Lecture 3

Frits Beukers

Arithmetic of values of E- and G-function

G-functions, definition

Definition

An analytic function $f(z)$ given by a powerseries

$$\sum_{k=0}^{\infty} a_k z^k$$

with $a_k \in \overline{\mathbb{Q}}$ for all k and positive radius of convergence, is called a G-function if

1. $f(z)$ satisfies a linear differential equation with coefficients in $\overline{\mathbb{Q}}(z)$.
2. Both $\overline{|a_k|}$ and the common denominators $\mathrm{den}(a_0, \dots, a_k)$ are bounded by an exponential bound of the form C^k, where $C > 0$ depends only on f.

G-functions, examples

1. $f(z)$ is algebraic over $\overline{\mathbb{Q}}(z)$ (Eisenstein theorem).

G-functions, examples

1. $f(z)$ is algebraic over $\overline{\mathbb{Q}}(z)$ (Eisenstein theorem).
2. $$f(z) = {}_2F_1\left(\begin{matrix}\alpha, \beta \\ \gamma\end{matrix}\,\middle|\, z\right)$$

   Gauss hypergeometric series with $\alpha, \beta, \gamma \in \mathbb{Q}$.

G-functions, examples

1. $f(z)$ is algebraic over $\overline{\mathbb{Q}}(z)$ (Eisenstein theorem).
2. $$f(z) = {}_2F_1\left(\begin{matrix}\alpha, \beta \\ \gamma\end{matrix}\middle| z\right)$$

   Gauss hypergeometric series with $\alpha, \beta, \gamma \in \mathbb{Q}$.
3. $f(z) = L_k(z) = \sum_{n\geq 1} \frac{z^n}{n^k}$, the k-th polylogarithm.

G-functions, examples

1. $f(z)$ is algebraic over $\overline{\mathbb{Q}}(z)$ (Eisenstein theorem).
2. $$f(z) = {}_2F_1\left(\begin{matrix}\alpha, \beta \\ \gamma\end{matrix}\,\middle|\, z\right)$$

   Gauss hypergeometric series with $\alpha, \beta, \gamma \in \mathbb{Q}$.
3. $f(z) = L_k(z) = \sum_{n \geq 1} \frac{z^n}{n^k}$, the k-th polylogarithm.
4. $f(z) = \sum_{k=0}^{\infty} a_k z^k$ where
   $a_0 = 1, a_1 = 3, a_2 = 19, a_3 = 147, \ldots$ are the Apéry numbers corresponding to Apéry's irrationality proof of $\zeta(2)$. They are determined by

   $$a_k = \sum_{r=0}^{k} \binom{k}{r}^2 \binom{r+k}{r}$$

   and satisfy the recurrence relation
   $(n+1)^2 a_{n+1} = (11n^2 - 11n + 3)a_n - n^2 a_{n-1}$.

Periods

Consider a family of algebraic varieties parametrised by z and consider a relative differential r-form Ω_z. We assume everything defined over $\overline{\mathbb{Q}}$. Take a continuous family of suitable cycles γ_z and consider the integral

$$w(z) = \int_{\gamma_z} \Omega_z.$$

Then, by a theorem of N.Katz $w(z)$ is a $\mathbb{C}$-linear combination of G-functions.

Periods

Consider a family of algebraic varieties parametrised by z and consider a relative differential r-form Ω_z. We assume everything defined over $\overline{\mathbb{Q}}$. Take a continuous family of suitable cycles γ_z and consider the integral

$$w(z) = \int_{\gamma_z} \Omega_z.$$

Then, by a theorem of N.Katz $w(z)$ is a $\mathbb{C}$-linear combination of G-functions.

Example: Euler integral for the hypergeometric function

$${}_2F_1\left(\begin{matrix}1/5, 4/5\\ 8/5\end{matrix}\,\middle|\, z\right) = \frac{1}{B(4/5,4/5)}\int_0^1 \frac{dx}{x^{1/5}(1-x)^{1/5}(1-zx)^{1/5}}.$$

This integral can be interpreted as a period (integral over a closed loop) of the differential form dx/y on the algebraic curve $y^5 = x(1-x)(1-zx)$.

