + A_{4300}(A_{\alpha 300}) + A_{6200}(A_{\alpha 300}) + \cdots + A_{\alpha 300}(A_{\alpha 300})\} \end{aligned}$$

Now, let us compute the following representation densities by using Corollary 2.23.

$$\begin{cases} A_{4300}(A_{\alpha 300}) = A_1(A_{\alpha-3}) = 1, \\ \vdots \\ A_{\alpha 300}(A_{\alpha 300}) = 1, \end{cases}$$

$$\begin{cases} A_{3200}(A_{\alpha 300}) = A_{10}(A_{(\alpha-2)1}) = A_{21}(A_{(\alpha-2)1}) = A_{10}(A_{(\alpha-3)0}) = 1, \\ A_{5200}(A_{\alpha 300}) = A_{30}(A_{(\alpha-2)1}) = q^2 A_{32}(A_{(\alpha-2)1}) = 0, \\ \vdots \\ A_{(\alpha-1)200}(A_{\alpha 300}) = A_{(\alpha-3)0}(A_{(\alpha-2)1}) = q^2 A_{(\alpha-3)2}(A_{(\alpha-2)1}) = 0, \end{cases}$$

$$\begin{cases} A_{2100}(A_{\alpha 300}) = A_{10}(A_{(\alpha-1)2}) = 1, \\ A_{4100}(A_{\alpha 300}) = A_{30}(A_{(\alpha-1)2}) = q^2 A_{32}(A_{(\alpha-1)2}) = q^2 A_{10}(A_{(\alpha-3)0}) = q^2, \\ \vdots \\ A_{\alpha 100}(A_{\alpha 300}) = A_{(\alpha-1)0}(A_{(\alpha-1)2}) = q^2 A_{(\alpha-1)2}(A_{(\alpha-1)2}) = q^2, \end{cases}$$

$$\left\{ \begin{array}{l} A_{1000}(A_{\alpha 300}) = A_{10}(A_{\alpha 3}) = 1, \\ A_{3000}(A_{\alpha 300}) = A_{30}(A_{\alpha 3}) = q^2 A_{32}(A_{\alpha 3}) = q^2 A_{10}(A_{(\alpha-2)1}) = q^2, \\ A_{5000}(A_{\alpha 300}) = A_{50}(A_{\alpha 3}) = q^2 A_{52}(A_{\alpha 3}) \\ \quad = q^2 A_{30}(A_{(\alpha-2)1}) = q^4 A_{32}(A_{(\alpha-2)1}) = 0, \\ \quad \vdots \\ A_{(\alpha-1)000}(A_{\alpha 300}) = A_{(\alpha-1)0}(A_{\alpha 3}) = q^2 A_{(\alpha-1)2}(A_{\alpha 3}) \\ \quad = q^2 A_{(\alpha-3)0}(A_{(\alpha-2)1}) = q^4 A_{(\alpha-3)2}(A_{(\alpha-2)1}) = 0 \end{array} \right.$$

Therefore, we have that

$$\begin{aligned} A'_{0000}(A_{\alpha 300}) &= 1 + q^2 + (1+q)q^2\left\{\frac{\alpha}{2} - 1\right\} + (1+q) + (1+q) + (1+q)\left\{\frac{\alpha}{2} - 1\right\} \\ &= (q+1)(q^2+1)\frac{\alpha}{2} - q^3 + q + 2. \end{aligned}$$

Also, we have that

$$A'_{0000}(A_{\alpha 300}) - A'_{0000}(A_{\alpha 100}) = q^2(q+1)\frac{\alpha}{2} - q^3 + q + 1.$$

(4) ($L_{\beta\gamma\delta} = L_{311}$) case) By (3.2.4) we have that

$$\begin{aligned} &A'_{0000}(A_{\alpha 311}) \\ &= A_{1000}(A_{\alpha 311}) + \cdots + A_{(\alpha-1)000}(A_{\alpha 311}) \\ &+ (1+q)\{A_{2100}(A_{\alpha 311}) + \cdots + A_{\alpha 100}(A_{\alpha 311}) \\ &\quad + A_{3200}(A_{\alpha 311}) + A_{5200}(A_{\alpha 311}) + \cdots + A_{(\alpha-1)200}(A_{\alpha 311}) \\ &\quad + A_{4300}(A_{\alpha 311}) + A_{6200}(A_{\alpha 311}) + \cdots + A_{\alpha 300}(A_{\alpha 311})\} \\ &+ (1+q)(1-q^2)\{A_{1110}(A_{\alpha 311}) + \cdots + A_{(\alpha-1)110}(A_{\alpha 311}) \\ &\quad + A_{2210}(A_{\alpha 311}) + \cdots + A_{\alpha 210}(A_{\alpha 311}) \\ &\quad + A_{3310}(A_{\alpha 311}) + \cdots + A_{(\alpha-1)310}(A_{\alpha 311})\} \\ &+ (1+q)(1-q^2)(1+q^3)\{A_{2111}(A_{\alpha 311}) + \cdots + A_{\alpha 111}(A_{\alpha 311}) \\ &\quad + A_{3211}(A_{\alpha 311}) + \cdots + A_{(\alpha-1)211}(A_{\alpha 311}) \\ &\quad + A_{4311}(A_{\alpha 311}) + \cdots + A_{\alpha 311}(A_{\alpha 311})\} \end{aligned}$$

