Situation

\[ X = V(I) \subseteq \mathbb{P}^n \]
\[ I \subseteq S = k[x_0, \ldots, x_n], \quad R = S/I \]

\( M \) graded \( R \)-module

(finitely generated)

\( \mathfrak{I} = \widetilde{M} \) coherent sheaf on \( X \)

want to find:

- \( h^i(\mathfrak{I}) := \dim_k H^i(X, \mathfrak{I}) \)
- \( H^i(X, \mathfrak{I}) \)
- \( H^i_+(\mathfrak{I}) \)

module structure:

\[ H^i_+(\mathfrak{I}) := \bigoplus_{d \in \mathbb{Z}} H^i(X, \mathfrak{I}(d)) \]

this is an \( R \)-module

- \( H^i_+(\mathfrak{I})(d) \)
Čech complex + Čech cohomology

\[ \mathcal{U} = \{ U_0, \ldots, U_m \} \]

open affine cover of \( X \)

\[ U_\lambda \coloneqq \bigcap_{i \in \lambda} U_i \quad \lambda = \{ 0, \ldots, m \} \]

Define

\[ C^p(\mathcal{F}) := \bigoplus_{\lambda \subseteq \{ 0, 1, \ldots, p \}} \mathcal{F}(U_\lambda) \]

\[ \sigma_p : C^p(\mathcal{F}) \to C^{p+1}(\mathcal{F}) \]

\[ (f_0 \ldots f_p) \mapsto (g_{i_0} \ldots g_{i_{p+1}}) \]

\[ g_{i_0} \ldots g_{i_{p+1}} = \sum_{i=0}^{p+1} (-1)^i f_{i_0} \ldots \hat{f}_i \ldots f_{i_{p+1}} \]
$\mathcal{C}(\mathcal{F})$ is the complex

$$0 \rightarrow \mathcal{C}^0(\mathcal{F}) \xrightarrow{\delta_0} \mathcal{C}^1(\mathcal{F}) \xrightarrow{\delta_1} \cdots \xrightarrow{\delta_{n-1}} \mathcal{C}^n(\mathcal{F})$$

Def \quad H^i(\mathcal{F}) = H^i(X, \mathcal{F})

$$:= H^i(\mathcal{C}(\mathcal{F}))$$

Theorem \quad This is independent of the open cover (as long as it is affine)
$H^i(\overline{M})$:

Let $M$ be a f.g. graded $S$-module $\overline{M}$ coherent sheaf on $\mathbb{P}^n$

Def Let $L^p(M) := \bigoplus_{1 = p+1} M \otimes_S S[x_{x^\lambda}]$

where $\lambda = \{x_0, ..., x_p\} \subseteq \{0, ..., n\}$

and $x_{x^\lambda} = x_{x_0} x_{x_1} ... x_{x_p} \in S$

Define $\sigma_p : L^p(M) \to L^{p+1}(M)$

as above

$L^p(M) : 0 \to L^p(M) \to ... \to L^n(M)$
Proposition

\[ H^i_*(\tilde{M}) = H^i_*(C(M)) \]
\[ H^i_*(\tilde{M}) = H^i_*(C(M)_{deg=0}) \]

"proof"

Let \( U_i = \mathbb{P}^n \setminus V(x_i) \quad i = 0, \ldots, n \)

then \( C(M)_d = C(\tilde{M}(d)) \)

Note: this complex + cohomology can be used to define the module structure on \( H^i_*(\tilde{M}) \)
cohomology of \( \Omega_{\mathbb{P}^n} \Rightarrow \mathbb{S} \) (Serre)

\[
H^i(\Omega_{\mathbb{P}^n}) = \begin{cases} 
\mathbb{S} & i = 0 \\
0 & 1 \leq i \leq n \\
\frac{1}{x_0 \cdots x_n} k[x_0^{-1}, \ldots, x_n^{-1}] & i = n 
\end{cases}
\]

proof idea: good exercise!

\( C(S) \) is \( \mathbb{Z}^{n+1} \)-graded

\( C(S)_m \) is recognizable \((m \in \mathbb{Z}^{n+1})\)
Def if \( M = \bigoplus_{d \in \mathbb{Z}} M_d \)

is a graded \( S \)-module

the graded \( k \)-dual \( M^\vee \) is

\[
M^\vee = \bigoplus_{d \in \mathbb{Z}} M_{-d}
\]

\( k \)-vector space dual

So:

\[
H^*_S(S) \cong \left[ S(-n-1) \right]^\vee
\]

\( \omega_{pr} \)

dualizing sheaf
Local duality (Serre) \( S = k[x_0, \ldots, x_n] \)

(a) for \( i \geq 1 \)

\[
H^i_*(\widetilde{M}) = \text{Ext}^{n-i}_S(M, S(-n-1))
\]

and so

\[
H^i_*(\widetilde{M}) = \text{Ext}^{n-i}_S(M, S)_{-n-1}
\]

(b)

\[
0 \rightarrow \text{Ext}^{n+1}_S(M, S(-n-1)) \rightarrow M \rightarrow H^0_*(\widetilde{M}) \rightarrow \text{Ext}^n_S(M, S(-n-1)) \rightarrow
\]

is exact.
simple, yet useful

corollary of local duality:

Let $M$ be a f.g. graded $S$-module.