Irrationality results

A typical result for G-functions,

Galochkin, 1972

Let $(f_1(z), \ldots, f_n(z))$ be a solution vector of a system of first order equations of the form $\mathbf{y}' = G\mathbf{y}$ and suppose that the $f_i(z)$ are G-functions with coefficients in $\mathbb{Q}$. Suppose also that $f_1(z), \ldots, f_n(z)$ are linearly independent over $\mathbb{Q}(z)$ and that the system satisfies the so-called Galochkin condition. Then there exists $C > 0$ such that $f_1(a/b), \ldots, f_n(a/b)$ are $\mathbb{Q}$-linear independent whenever $a, b \in \mathbb{Z}$ and $b > C|a|^{n+1} > 0$.

Wolfart's examples

Theorem, Wolfart 1988

The functions ${}_2F_1\left(\begin{matrix}1/12,\ 5/12\\ 1/2\end{matrix}\,\middle|\, z\right)$ and ${}_2F_1\left(\begin{matrix}1/12,\ 7/12\\ 2/3\end{matrix}\,\middle|\, z\right)$ assume algebraic values for a dense set of algebraic arguments in the unit disk.

Wolfart's examples

Theorem, Wolfart 1988

The functions ${}_2F_1\left(\begin{matrix}1/12,\ 5/12\\ 1/2\end{matrix}\,\middle|\, z\right)$ and ${}_2F_1\left(\begin{matrix}1/12,\ 7/12\\ 2/3\end{matrix}\,\middle|\, z\right)$ assume algebraic values for a dense set of algebraic arguments in the unit disk.

Theorem, Wolfart+FB, 1989

We have

$$ {}_2F_1\left(\frac{1}{12},\frac{5}{12},\frac{1}{2}\,\middle|\,\frac{1323}{1331}\right)=\frac{3}{4}\sqrt[4]{11}. $$

$$ {}_2F_1\left(\frac{1}{12},\frac{7}{12},\frac{2}{3}\,\middle|\,\frac{64000}{64009}\right)=\frac{2}{3}\sqrt[6]{253}. $$

Galochkin condition

Start with the system

$$\mathbf{y}' = G\mathbf{y}$$

For $s = 1, 2, 3, \ldots$ define the iterated $n \times n$-matrices G_s by

$$\frac{1}{s!}\mathbf{y}^{(s)} = G_s\mathbf{y}.$$

Galochkin condition

Start with the system

$$\mathbf{y}' = G\mathbf{y}$$

For $s = 1, 2, 3, \ldots$ define the iterated $n \times n$-matrices G_s by

$$\frac{1}{s!}\mathbf{y}^{(s)} = G_s\mathbf{y}.$$

Let $T(z)$ be the common denominator of all entries of G. Then, for every s, the entries of $T(z)^s G_s$ are polynomials. Denote the least common denominator of all coefficients of all entries of $T(z)^m G_m/m!$ $(m = 1, \ldots, s)$ by q_s.

Definition

With notation as above, we say that the system $\mathbf{y}'(z) = G(z)\mathbf{y}(z)$ satisfies Galochkin's condition if there exists $C > 0$ such that $q_s < C^s$ for all $s \geq 1$.

Why Galochkin's condition?

Recall that in Siegel's method we construct polynomials P_i of degree $\leq N$ such that

$$P_1 f_1 + \cdots + P_n f_n = O(z^{N(n-\epsilon)}).$$

In vector notation $\mathbf{P} \cdot \mathbf{f} = O(z^{N(n-\epsilon)})$. We also need the derivatives

$$\frac{1}{m!}(\mathbf{P} \cdot \mathbf{f})^{(m)} = O(z^{N(n-\epsilon)-m}), \quad m = 0, 1, \ldots, N\epsilon + \gamma.$$

Why Galochkin's condition?