Now, let us compute the following representation densities by using Corollary 2.23.

$$\left\{ \begin{array}{l} A_{4311}(A_{\alpha 311}) = 1, \\ \quad \vdots \\ A_{\alpha 311}(A_{\alpha 311}) = 1, \end{array} \right.$$

$$\begin{cases} A_{3211}(A_{\alpha 311}) = A_{10}(A_{(\alpha-2)1}) = 1, \\ A_{5211}(A_{\alpha 311}) = A_{30}(A_{(\alpha-2)1}) = q^2 A_{32}(A_{(\alpha-2)1}) = 0, \\ \quad \vdots \\ A_{(\alpha-1)211} = 0 \text{ (since } A_{(\alpha-1)211}(A_{\alpha 311}) \leq A_{5211}(A_{\alpha 311}) \text{)}, \end{cases}$$

$$\begin{cases} A_{2111}(A_{\alpha 311}) = A_{10}(A_{(\alpha-1)2}) = 1, \\ A_{4111}(A_{\alpha 311}) = A_{30}(A_{(\alpha-1)2}) = q^2 A_{32}(A_{(\alpha-1)2}) = q^2, \\ \quad \vdots \\ A_{\alpha 111}(A_{\alpha 311}) = A_{(\alpha-1)0}(A_{(\alpha-1)2}) = q^2 A_{(\alpha-1)2}(A_{(\alpha-1)2}) = q^2, \end{cases}$$

$$\begin{cases} A_{3310}(A_{\alpha 311}) = q^4 A_{3321}(A_{\alpha 311}) = 0, \\ \quad \vdots \\ A_{(\alpha-1)310}(A_{\alpha 311}) = 0, \end{cases}$$

$$\begin{cases} A_{2210}(A_{\alpha 311}) = q^2(q^2 - q + 1)A_{2221}(A_{\alpha 311}) = 0, \\ \quad \vdots \\ A_{\alpha 210}(A_{\alpha 311}) = 0, \end{cases}$$

$$\begin{cases} A_{1110}(A_{\alpha 311}) = A_{2111}(A_{\alpha 311}) = 1, \\ A_{3110}(A_{\alpha 311}) = q^2 A_{3211}(A_{\alpha 311}) = q^2, \\ A_{5110}(A_{\alpha 311}) = q^2 A_{5211}(A_{\alpha 311}) = 0 \\ \quad \vdots \\ A_{(\alpha-1)110}(A_{\alpha 311}) = 0, \end{cases}$$

$$\begin{cases} A_{4300}(A_{\alpha 311}) = q^4(q+1)A_{4311}(A_{\alpha 311}) = q^4(q+1), \\ \quad \vdots \\ A_{\alpha 300}(A_{\alpha 311}) = q^4(q+1)A_{\alpha 311}(A_{\alpha 311}) = q^4(q+1), \end{cases}$$

$$\begin{cases} A_{3200}(A_{\alpha 311}) = q^4(q+1)A_{3211}(A_{\alpha 311}) = q^4(q+1), \\ A_{5200}(A_{\alpha 311}) = q^4(q+1)A_{5211}(A_{\alpha 311}) = 0, \\ \quad \vdots \\ A_{(\alpha-1)200}(A_{\alpha 311}) = 0, \end{cases}$$

$$\left\{ \begin{array}{l} A_{2100}(A_{\alpha 311}) = q^2(q^3 + 1)A_{2111}(A_{\alpha 311}) = q^2(q^3 + 1), \\ A_{4100}(A_{\alpha 311}) = q^2(q^3 + 1)A_{4111}(A_{\alpha 311}) = q^4(q^3 + 1), \\ \quad \vdots \\ A_{\alpha 100}(A_{\alpha 311}) = q^2(q^3 + 1)A_{\alpha 111}(A_{\alpha 311}) = q^4(q^3 + 1), \end{array} \right.$$

$$\left\{ \begin{array}{l} A_{1000}(A_{\alpha 311}) = (q^3 + 1)A_{2111}(A_{\alpha 311}) = (q^3 + 1), \\ A_{3000}(A_{\alpha 311}) = q^4(q + 1)A_{3211}(A_{\alpha 311}) = q^4(q + 1), \\ A_{5000}(A_{\alpha 311}) = q^4(q + 1)A_{5211}(A_{\alpha 311}) = 0, \\ \quad \vdots \\ A_{(\alpha-1)000}(A_{\alpha 311}) = q^4(q + 1)A_{(\alpha-1)211}(A_{\alpha 311}) = 0, \end{array} \right.$$