Then

$$\text{pdim}_S(M) \leq n-1$$

$$\iff M = H^0(\tilde{M})$$

[in particular, in this case

$$M_0 = H^0(\tilde{M})$$]
Corollary of local duality

\[ H^0_\mathfrak{m}(M) \text{ is f.g.} \]

\[ \iff \text{every associated component of } M \]

\[ \text{has dimension } \geq 1 \text{ in } \mathbb{P}^n \]

**Proof**

\[ H^0_\mathfrak{m}(M) \text{ f.g.} \]

\[ \iff \text{Ext}^n_\mathfrak{m}(M, S) \text{ has finite dim over } k \]

\[ \iff \text{codim Ext}^n_\mathfrak{m}(M, S) = n+1 \]

But (Eisenbud-Huneke-Vasconcelos)

\[ \text{codim Ext}^i_\mathfrak{m}(M, S) \geq i \]

and equality holds iff \( M \) has an associated prime of codim \( i \)
Important example:

Sheaf \( \Omega'_x \) of differential forms on \( X \in \mathbb{P}^n \).

Two useful exact sequences:

\[
0 \to \Omega'_{P^n} \to \Theta_{P^n}(-1) \to \Theta_{P^n}^{n+1}(x_0, \ldots, x_n) \to 0
\]

[think: \( dx_0, \ldots, dx_n \) on \( U_i \):

generated by \( dx_0, \ldots, dx_1, \ldots, dx_n \)]

\[
\tilde{I}_x \to \Omega'_{P^n} \oplus \Theta_x \to \Omega'_x \to 0
\]

\( g \mapsto dg \)
unwind these:

proposition  Let \( X = V(I) \subset \mathbb{P}^n \)

\[ R = S/I \]. The cotangent sheaf \( \Omega^1_X \) is the sheaf associated to the homology module of

\[ F \otimes_R \overset{dj}{\rightarrow} R(-1)^{\text{dim} I} \overset{1 \otimes \text{generator matrix of } I}{\rightarrow} R \]

where if \( j: F \rightarrow \mathbb{R}S \) is the generator matrix of \( I \)

then \( dj \) is the Jacobian of \( j \).
Example Fermat quartic

\[ X = V(a^4 + b^4 + c^4 + d^4) \subseteq \mathbb{P}^3 \]

K3 surface

Let's find \( \Omega_x^1 = \mathcal{M}, \quad h^1(\Omega_x^1) \)

(1)

\[ R(-4) \xrightarrow{\partial} R(-1)^4 \xrightarrow{\partial} R \]

\[ \begin{pmatrix}
  a^3 \\
  b^3 \\
  c^3 \\
  d^3
\end{pmatrix} \]

(2) gives \( \mathcal{M} \).

\[ 0 \rightarrow \mathcal{O}(-4) \rightarrow \mathcal{O}(-3)^4 \rightarrow \mathcal{O}(-2)^6 \rightarrow \mathcal{O} \rightarrow 0 \]

\[ \begin{pmatrix}
  \mathcal{O}(-8) \\
  \mathcal{O}(-5)^4
\end{pmatrix} \]

So \( h^0(\Omega_x^1) = \mathcal{M} \)

\( h^0(\Omega_x^1) = 0 \)
\[ h'(\Omega'_x) = 20 \]
\[ h^2(\Omega'_x) = 0 \]

**Question**

\[ X = V(I) \subset \mathbb{P}^n \] smooth, say

From above, get \( M \)

\[ \tilde{M} = \Omega'_x \]

when is \( \text{pdim}_s(M) \leq n-1 \)

ie:

\[ H^0(\Omega'_x) = M \]?
Given \( G \xrightarrow{\varphi} F \xrightarrow{\epsilon} M \xrightarrow{\partial} 0 \)

then \( G \Theta \Lambda \Phi' F \xrightarrow{\partial} \Lambda F \xrightarrow{\epsilon} \Lambda P M \xrightarrow{\partial} 0 \)

is a presentation of \( \Lambda P M \)

and: \( \Omega^p_x = \Lambda P M \)

if \( \Omega'_x = \sim M \)
Example: Hodge diamond

$X \subseteq \mathbb{P}^n$ smooth

dimension $d$ (say $= 3$)

$h^{p-q} := \dim \mathbb{H}^q(\Omega^p_X)$

have:

$$h^{p-q} = h^{d-p, d-q}$$

$$h^{p-q} = h^{q, p}$$

$$H^i(X; \mathbb{C}) = \bigoplus_{p+q = i} H^q(\Omega^p_X)$$

$h^0(\Omega^p_X)$
$h^1(\Omega^p_X)$
$h^2(\Omega^p_X)$
$h^3(\Omega^p_X)$
Want: to compute as few of these as possible.

\[ \chi(\tilde{M}) := \sum_{i=0}^{\eta} (-1)^i h^i(\tilde{M}) \]

is \( P_m(0) \), where \( P_m(d) := \) Hilbert poly of \( M \)

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first row: easiest

second row: \( \tilde{M} = \Omega'_{x} \)

need only: \( \chi(\Omega'_{x}) \)

\( h^i(\Omega'_{x}) \)

(for \( \text{dim } X = 2 \) or \( 3 \))
Hodge diamond

$X \in \mathbb{P}^4$ quintic 3-fold

$$
\begin{bmatrix}
1 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 \\
0 & 1 & 0 & 1 \\
1 & 0 & 0 & 1
\end{bmatrix}
$$

Note:

$h'(\tilde{\mathcal{M}}) = \dim \text{Ext}^{n-1}_S(M,S)_{n-1}

= \dim \text{Ext}^3_S(M,S)_{-5}$
Def \( X \subseteq \mathbb{P}^n \), smooth, is called rationally connected if \( \forall \, \, p \neq q \in X \), there is a rational curve on \( X \) containing \( p, q \).

Conjecture \( X \) is RC

\[ \Rightarrow \, H^0((\Omega^1_X)^\otimes m) = 0 \]

for all \( m \geq 1 \)

[Mumford, Mori?]