Recall that in Siegel's method we construct polynomials P_i of degree $\leq N$ such that

$$P_1 f_1 + \cdots + P_n f_n = O(z^{N(n-\epsilon)}).$$

In vector notation $\mathbf{P} \cdot \mathbf{f} = O(z^{N(n-\epsilon)})$. We also need the derivatives

$$\frac{1}{m!}(\mathbf{P} \cdot \mathbf{f})^{(m)} = O(z^{N(n-\epsilon)-m}), \quad m = 0, 1, \ldots, N\epsilon + \gamma.$$

Notice

$$\begin{aligned}
\frac{1}{m!}(\mathbf{P} \cdot \mathbf{f})^{(m)} &= \sum_{s=0}^{m} \frac{1}{s!(m-s)!}(\mathbf{P})^{(m-s)} \cdot (\mathbf{f})^{(s)} \\
&= \sum_{s=0}^{m} \frac{1}{(m-s)!}(\mathbf{P})^{(m-s)} \cdot G_s \mathbf{f}
\end{aligned}$$

Galochkin implies G-property

Lemma

Suppose we have an $n \times n$-system satisfying Galochkin. Then, at any nonsingular point a the system has a basis of solutions consisting of G-functions in $z - a$.

Galochkin implies G-property

Lemma

Suppose we have an $n \times n$-system satisfying Galochkin. Then, at any nonsingular point a the system has a basis of solutions consisting of G-functions in $z - a$.

Proof Put G_0 equal to the $n \times n$ identity matrix and consider the matrix

$$Y = \sum_{s \geq 0} \frac{1}{s!} G_s(a)(z-a)^s.$$

Galochkin implies G-property

Lemma

Suppose we have an $n \times n$-system satisfying Galochkin. Then, at any nonsingular point a the system has a basis of solutions consisting of G-functions in $z - a$.

Proof Put G_0 equal to the $n \times n$ identity matrix and consider the matrix

$$Y = \sum_{s \geq 0} \frac{1}{s!} G_s(a)(z-a)^s.$$

It satisfies

$$Y' = GY$$

hence its columns satisfy the linear differential system and since $\det(Y) \neq 0$ they form a basis. The G-function property of $G_s(a)/s!$ follows directly from Galochkin's condition.

Chudnovsky's theorem

Chudnovsky, 1984

Let $(f_1(z), \ldots, f_n(z))$ be a solution vector consisting of G-functions of a system of first order equations of the form $\mathbf{y}' = G\mathbf{y}$. Suppose that $f_1(z), \ldots, f_n(z)$ are linearly independent over $\overline{\mathbb{Q}}(z)$. Then the system satisfies Galochkin's condition.

Chudnovsky's theorem

Chudnovsky, 1984

Let $(f_1(z), \dots, f_n(z))$ be a solution vector consisting of G-functions of a system of first order equations of the form $\mathbf{y}' = G\mathbf{y}$. Suppose that $f_1(z), \dots, f_n(z)$ are linearly independent over $\overline{\mathbb{Q}}(z)$. Then the system satisfies Galochkin's condition.

Idea of proof: Construct $Q, P_1, \dots, P_n \in \overline{\mathbb{Q}}[z]$ of degrees $\le N$ such that

$$Qf_i - P_i = O(z^{N(1+1/n-\epsilon)}).$$

In vector notation:

$$Q\mathbf{f} - \mathbf{P} = O(z^{N(1+1/n-\epsilon)}).$$

Chudnovsky's theorem

Chudnovsky, 1984

Let $(f_1(z),\dots,f_n(z))$ be a solution vector consisting of G-functions of a system of first order equations of the form $\mathbf{y}' = G\mathbf{y}$. Suppose that $f_1(z),\dots,f_n(z)$ are linearly independent over $\overline{\mathbb{Q}}(z)$. Then the system satisfies Galochkin's condition.

Idea of proof: Construct $Q, P_1, \dots, P_n \in \overline{\mathbb{Q}}[z]$ of degrees $\le N$ such that

$$Qf_i - P_i = O(z^{N(1+1/n-\epsilon)}).$$

In vector notation:

$$Q\mathbf{f} - \mathbf{P} = O(z^{N(1+1/n-\epsilon)}).$$

Differentiate,

$$Q'\mathbf{f} + Q\mathbf{f}' - \mathbf{P}' = O(z^{N(1+1/n-\epsilon)-1}).$$

Proof sketch of Chudnovsky's theorem

Use $\mathbf{f}' = G\mathbf{f}$ to get

$$Q'\mathbf{f} + QG\mathbf{f} - D\mathbf{P} = O(z^{N(1+1/n-\epsilon)-1})$$

where $\mathbf{P}' = D\mathbf{P}$.