Therefore, we have that

$$\begin{aligned} & A'_{0000}(A_{\alpha 311}) \\ &= (q^3 + 1) + q^4(q + 1) \\ &+ (1 + q)\left\{ \left(\frac{\alpha}{2} - 1\right)q^4(q^3 + 1) + q^2(q^3 + 1) + q^4(q + 1) + \left(\frac{\alpha}{2} - 1\right)q^4(q + 1) \right\} \\ &+ (1 + q)(1 - q^2)\{q^2 + 1\} \\ &+ (1 + q)(1 - q^2)(1 + q^3)\left\{ \left(\frac{\alpha}{2} - 1\right)q^2 + 1 + 1 + \left(\frac{\alpha}{2} - 1\right) \right\} \\ &= \frac{\alpha}{2}(q + 1)(q^5 + q^4 + q^3 + 1) - (q + 1)(q^5 - q^3 + q - 3). \end{aligned}$$

On the other hand, we have that

$$\left\{ \begin{array}{l} A_{100}(A_{311}) = (q^3 + 1)A_{111}(A_{311}) = (q^3 + 1), \\ A_{300}(A_{311}) = q^2(q + 1)A_{311}(A_{311}) = q^2(q + 1), \\ A_{111}(A_{111}) = 1, \\ A_{311}(A_{311}) = 1, \end{array} \right.$$

and

$$\left\{ \begin{array}{l} A'_{0000}(A_{\alpha 100}) = (q + 1)\frac{\alpha}{2} + 1, \\ A'_{0000}(A_{\alpha 300}) - A'_{0000}(A_{\alpha 100}) = q^2(q + 1)\frac{\alpha}{2} - q^3 + q + 1, \\ A_{1110}(A_{\alpha 111}) = 1, \\ A_{1110}(A_{\alpha 311}) = 1. \end{array} \right.$$

These imply that

$$\begin{aligned}
& A_{100}(A_{311})A'_{0000}(A_{\alpha 100}) + A_{300}(A_{311})\{A'_{0000}\{A_{\alpha 300}\} - A'_{0000}(A_{\alpha 100})\} \\
& + (1+q)(1-q^2)A_{111}(A_{311})A_{1110}(A_{\alpha 111}) + (1+q)(1-q^2)A_{311}(A_{311})A_{1110}(A_{\alpha 311}) \\
& = (q^3+1)\{(q+1)\frac{\alpha}{2}+1\} + q^2(q+1)\{q^2(q+1)\frac{\alpha}{2} - q^3+q+1\} \\
& + (1+q)(1-q^2) + (1+q)(1-q^2) \\
& = \frac{\alpha}{2}(q+1)(q^5+q^4+q^3+1) - (q+1)(q^5-q^3+q-3).
\end{aligned}$$

Therefore, (3.2.3) holds in this case.

(5) ($L_{\beta\gamma\delta} = L_{210}$) case) By (3.2.4) we have that

$$\begin{aligned}
& A'_{0000}(A_{\alpha 210}) \\
& = A_{1000}(A_{\alpha 210}) + \cdots + A_{(\alpha-1)210}(A_{\alpha 210}) \\
& + (1+q)\{A_{2100}(A_{\alpha 210}) + \cdots + A_{\alpha 100}(A_{\alpha 210}) \\
& \quad + A_{3200}(A_{\alpha 210}) + A_{5200}(A_{\alpha 210}) + \cdots + A_{(\alpha-1)200}(A_{\alpha 210})\} \\
& + (1+q)(1-q^2)\{A_{1110}(A_{\alpha 210}) + \cdots + A_{(\alpha-1)110}(A_{\alpha 210}) \\
& \quad + A_{2210}(A_{\alpha 210}) + \cdots + A_{\alpha 210}(A_{\alpha 210})\}
\end{aligned}$$

Now, let us compute the following representation densities by using Corollary 2.23.

$$\left\{ \begin{array}{l} A_{2210}(A_{\alpha 210}) = A_2(A_{\alpha}) = 1, \\ \vdots \\ A_{\alpha 210}(A_{\alpha 210}) = 1, \end{array} \right.$$

$$\left\{ \begin{array}{l} A_{1110}(A_{\alpha 210}) = A_{111}(A_{\alpha 21}) = A_{000}(A_{(\alpha-1)10}) = A_{00}(A_{(\alpha-1)1}) = (q+1), \\ A_{3110}(A_{\alpha 210}) = A_{311}(A_{\alpha 21}) = A_{20}(A_{(\alpha-1)1}) = q(q-1)A_{22}(A_{(\alpha-1)1}) = 0, \\ \vdots \\ A_{(\alpha-1)110}(A_{\alpha 210}) = 0, \end{array} \right.$$

$$\left\{ \begin{array}{l} A_{3200}(A_{\alpha 210}) = A_{320}(A_{\alpha 21}) = q^3(q-1)A_{322}(A_{\alpha 21}) = 0, \\ \vdots \\ A_{(\alpha-1)200}(A_{\alpha 210}) = 0, \end{array} \right.$$

