Let \( p = 2 \). Consider the modular curve \( X_0(2) = \left( \Gamma_0(2) \backslash \mathbb{H} \right)^* \) which parametrizes pairs \((E, C)\) where \( C \) is a subgroup of order 2. As an affine curve, \( X_0(2) \) is given by \( xy = 2^{12} \). And we define

\[
f := 1/x = \frac{\Delta(q^2)}{\Delta(q)} = q \prod_{n=1}^{\infty} (1 + q^n)^{24}
\]

Since \( \Delta(q) \), \( \Delta(q^2) \) are weight 12 level 2 modular forms, \( f \) is a modular function of weight 0 level 2. We have a canonical map of degree 3 between \( X_0(2) \) and \( X_0(1) \), which maps the two cusps \( 0, \infty \) of \( X_0(2) \) to the one cusp \( \infty \) of \( X_0(1) \) and preserves \( q \)-expansions. (Picture was drawn here during the talk.) Since \( \Delta(q) \) has a simple zero at \( \infty \) and a double zero at \( 0 \) on \( X_0(2) \) and \( \Delta(q^2) \) has a double zero at \( \infty \) and a single zero at \( 0 \) on \( X_0(2) \), \( f \) has a simple zero at \( \infty \) and a simple pole at \( 0 \). Therefore,

\[
f : X_0(2) \to \mathbb{P}^1
\]

Now if we consider \( X_0(2) \) as a rigid analytic space, the collection \( \{1, f, f^2, f^3, \ldots\} \) forms a basis of the space of functions on \( \{x \in X_0(2) : |f(x)| \leq 1\} \) such that \( \sum_{n=0}^{\infty} a_n T^n, |a_n|_2 \to 0 \) as \( n \to \infty \), and where \( |T| < 1 \). The reduction of \( X_0(2)_{\mathbb{Q}_2} \) to \( X_0(2)_{\overline{\mathbb{Q}_2}} \) takes an 2-adic annulus of supersingular points \( 2^{-12} > |x| > 1 \) to the singular point of \( xy = 0 \). (Picture could be drawn here.)

If we write \( f(q) = \sum_{n=1}^{\infty} a_n q^n \), then our goal is to compute what \( U_2 \) looks like on \( \{1, f, f^2, f^3, \ldots\} \). If we let \( \alpha = f(\sqrt{q}) \) and \( \beta = f(-\sqrt{q}) \), then we have

\[
\alpha + \beta = \sum a_n q^{n/2} + \sum (-1)^n a_n q^{n/2} = 2 \sum a_n q^{n/2} + 2 \sum a_{2n} q^n = 2U_2 f
\]

Therefore \( U_2 f = \frac{1}{2}(\alpha + \beta) \), and \( U_2 f : X_0(2) \to \mathbb{P}^1_{\overline{\mathbb{Q}_2}} \) is a map of degree 2.

If \( f(q) \) is a modular form for \( \Gamma \), then \( f(\delta q) \) is a modular form for \( \delta^{-1}\Gamma\delta \). So \( f(\sqrt{q}) = f(\tau/2) = f\left(\left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right], \tau\right) \) is a modular form on \( \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]^{-1} \Gamma_0(2) \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right] \supseteq \Gamma(2) = \left\{ \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right] \right\} \) mod 2

Note that we know the degree of \( \alpha, \beta \) are 2, and looking at where our maps factor through we have

\[
\alpha, \beta : X(2) \to \mathbb{P}^1 \quad \text{with} \quad \tau \mapsto \tau/2 \quad \text{and} \quad f \mapsto 1/2
\]

and

\[
\alpha, \beta : X_0(2) \to \mathbb{P}^1 \quad \text{with} \quad \tau \mapsto \tau/2
\]
Thus $\deg(U_2 f) \leq 4$ on $X(2)$ and we have the following diagram

$$U_2 f = 1/2(\alpha + \beta) : X(2) \to \mathbb{P}^1$$

$$\leq 4$$

$$\leq 2$$

$$X_0(2)$$

So $U_2 f$ will factor through $X_0(2)$. Now the question becomes how far out do we need to look for $f$ expansions?

By the above discussion we see that $\deg(U_2 f) = 2$ on $X_0(2)$, so

$$U_2 f = \frac{\alpha + f + f^2}{\alpha + f + f^2}$$

and we need to figure out what should go in the blanks. Since $f : 0 \to \infty$ and $f : \infty \to 0$, $U_2 f = \frac{\alpha + f + f^2}{\alpha + f + f^2}$. By a computer calculation we see that $U_2 f = 24f + 2084f^2$.

The matrix for $2U_2$ with respect to $\{1, f, f^2, f^3, \ldots\}$ will be denoted by $(c_{ij})$, where $2U_2 f^j = \sum c_{ij} f^i$. And this matrix actually begins

$$\begin{pmatrix}
2 & 0 & 0 \\
0 & 48 & 2 \\
0 & 2^{12} & 2304 \\
0 & 0 & \vdots \\
1 & \vdots & \\
\end{pmatrix}$$

So we are interested in a generating function. Let $g = \sum b_n q^n$, then $2U_2 g = 2 \sum b_{2n} q^n = g(\sqrt{q}) + g(-\sqrt{q})$. So if $g = f^j$, $2u_2 f^j = \alpha^j + \beta^j = \sum c_{ij} f^i$. Now

$$\sum (\alpha y)^j + (\beta y)^j = \sum c_{ij} f^i y^j$$

$$= \frac{1}{1 - \alpha y} + \frac{1}{1 - \beta y}$$

$$= \frac{2 - (\alpha + \beta)y}{2 - (\alpha + \beta)y + \alpha \beta y^2}$$

In fact
\[\alpha \beta = \sqrt{q} \prod_n (1 + (\sqrt{q})^n)^{24} - \sqrt{q} \prod_n (1 + (\sqrt{q})^n)^{24} \]
\[= -q \prod_{n \text{ even}} \left( (1 + q^n)^{24} \right) \cdot \prod_{n \text{ odd}} (1 - q^n)^{24} \]
\[= -q \prod_n \frac{(1 + q^n)^{24} (1 - q^n)^{24}}{(1 - q^{2n})^{24}} \]
\[= -q \prod_n (1 + q^n)^{24} \]
\[= -f \]

So we have a generating function

\[\frac{2 - (48f + 2^{12}f^2)y}{1 - (48f + 2^{12}f^2)y - fy^2} \]

Can you figure out the \(c_{ij}\) from this generating function?