Proof sketch of Chudnovsky's theorem

Use $\mathbf{f}' = G\mathbf{f}$ to get

$$Q'\mathbf{f} + QG\mathbf{f} - D\mathbf{P} = O(z^{N(1+1/n-\epsilon)-1})$$

where $\mathbf{P}' = D\mathbf{P}$.
Substract G times the original form

$$Q'\mathbf{f} - (D - G)\mathbf{P} = O(z^{N(1+1/n-\epsilon)-1}).$$

Proof sketch of Chudnovsky's theorem

Use $\mathbf{f}' = G\mathbf{f}$ to get

$$Q'\mathbf{f} + QG\mathbf{f} - D\mathbf{P} = O(z^{N(1+1/n-\epsilon)-1})$$

where $\mathbf{P}' = D\mathbf{P}$.
Substract G times the original form

$$Q'\mathbf{f} - (D - G)\mathbf{P} = O(z^{N(1+1/n-\epsilon)-1}).$$

Repeating the argument s times and divide by $s!$,

$$\frac{1}{s!}Q^{(s)}\mathbf{f} - \frac{1}{s!}(D - G)^s\mathbf{P} = O(z^{N(1+1/n-\epsilon)-s}).$$

Proof sketch of Chudnovsky's theorem

Use $\mathbf{f}' = G\mathbf{f}$ to get

$$Q'\mathbf{f} + QG\mathbf{f} - D\mathbf{P} = O(z^{N(1+1/n-\epsilon)-1})$$

where $\mathbf{P}' = D\mathbf{P}$.
Substract G times the original form

$$Q'\mathbf{f} - (D-G)\mathbf{P} = O(z^{N(1+1/n-\epsilon)-1}).$$

Repeating the argument s times and divide by $s!$,

$$\frac{1}{s!}Q^{(s)}\mathbf{f} - \frac{1}{s!}(D-G)^s\mathbf{P} = O(z^{N(1+1/n-\epsilon)-s}).$$

Lemma

We have for any vector $\mathbf{P} \in \overline{\mathbb{Q}}(z)^n$,

$$G_s\mathbf{P} = \sum_{m=0}^{s} \frac{(-1)^m}{(s-m)!m!} D^{s-m}(D-G)^m\mathbf{P}.$$

Galochkin's condition for equations

Consider the differential

$$T(z)y^{(n)} = Q_{n-1}(z)y^{(n-1)} + \cdots + Q_1(z)y' + Q_0 y$$

where $T(z), Q_0(z), \ldots, Q_{n-1}(z)$ are polynomials in $\overline{\mathbb{Q}}[z]$.

Galochkin's condition for equations

Consider the differential

$$T(z)y^{(n)} = Q_{n-1}(z)y^{(n-1)} + \cdots + Q_1(z)y' + Q_0 y$$

where $T(z), Q_0(z), \ldots, Q_{n-1}(z)$ are polynomials in $\overline{\mathbb{Q}}[z]$.
By recursion on m find polynomials $Q_{m,r} \in \overline{\mathbb{Q}}(z)$ for $r = 0, 1, \ldots, n-1$ such that

$$T(z)^{m-n+1}y^{(m)} = Q_{m,n-1}(z)y^{(n-1)} + \cdots + Q_{m,1}(z)y' + Q_{m,0}(z)y.$$

In particular $Q_{n,r}(z) = Q_r(z)$.

Galochkin's condition for equations

Consider the differential

$$T(z)y^{(n)} = Q_{n-1}(z)y^{(n-1)} + \cdots + Q_1(z)y' + Q_0 y$$

where $T(z), Q_0(z), \ldots, Q_{n-1}(z)$ are polynomials in $\overline{\mathbb{Q}}[z]$.
By recursion on m find polynomials $Q_{m,r} \in \overline{\mathbb{Q}}(z)$ for $r = 0, 1, \ldots, n-1$ such that

$$T(z)^{m-n+1}y^{(m)} = Q_{m,n-1}(z)y^{(n-1)} + \cdots + Q_{m,1}(z)y' + Q_{m,0}(z)y.$$

In particular $Q_{n,r}(z) = Q_r(z)$.