$$\left\{ \begin{array}{l} A_{2100}(A_{\alpha 210}) = A_{210}(A_{\alpha 21}) = q(q-1)A_{221}(A_{\alpha 21}) = q(q-1), \\ A_{4100}(A_{\alpha 210}) = A_{410}(A_{\alpha 21}) = q^2A_{421}(A_{\alpha 21}) = q^2, \\ \vdots \\ A_{\alpha 100}(A_{\alpha 210}) = A_{\alpha 10}(A_{\alpha 21}) = q^2A_{\alpha 21}(A_{\alpha 21}) = q^2, \end{array} \right.$$

$$\left\{ \begin{array}{l} A_{1000}(A_{\alpha 210}) = A_{100}(A_{\alpha 21}) = (q^3 + 1)A_{111}(A_{\alpha 21}) - qA_{221}(A_{\alpha 21}) \\ \quad = (q^3 + 1)(q + 1) - q, \\ A_{3000}(A_{\alpha 210}) = A_{300}(A_{\alpha 21}) = q^2(q + 1)A_{311}(A_{\alpha 21}) - q^5 A_{322}(A_{\alpha 21}) = 0, \\ \quad \vdots \\ A_{(\alpha-1)000}(A_{\alpha 210}) = 0. \end{array} \right.$$

Therefore, we have that

$$\begin{aligned} A'_{0000}(A_{\alpha 210}) &= (q^3 + 1)(q + 1) - q + (1 + q)\left\{\left(\frac{\alpha}{2} - 1\right)q^2 + q(q - 1)\right\} \\ &\quad + (1 + q)(1 - q^2)\left\{(q + 1) + \frac{a}{2}\right\} \\ &= \frac{a}{2}(q + 1) - q^3 - q^2 + q + 2. \end{aligned}$$

On the other hand, we have that $A_{100}(A_{210}) = A_{10}(A_{21}) = 1$, and hence

$$\begin{aligned} &A_{100}(A_{210})A'_{0000}(A_{\alpha 100}) + (1 - q^2)A_{210}(A_{210})A_{1110}(A_{\alpha 210}) \\ &= (q + 1)\frac{\alpha}{2} + 1 + (1 - q^2)(q + 1) = \frac{\alpha}{2}(q + 1) - q^3 - q^2 + q + 2. \end{aligned}$$

This shows that (3.2.3) holds in this case.

(6) ($L_{\beta\gamma\delta} = L_{421}$) case) By (3.2.4) we have that
(3.2.5)

$$\begin{aligned} &A'_{0000}(A_{\alpha 421}) \\ &= A_{1000}(A_{\alpha 421}) + \cdots + A_{(\alpha-1)000}(A_{\alpha 421}) \\ &+ (1 + q)\{A_{2100}(A_{\alpha 421}) + \cdots + A_{\alpha 100}(A_{\alpha 421}) \\ &\quad + A_{3200}(A_{\alpha 421}) + A_{5200}(A_{\alpha 421}) + \cdots + A_{(\alpha-1)200}(A_{\alpha 421}) \\ &\quad + A_{4300}(A_{\alpha 421}) + A_{6200}(A_{\alpha 421}) + \cdots + A_{\alpha 300}(A_{\alpha 421}) \\ &\quad + A_{5400}(A_{\alpha 421}) + A_{7400}(A_{\alpha 421}) + \cdots + A_{(\alpha-1)400}(A_{\alpha 421})\} \\ &+ (1 + q)(1 - q^2)\{A_{1110}(A_{\alpha 421}) + \cdots + A_{(\alpha-1)110}(A_{\alpha 421}) \\ &\quad + A_{2210}(A_{\alpha 421}) + \cdots + A_{\alpha 210}(A_{\alpha 421}) \\ &\quad + A_{3310}(A_{\alpha 421}) + \cdots + A_{(\alpha-1)310}(A_{\alpha 421}) \\ &\quad + A_{4410}(A_{\alpha 421}) + \cdots + A_{\alpha 410}(A_{\alpha 421}) \\ &\quad + A_{3220}(A_{\alpha 421}) + \cdots + A_{(\alpha-1)220}(A_{\alpha 421}) \\ &\quad + A_{4320}(A_{\alpha 421}) + \cdots + A_{\alpha 320}(A_{\alpha 421}) \\ &\quad + A_{5420}(A_{\alpha 421}) + \cdots + A_{(\alpha-1)420}(A_{\alpha 421})\} \\ &+ (1 + q)(1 - q^2)(1 + q^3)\{A_{2111}(A_{\alpha 421}) + \cdots + A_{\alpha 111}(A_{\alpha 421}) \\ &\quad + A_{3211}(A_{\alpha 421}) + \cdots + A_{(\alpha-1)211}(A_{\alpha 421}) \\ &\quad + A_{4311}(A_{\alpha 421}) + \cdots + A_{\alpha 311}(A_{\alpha 421}) \\ &\quad + A_{5411}(A_{\alpha 421}) + \cdots + A_{(\alpha-1)411}(A_{\alpha 421}) \\ &\quad + A_{2221}(A_{\alpha 421}) + \cdots + A_{\alpha 221}(A_{\alpha 421}) \\ &\quad + A_{3321}(A_{\alpha 421}) + \cdots + A_{(\alpha-1)321}(A_{\alpha 421}) \\ &\quad + A_{4421}(A_{\alpha 421}) + \cdots + A_{\alpha 421}(A_{\alpha 421})\}. \end{aligned}$$