In order to make our argument rigorous, we need to show that \(2U_2\) is a compact operator on the \(\rho\)-overconvergent modular forms with \(|\rho|_2 < 1\). We know that if \(M = (m_{ij})\) is an infinite matrix giving an operator, then \(M\) is compact if and only if for \(\gamma_i = \sup_j |m_{ij}|_2\), we have that \(\gamma_i \to 0\) as \(i \to \infty\). The problem we face is that \(c_{i,2i} = 2\), so we do not have a compact operator for the basis \(\{1, f, f^2, f^3, \ldots\}\).

Since we do not have a compact operator on the space of all 2-adic modular forms, we must restrict to \(\rho\)-overconvergent modular forms. Pick \(\omega \in \mathbb{Q}_2\) such that \(|\omega|_2 < 1\). We now adjust our basis: \(\{1, (\omega f), (\omega f)^2, \ldots\}\) and look at \(2U_2\) as a matrix with respect to this new basis

\[2U_2(\omega f)^i = \omega^i \sum_i c_{ij} f^i \]
\[= \sum_i c_{ij} \omega^{i-j} (\omega f)^i \]

So in terms of the new basis \(2U_2 = (d_{ij}) = (c_{ij} \omega^{j-i})\). Now letting \(\omega = 2^l\) with \(l\) a positive rational and recalling the generating function for \((c_{ij})\), we have

\[\sum d_{ij} f^i y^j = \sum c_{ij} \omega^{j-i} (\omega y)^j \]
\[= \frac{2 - (2^4 \cdot 3f + 2^{12}f^2, \omega^{-1})y}{1 - (2^4 \cdot 3f + 2^{12}f^2, \omega^{-1})y - \omega fy^2} \]
\[= \frac{2 - (2^4 \cdot 3f + 2^{12}f^2) y}{1 - (2^4 \cdot 3f + 2^{12}f^2) y - 2f y^2} \]

which has a power series in \(\mathcal{O}_{\mathbb{Q}_2}\). Thus \(|d_{ij}|_2 < |2^l|_2 = 2^{-i} \to 0\) as \(i \to 0\). So \(2U_2\) is compact on \(\{1, (\omega f), (\omega f)^2, \ldots\}\) as desired.
We have been interested in computing \((c_{ij})\) and a result of Frank Calegari shows this matrix is computable for \(U_2\).

**Theorem.** \(c_{ij} = \frac{2^{8i-4j}3j(i+j-1)!}{(2i-j)!(2j-i)!}\) where \(c_{ij} = 0\) whenever not well-defined.

**Proof.** There is a recurrence relation

\[
\begin{align*}
F_1 &= 2U_2 f = \alpha + \beta = 48f + 4096f^2 \\
F_n &= 2U_2 f^n = \alpha^n + \beta^n \\
F_{n+1} &= 2U_2 f^{n+1} = \alpha^{n+1} + \beta^{n+1} = (\alpha + \beta)(\alpha^n + \beta^n) - \alpha\beta(\alpha^{n-1} + \beta^{n-1})
\end{align*}
\]

then

\[
F_{n+1} = 2U_2 f \cdot 2U_2 f^n + f(2U_2)(f^{n-1}) = (48f + 4096f^2)F_n + fF_{n-1}
\]

so \(c_{i+1,j+1} = 48c_{i,j} + 4096c_{i-1,j} + c_{i,j-1}\) and the rest follows by an ugly calculation.

Ideally we would like to have a closed form, which we hope to derive from

\[
\frac{2 - (\alpha + \beta)y}{1 - (\alpha + \beta)y + \alpha\beta y^2}
\]

but what we have been able to compute is that

\[
c_{ij} = 2^{8i-4j}3^{2j-1} \left( \binom{j}{2j-i} + j \sum_{k=1}^{\frac{j-1}{2}} \binom{3-3k}{k} \binom{j - (k + 1)}{j - 2k} \binom{j - 2k}{2j - i - 3k} \right)
\]

but we have been unable to reduce this to Frank’s result.

There are many related open and fun problems. We can try the same calculations with \(E_2(q) = \text{weight 2, level 2 Eisenstein series, then } E_2(q), E_2(q^2)\) are overconvergent modular forms of weight 2 and level 1. Let \(g = \frac{E_2(q)/E_2(q^2) - 1}{24}\) (which is overconvergent of weight 0 and level 2) try to compute the matrix in terms of the parameters

\[
\begin{align*}
U_2(g) &= 0 \\
U_2(g^2) &= \sum ((2i+1)3^{i-2}2^{3(j-1)}(-1)^{i+1})g^i \\
U_2(g^3) &= 0 \\
U_2(g^4) &= \pm (U_2(g^2))^2
\end{align*}
\]

There are many interesting problems for \(p = 3, 5, 7, 13, \ldots\). What are nice parameters ... try to find nice formulae.