Definition

The equation satisfies Galoschkin's condition if there exists $C > 0$ such that for every integer s the common denominator of all coefficients of all polynomials $\frac{1}{m!}Q_{m,r}$ with $n \leq m \leq s, 0 \leq r \leq n-1$ is bounded by C^s.

Chudnovski's Theorem, bis

Theorem, Chudnovsky 1984

Let f be a G-function and let $Ly = 0$ be its minimal differential equation. Then $Ly = 0$ satisfies Galochkin's condition.

Regular singularities

Consider a linear differential equation

$$p_n y^{(n)} + p_{n-1} y^{(n-1)} + \cdots + p_1 y' + p_0 y = 0$$

where $p_i \in \mathbb{C}[z]$ for all i. Suppose that $p_n(0) = 0$. Then $z = 0$ is a singular point.

Regular singularities

Consider a linear differential equation

$$p_n y^{(n)} + p_{n-1} y^{(n-1)} + \cdots + p_1 y' + p_0 y = 0$$

where $p_i \in \mathbb{C}[z]$ for all i. Suppose that $p_n(0) = 0$. Then $z = 0$ is a singular point.

Criterion for regular singular points

The point $z = 0$ is regular or a regular singular point if and only if the pole order of p_i/p_n at $z = 0$ is at most $n - i$.

Regular singularities

Consider a linear differential equation

$$p_n y^{(n)} + p_{n-1} y^{(n-1)} + \cdots + p_1 y' + p_0 y = 0$$

where $p_i \in \mathbb{C}[z]$ for all i. Suppose that $p_n(0) = 0$. Then $z = 0$ is a singular point.

Criterion for regular singular points

The point $z = 0$ is regular or a regular singular point if and only if the pole order of p_i/p_n at $z = 0$ is at most $n - i$.

Alternative criterion

The point $z = 0$ is regular or a regular singular point if and only if the equation can be rewritten as

$$z^n q_n y^{(n)} + z^{n-1} q_{n-1} y^{(n-1)} + \cdots + z q_1 y' + q_0 y = 0$$

where $q_i \in \mathbb{C}[z]$ and $q_n(0) \neq 0$.

Regular singularities in general

Consider a differential equation

$$p_n y^{(n)} + p_{n-1} y^{(n-1)} + \cdots + p_1 y' + p_0 y = 0$$

with $p_i \in \overline{\mathbb{Q}}[z]$ and let a be any point in $\mathbb{C} \cup \infty$.

Regular singularities in general

Consider a differential equation

$$p_n y^{(n)} + p_{n-1} y^{(n-1)} + \cdots + p_1 y' + p_0 y = 0$$

with $p_i \in \overline{\mathbb{Q}}[z]$ and let a be any point in $\mathbb{C} \cup \infty$.

- Rewrite the equation in terms of a local parameter t at a (which comes down to putting $z = t + a$ and $z = 1/t$ if $a = \infty$).

Regular singularities in general

Consider a differential equation

$$p_n y^{(n)} + p_{n-1} y^{(n-1)} + \cdots + p_1 y' + p_0 y = 0$$

with $p_i \in \overline{\mathbb{Q}}[z]$ and let a be any point in $\mathbb{C} \cup \infty$.

- Rewrite the equation in terms of a local parameter t at a (which comes down to putting $z = t + a$ and $z = 1/t$ if $a = \infty$).
- If $t = 0$ is a regular singularity of the resulting equation, we say that a is a regular singularity of the orginal equation.

Regular singularities in general

Consider a differential equation

$$p_n y^{(n)} + p_{n-1} y^{(n-1)} + \cdots + p_1 y' + p_0 y = 0$$

with $p_i \in \overline{\mathbb{Q}}[z]$ and let a be any point in $\mathbb{C} \cup \infty$.

- Rewrite the equation in terms of a local parameter t at a (which comes down to putting $z = t + a$ and $z = 1/t$ if $a = \infty$).
- If $t = 0$ is a regular singularity of the resulting equation, we say that a is a regular singularity of the orginal equation.