Now, let us compute the following representation densities by using Corollary 2.23.

$$\begin{cases}
A_{4421}(A_{\alpha 421}) = A_4(A_\alpha) = 1, \\
\vdots \\
A_{\alpha 421}(A_{\alpha 421}) = 1,
\end{cases}$$

$$\begin{cases}
A_{3321}(A_{\alpha 421}) = A_{00}(A_{(\alpha-3)1}) = (q+1), \\
A_{5321}(A_{\alpha 421}) = A_{53}(A_{\alpha 4}) = A_{20}(A_{(\alpha-3)1}) = q(q-1)A_{22}(A_{(\alpha-3)1}) = 0, \\
\vdots \\
A_{(\alpha-1)321}(A_{\alpha 421}) = 0,
\end{cases}$$

$$\begin{cases}
A_{2221}(A_{\alpha 421}) = A_{00}(A_{(\alpha-2)2}) = (q+1)A_{11}(A_{(\alpha-2)2}) - qA_{22}(A_{(\alpha-2)2}) \\
\quad = (q+1)A_{00}(A_{(\alpha-3)1}) - q = (q^2 + q + 1), \\
A_{4221}(A_{\alpha 421}) = A_{20}(A_{(\alpha-2)2}) = q(q-1)A_{22}(A_{(\alpha-2)2}) = q(q-1), \\
A_{6221}(A_{\alpha 421}) = A_{40}(A_{(\alpha-2)2}) = q^2A_{42}(A_{(\alpha-2)2}) = q^2, \\
\vdots \\
A_{\alpha 221}(A_{\alpha 421}) = A_{(\alpha-2)0}(A_{(\alpha-2)2}) = q^2A_{(\alpha-2)2}(A_{(\alpha-2)2}) = q^2,
\end{cases}$$

$$\begin{cases}
A_{5411}(A_{\alpha 421}) = A_{430}(A_{(\alpha-1)31}) = q^4A_{432}(A_{(\alpha-1)31}) = 0, \\
\vdots \\
A_{(\alpha-1)411}(A_{\alpha 421}) = 0,
\end{cases}$$

$$\begin{cases}
A_{4311}(A_{\alpha 421}) = A_{320}(A_{(\alpha-1)31}) = q^3(q-1)A_{332}(A_{(\alpha-1)31}) = 0, \\
\vdots \\
A_{\alpha 311}(A_{\alpha 421}) = 0,
\end{cases}$$

$$\begin{cases}
A_{3211}(A_{\alpha 421}) = A_{210}(A_{(\alpha-1)31}) = q(q-1)A_{221}(A_{(\alpha-1)31}) \\
\quad = q(q-1)A_{00}(A_{(\alpha-3)1}) = q(q^2 - 1), \\
A_{5211}(A_{\alpha 421}) = A_{521}(A_{\alpha 42}) = A_{410}(A_{(\alpha-1)31}) = q^2A_{421}(A_{(\alpha-1)31}) \\
\quad = q^2A_{20}(A_{(\alpha-3)1}) = 0, \\
\vdots \\
A_{(\alpha-1)211}(A_{\alpha 421}) = 0,
\end{cases}$$

$$\begin{cases}
A_{2111}(A_{\alpha 421}) = A_{100}(A_{(\alpha-1)31}) = (q^3 + 1)A_{111}(A_{(\alpha-1)31}) - qA_{221}(A_{(\alpha-1)31}) \\
\quad = (q^3 + 1)A_{00}(A_{(\alpha-2)2}) - qA_{00}(A_{(\alpha-3)1}) \\
\quad = (q^3 + 1)(q^2 + q + 1) - q(q + 1), \\
A_{4111}(A_{\alpha 421}) = A_{300}(A_{(\alpha-1)31}) = q^2(q+1)A_{311}(A_{(\alpha-1)31}) \\
\quad = q^2(q+1)A_{20}(A_{(\alpha-2)2}) = q^3(q^2 - 1)A_{22}(A_{(\alpha-2)2}) = q^3(q^2 - 1),
\end{cases}$$

$$\left\{ \begin{array}{l} A_{6111}(A_{\alpha 421}) = A_{500}(A_{(\alpha-1)31}) = q^2(q+1)A_{511}(A_{(\alpha-1)31}) \\ \quad = q^2(q+1)A_{40}(A_{(\alpha-2)2}) = q^4(q+1)A_{42}(A_{(\alpha-2)}) = q^4(q+1), \\ \quad \vdots \\ A_{\alpha 111}(A_{\alpha 421}) = A_{(\alpha-1)00}(A_{(\alpha-1)31}) = q^2(q+1)A_{(\alpha-1)11}(A_{(\alpha-1)31}) \\ \quad = q^2(q+1)A_{(\alpha-2)0}(A_{(\alpha-2)2}) = q^4(q+1)A_{(\alpha-2)2}(A_{(\alpha-2)}) = q^4(q+1), \end{array} \right.$$

$$\left\{ \begin{array}{l} A_{5420}(A_{\alpha 421}) = q^5(q-1)A_{5422}(A_{\alpha 421}) = 0, \\ \quad \vdots \\ A_{(\alpha-1)420}(A_{\alpha 421}) = 0, \end{array} \right.$$

$$\left\{ \begin{array}{l} A_{4320}(A_{\alpha 421}) = q^5(q-1)A_{4322}(A_{\alpha 421}) = 0, \\ \quad \vdots \\ A_{\alpha 320}(A_{\alpha 421}) = 0, \end{array} \right.$$

$$\left\{ \begin{array}{l} A_{3220}(A_{\alpha 421}) = q^4(q^2 - q + 1)A_{3222}(A_{\alpha 421}) = 0, \\ \quad \vdots \\ A_{(\alpha-1)220}(A_{\alpha 421}) = 0, \end{array} \right.$$

$$\left\{ \begin{array}{l} A_{4410}(A_{\alpha 421}) = q^4A_{4421}(A_{\alpha 421}) = q^4, \\ \quad \vdots \\ A_{\alpha 410}(A_{\alpha 421}) = q^4A_{\alpha 421}(A_{\alpha 421}) = q^4, \end{array} \right.$$

$$\left\{ \begin{array}{l} A_{3310}(A_{\alpha 421}) = q^4A_{3321}(A_{\alpha 421}) = q^4(q+1), \\ A_{5310}(A_{\alpha 421}) = q^4A_{5321}(A_{\alpha 421}) = 0, \\ \quad \vdots \\ A_{(\alpha-1)310}(A_{\alpha 421}) = 0, \end{array} \right.$$

$$\left\{ \begin{array}{l} A_{2210}(A_{\alpha 421}) = q^2(q^2 - q + 1)A_{2221}(A_{\alpha 421}) = q^2(q^2 - q + 1)(q^2 + q + 1), \\ A_{4210}(A_{\alpha 421}) = q^3(q-1)A_{4221}(A_{\alpha 421}) = q^4(q-1)^2, \\ A_{6210}(A_{\alpha 421}) = q^3(q-1)A_{6221}(A_{\alpha 421}) = q^5(q-1), \\ \quad \vdots \\ A_{\alpha 210}(A_{\alpha 421}) = q^3(q-1)A_{\alpha 221}(A_{\alpha 421}) = q^5(q-1), \end{array} \right.$$

$$\begin{cases}
A_{1110}(A_{\alpha 421}) = A_{2111}(A_{\alpha 421}) = (q^3 + 1)(q^2 + q + 1) - q(q + 1), \\
A_{3110}(A_{\alpha 421}) = q^2 A_{3211}(A_{\alpha 421}) = q^3(q^2 - 1), \\
A_{5110}(A_{\alpha 421}) = q^2 A_{5211}(A_{\alpha 421}) = 0, \\
\vdots \\
A_{(\alpha-1)110}(A_{\alpha 421}) = 0, \\
\\
A_{5400}(A_{\alpha 421}) = q^4(q + 1)A_{5411}(A_{\alpha 421}) = 0, \\
\vdots \\
A_{(\alpha-1)400}(A_{\alpha 421}) = 0, \\
\\
A_{4300}(A_{\alpha 421}) = q^4(q + 1)A_{4311}(A_{\alpha 421}) = 0, \\
\vdots \\
A_{\alpha 300}(A_{\alpha 421}) = 0, \\
\\
A_{3200}(A_{\alpha 421}) = q^4(q + 1)A_{3211}(A_{\alpha 421}) = q^5(q + 1)(q^2 - 1), \\
A_{5200}(A_{\alpha 421}) = q^4(q + 1)A_{5211}(A_{\alpha 421}) = 0, \\
\vdots \\
A_{(\alpha-1)200}(A_{\alpha 421}) = 0, \\
\\
A_{2100}(A_{\alpha 421}) = q^2(q^3 + 1)A_{2111}(A_{\alpha 421}) - q^3(q^2 - q + 1)A_{2221}(A_{\alpha 421}) \\
= q^2(q^3 + 1)\{(q^3 + 1)(q^2 + q + 1) - q(q + 1)\} - q^3(q^2 - q + 1)(q^2 + q + 1), \\
A_{4100}(A_{\alpha 421}) = q^2(q^3 + 1)A_{4111}(A_{\alpha 421}) - q^5 A_{4221}(A_{\alpha 421}) \\
= q^5(q^2 - 1)(q^3 + 1) - q^6(q - 1), \\
\\
A_{6100}(A_{\alpha 421}) = q^2(q^3 + 1)A_{6111}(A_{\alpha 421}) - q^5 A_{6221}(A_{\alpha 421}) \\
= q^6(q + 1)(q^3 + 1) - q^7, \\
\vdots \\
A_{\alpha 100}(A_{\alpha 421}) = q^2(q^3 + 1)A_{\alpha 111}(A_{\alpha 421}) - q^5 A_{\alpha 221}(A_{\alpha 421}) \\
= q^6(q + 1)(q^3 + 1) - q^7, \\
\\
A_{1000}(A_{\alpha 421}) = (q^3 + 1)A_{2111}(A_{\alpha 421}) - q^3 A_{2221}(A_{\alpha 421}) \\
= (q^3 + 1)\{(q^3 + 1)(q^2 + q + 1) - q(q + 1)\} - q^3(q^2 + q + 1), \\
A_{3000}(A_{\alpha 421}) = q^4(q + 1)A_{3211}(A_{\alpha 421}) = q^5(q + 1)(q^2 - 1), \\
\\
A_{5000}(A_{\alpha 421}) = q^4(q + 1)A_{5211}(A_{\alpha 421}) = 0, \\
\vdots \\
A_{(\alpha-1)000}(A_{\alpha 421}) = 0.
\end{cases}$$