Proposition

The point $z = \infty$ is a singular point if and only if $\deg(p_i) \leq \deg(p_n) - n + i$ for $i = 0, 1, \ldots, n$

Fuchsian equations

Definition

A linear differential equation is called *Fuchsian* if every point in $\mathbb{C} \cup \infty$ is regular or a regular singularity.

Fuchsian equations

Definition

A linear differential equation is called *Fuchsian* if every point in $\mathbb{C} \cup \infty$ is regular or a regular singularity.

Example: $y' = y$ is not a Fuchsian equation because ∞ is not a regular singularity (Replace $z = 1/t$ and we obtain $-t^2 \frac{dy}{dt} = y$).

Galochkin implies Fuchsian

Theorem

Suppose the differential equation $Ly = 0$ satisfies Galochkin's condition. Then $Ly = 0$ is Fuchsian.

Galochkin implies Fuchsian

Theorem

Suppose the differential equation $Ly = 0$ satisfies Galochkin's condition. Then $Ly = 0$ is Fuchsian.

For the experts: Galochkin's condition implies that the equation is globally nilpotent (Bombieri,Dwork). That is

$$D^{sp} \equiv M \circ L \pmod{p}$$

for almost all primes p and some integer s.
N.Katz showed that a globally nilpotent equation is Fuchsian with rational local exponents.

Galochkin to Fuchsian, proof sketch

By way of example consider the equation

$$z^2 y'' = zA_1 y' + A_0 y$$

where A_1, A_0 are rational functions.

Galochkin to Fuchsian, proof sketch

By way of example consider the equation

$$z^2y'' = zA_1y' + A_0y$$

where A_1, A_0 are rational functions.
Suppose the equation is not Fuchsian. By way of example assume that A_1, A_0 both have a first order pole in $z = 0$.

Galochkin to Fuchsian, proof sketch

By way of example consider the equation

$$z^2y'' = zA_1y' + A_0y$$

where A_1, A_0 are rational functions.
Suppose the equation is not Fuchsian. By way of example assume that A_1, A_0 both have a first order pole in $z = 0$.
By induction on m,

$$z^m y^{(m)} = zA_{m,1}y' + A_{m,0}y$$

and

$$\begin{aligned} A_{m+1,1} &= (1-m)A_{m,1} + zA'_{m,1} + A_{m,1}A_1 + A_{m,0} \\ A_{m+1,0} &= A_{m,1}A_0 + zA'_{m,0} - mA_{m,0} \end{aligned}$$

Galochkin to Fuchsian, continued

From the previous slide,

$$z^2 y'' = zA_1 y' + A_0 y$$

and that A_1, A_0 have pole order 1. By induction,

$$z^m y^{(m)} = zA_{m,1} y' + A_{m,0} y$$

Galochkin to Fuchsian, continued

From the previous slide,

$$z^2 y'' = zA_1 y' + A_0 y$$

and that A_1, A_0 have pole order 1. By induction,

$$z^m y^{(m)} = zA_{m,1} y' + A_{m,0} y$$

Suppose residue of A_1 at $z = 0$ is a. Then

$$A_{m,1} = \frac{a^{m-1}}{z^{m-1}} + \cdots$$

Galochkin to Fuchsian, continued

From the previous slide,

$$z^2 y'' = zA_1 y' + A_0 y$$

and that A_1, A_0 have pole order 1. By induction,

$$z^m y^{(m)} = zA_{m,1} y' + A_{m,0} y$$

Suppose residue of A_1 at $z = 0$ is a. Then

$$A_{m,1} = \frac{a^{m-1}}{z^{m-1}} + \cdots$$

So $A_{m,1}$ is a rational function whose numerator has a constant term which grows exponentially in m. Thus $A_{m,1}/m!$ cannot satisfy Galochkin's condition.

Chudnovski's Theorem, encore

Theorem, Chudnovsky 1984

Let f be a G-function and let $Ly = 0$ be its minimal differential equation. Then $Ly = 0$ satisfies Galochkin's condition.

Chudnovski's Theorem, encore

Theorem, Chudnovsky 1984

Let f be a G-function and let $Ly = 0$ be its minimal differential equation. Then $Ly = 0$ satisfies Galochkin's condition. Moreover, $z = 0$ is at worst a regular singularity of $Ly = 0$