Now, by (3.2.5), we have

$$\begin{aligned}
A'_{0000}(A_{\alpha 421}) &= (1+q)(1-q^2)(1+q^3) \\
&\quad \times \left\{ \left(\frac{\alpha}{2} - 1\right) + (q+1) + (q^2+q+1) + q(q-1) + q^2\left(\frac{\alpha}{2} - 2\right) \right. \\
&\quad \quad \left. + q(q^2-1) + (q^3+1)(q^2+q+1) - q(q+1) + q^3(q^2-1) \right. \\
&\quad \quad \left. + q^4(q+1)\left(\frac{\alpha}{2} - 2\right) \right\} \\
&\quad + (1+q)(1-q^2) \\
&\quad \times \left\{ q^4\left(\frac{\alpha}{2} - 1\right) + q^4(q+1) + q^2(q^2-q+1)(q^2+q+1) + q^4(q-1)^2 \right. \\
&\quad \left. + q^5(q-1)\left(\frac{\alpha}{2} - 2\right) + (q^3+1)(q^2+q+1) - q(q+1) + q^3(q^2-1) \right\} \\
&\quad + (1+q) \\
&\quad \times \left\{ q^5(q+1)(q^2-1) + q^2(q^3+1)\{(q^3+1)(q^2+q+1) - q(q+1)\} \right. \\
&\quad \quad \left. - q^3(q^2-q+1)(q^2+q+1) + q^5(q^2-1)(q^3+1) - q^6(q-1) \right. \\
&\quad \quad \left. + \{q^6(q+1)(q^3+1) - q^7\}\left(\frac{\alpha}{2} - 2\right) \right\} \\
&\quad + (q^3+1)\{(q^3+1)(q^2+q+1) - q(q+1)\} \\
&\quad \quad - q^3(q^2+q+1) + q^5(q+1)(q^2-1) \\
&= (1+q)(1-q^2)(1+q^3)\left\{\frac{\alpha}{2}(q^5+q^4+q^2+1) - q^4+q^3+2\right\} \\
&\quad + (1+q)(1-q^2)\left\{\frac{\alpha}{2}(q^6-q^5+q^4) + 3q^5+3q^4+q^2+1\right\} \\
&\quad + (1+q)\left\{\frac{\alpha}{2}(q^{10}+q^9+q^6) - q^9+q^8+q^7 - q^6 - q^5 - q^3 + q^2\right\} \\
&\quad \quad + 2q^8 + 2q^7 - q^5 + q^3 + 1 \\
&= \frac{a}{2}(q+1)(q^4+q^3+1) - q^8 - 3q^7 - 3q^6 - q^5 + 3q^4 + 2q^3 - q^2 + 3q + 4.
\end{aligned}$$

On the other hand, we have that

$$\begin{cases} A_{100}(A_{421}) = (q^3+1)A_{111}(A_{421}) - qA_{221}(A_{421}) = q^4+q^3+1, \\ A'_{0000}(A_{\alpha 100}) = \frac{a}{2}(q+1)+1, \end{cases}$$

$$\{A_{300}(A_{421}) = q^2(q+1)A_{311}(A_{421}) = 0,$$

$$\begin{cases} A_{210}(A_{421}) = q(q-1)A_{221}(A_{421}) = q(q-1), \\ A_{1110}(A_{\alpha 210}) = A_{111}(A_{\alpha 21}) = A_{00}(A_{(\alpha-1)1}) = (q+1), \end{cases}$$

$$\begin{cases} A_{410}(A_{421}) = q^2A_{421}(A_{421}) = q^2, \\ A_{1110}(A_{\alpha 410}) = A_{111}(A_{\alpha 41}) = A_{00}(A_{(\alpha-1)3}) = (q^3+q^2+q+1), \end{cases}$$

$$\begin{cases} A_{111}(A_{421}) = A_{00}(A_{31}) = q+1, \\ A_{1110}(A_{\alpha 111}) = A_{2111}(A_{\alpha 111}) = 1, \end{cases}$$

$$\begin{cases} A_{221}(A_{421}) = 1, \\ A_{1110}(A_{\alpha 221}) = A_{2111}(A_{\alpha 221}) = A_{100}(A_{(\alpha-1)11}) = (q^3 + 1)A_{111}(A_{(\alpha-1)11}) = (q^3 + 1), \end{cases}$$

$$\begin{cases} A_{311}(A_{421}) = 0, \end{cases}$$

$$\begin{cases} A_{421}(A_{421}) = 1, \\ A_{1110}(A_{\alpha 421}) = A_{2111}(A_{\alpha 421}) = (q^3 + 1)(q^2 + q + 1) - q(q + 1). \end{cases}$$

Therefore, we have that

$$\begin{aligned} & A_{100}(A_{421})A'_{0000}(A_{\alpha 100}) + A_{300}(A_{421})\{A'_{0000}(A_{\alpha 300}) - A'_{0000}(A_{\alpha 100})\} \\ & + (1 - q^2)A_{210}(A_{421})A_{1110}(A_{\alpha 210}) + (1 - q^2)A_{410}(A_{421})A_{1110}(A_{\alpha 410}) \\ & + (1 + q)(1 - q^2)A_{111}(A_{421})A_{1110}(A_{\alpha 111}) + (1 + q)(1 - q^2)A_{221}(A_{421})A_{1110}(A_{\alpha 221}) \\ & + (1 + q)(1 - q^2)A_{311}(A_{421})A_{1110}(A_{\alpha 311}) + (1 + q)(1 - q^2)A_{421}(A_{421})A_{1110}(A_{\alpha 421}) \\ & = \frac{\alpha}{2}(q + 1)(q^4 + q + 1) - q^8 - 3q^7 - 3q^6 - q^5 + 3q^4 + 2q^3 - q^2 + 3q + 4 \end{aligned}$$

This shows that (3.2.3) holds in this case.

From the example 3.4, we conjecture that the following formula holds in general.

Conjecture 3.5 ($\mathcal{N}_{E/F}^0(1, 3)$). *Assume that $\alpha > \beta \geq \gamma \geq \delta \geq 0$. Then we have*

$$\begin{aligned} A'_{0000}(A_{\alpha\beta\gamma\delta}) &= A_{100}(A_{\beta\gamma\delta})A'_{0000}(A_{\alpha 100}) \\ &+ \sum_{\lambda \geq 3} A_{\lambda 00}(A_{\beta\gamma\delta})\{A'_{0000}(A_{\alpha\lambda 00}) - A'_{0000}(A_{\alpha(\lambda-2)00})\} \\ &+ (1 - q^2) \sum_{\lambda \geq \kappa \geq 1} A_{\lambda\kappa 0}(A_{\beta\gamma\delta})A_{1110}(A_{\alpha\lambda\kappa 0}) \\ &+ (1 + q)(1 - q^2) \sum_{\lambda \geq \kappa \geq \mu \geq 1} A_{\lambda\kappa\mu}(A_{\beta\gamma\delta})A_{1110}(A_{\alpha\lambda\kappa\mu}). \end{aligned}$$

Remark 3.6. This Conjecture 3.5 suggests that the arithmetic intersection numbers of special cycles on unitary Rapoport-Zink spaces $\mathcal{N}_{E/F}^0(1, 3)$ are closely related to the counting of vertex lattices. Let $L_{\beta\gamma\delta}$ be a lattice of rank 3 with the hermitian matrix $A_{\beta\gamma\delta}$. By [LZ22, Theorem 4.2.1], $A'_{0000}(A_{\alpha\lambda 00})$ terms in the conjecture are related to the horizontal part of $\mathcal{Z}(L_{\beta\gamma\delta})$. Also, even though it is not clear yet (for example, we do not know why we have $(1 + q)(1 - q^2)$ in the sum instead of $(1 - q^2)$), we believe that this conjecture has information about the multiplicities of the vertical components attached to vertex lattices of type 3 in the special cycle $\mathcal{Z}(L_{\beta\gamma\delta})$.

Based on the fact that the irreducible components of the reduced subscheme of $\mathcal{N}_{E/F}^h(1, n - 1)$ are closely related to vertex lattices ([VW11], [Cho18]), it is reasonable to ask the following question: for the conjectural formula for the arithmetic intersection numbers of special cycles on

$\mathcal{N}_{E/F}^h(1, n-1)$ in [Cho22a, Conjecture 3.17, Conjecture 3.25], can we write it in terms of vertex lattice counting plus some additional parts like horizontal parts? We believe that this will give us important information about special cycles. We already have a version of the Cho-Yamauchi's formula in [Cho22b, Conjecture 4.9], and we have an efficient computational method by Theorem 2.5. Therefore, we think that we have enough tools to consider this question. We hope to return to this in the future.

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