Transcendence in Positive Characteristic

Difference Equations and Linear Independence

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Outline

Functions on curves

- Rational functions
- Analytic and entire functions
- Frobenius twisting

Scalar quantities

Let p be a fixed prime; q a fixed power of p.

$$A := \mathbb{F}_q[\theta] \longleftrightarrow \mathbb{Z}$$

$$k := \mathbb{F}_q(\theta) \longleftrightarrow \mathbb{Q}$$

$$\overline{k} \longleftrightarrow \overline{\mathbb{Q}}$$

$$k_\infty := \mathbb{F}_q((1/\theta)) \longleftrightarrow \mathbb{R}$$

$$\mathbb{C}_\infty := \widehat{\overline{k_\infty}} \longleftrightarrow \mathbb{C}$$

$$|f|_\infty = q^{\deg f} \longleftrightarrow |\cdot|$$

Rational functions

- We select a variable t that is independent from θ. The rational function field $\mathbb{F}_q(t)$ is taken to be the function field of $\mathbb{P}^1/\mathbb{F}_q$:

$$\mathbb{F}_q(t) \longleftrightarrow \mathbb{P}^1/\mathbb{F}_q.$$

- Moreover, for any field $K \supseteq \mathbb{F}_q$,

$$K(t) \longleftrightarrow \mathbb{P}^1/K.$$

- We will often take $K = \overline{k}$ or $K = \mathbb{C}_\infty$.

Anayltic functions

The Tate algebra

- The *Tate algebra* is defined to be the ring of functions in $\mathbb{C}_\infty[[t]]$ that are analytic on the closed unit disk:

$$\mathbb{T} := \left\{ \sum_{i \geq 0} a_i t^i \in \mathbb{C}_\infty[[t]] \;\middle|\; |a_i|_\infty \to 0 \right\}.$$

- $\mathbb{T}$ is a p.i.d. with maximal ideals generated by $t - a$ for $|a|_\infty \leq 1$.
- Useful fact:

$$\mathbb{T} \cap \mathbb{F}_q[[t]] = \mathbb{F}_q[t].$$

- We will take $\mathbb{L} \subseteq \mathbb{C}_\infty((t))$ to be the fraction field of $\mathbb{T}$.

Entire functions

- The ring $\mathbb{E}$ of *entire functions* is defined to be

$$\mathbb{E} := \left\{ \sum_{i \geq 0} a_i t^i \in \mathbb{C}_\infty[[t]] \;\middle|\; \begin{array}{c} \sqrt[i]{|a_i|_\infty} \to 0, \\ [k_\infty(a_0, a_1, a_2, \dots) : k_\infty] < \infty \end{array} \right\}.$$

- The first condition implies that a given $f \in \mathbb{E}$ converges on all of $\mathbb{C}_\infty$. It is equivalent to having

$$\lim_{i \to \infty} \frac{1}{i} \operatorname{ord}_\infty(a_i) = \infty.$$

- The second condition implies that $f(\overline{k_\infty}) \subseteq \overline{k_\infty}$.

Frobenius twisting

- Let $f = \sum a_i t^i \in \mathbb{C}_\infty((t))$. For any $n \in \mathbb{Z}$, we set

$$f^{(n)} := \sum a_i^{q^n} t^i \in \mathbb{C}_\infty((t)).$$

  Thus $f \mapsto f^{(n)}$ has the effect of simply raising the coefficients of f to the q^n-th power.

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- These maps are automorphism

$$f \mapsto f^{(n)} : \mathbb{C}_\infty((t)) \xrightarrow{\sim} \mathbb{C}_\infty((t)),$$

  which induce automorphisms of each of the following rings and fields:

$$\overline{k}[t], \quad \mathbb{T}, \quad \overline{k}(t), \quad \mathbb{L}, \quad \mathbb{E}.$$

The automorphism σ

$\sigma : f \mapsto f^{(-1)}$

- When $n = -1$, we call this automorphism σ: for $f = \sum_i a_i t^i$,

$$\sigma(f) = f^{(-1)} = \sum_i a_i^{1/q} t^i.$$

- Moreover, σ has the following fixed rings and fields:

$$\mathbb{C}_\infty((t))^\sigma = \mathbb{F}_q((t)), \quad \overline{k}(t)^\sigma = \mathbb{F}_q(t), \quad \mathbb{T}^\sigma = \mathbb{F}_q[t], \quad \mathbb{L}^\sigma = \mathbb{F}_q(t).$$

The function $\Omega(t)$

- Fix $\zeta_\theta := \sqrt[q-1]{-\theta} = \exp_C(\pi_q/\theta)$.
- We define an infinite product,

$$\Omega(t) := \zeta_\theta^{-q} \prod_{i=1}^{\infty} \left(1 - \frac{t}{\theta^{q^i}}\right) \in \mathbb{E} \cap k_\infty(\zeta_\theta)[[t]].$$

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$$\Omega^{(-1)}(t) = (t - \theta)\Omega(t).$$

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$$\Omega^{(-1)}(t) = (t - \theta)\Omega(t).$$

$$\left[\Omega^{(-1)}(t) = \zeta_\theta^{-1}\left(1 - \frac{t}{\theta}\right) \prod_{i=1}^{\infty} \left(1 - \frac{t}{\theta^{q^i}}\right)\right]$$

The function $1/\Omega(t)$

- Recall $\Omega(t) = \zeta_\theta^{-q} \prod_{i=1}^{\infty} (1 - t/\theta^{q^i})$
- The zeros of $\Omega(t)$ in $\mathbb{C}_\infty$ are precisely $t = \theta^q$, $t = \theta^{q^2}, \ldots,$ each of which has absolute value > 1. Therefore,

$$\frac{1}{\Omega(t)} \in \mathbb{T},$$

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$$\frac{1}{\Omega(t)} \in \mathbb{T},$$

  and in fact $1/\Omega(t)$ converges on $|\alpha|_\infty < |\theta^q|_\infty$.
- If we compare with the Carlitz period,

$$\pi_q = \theta \zeta_\theta \prod_{i=1}^{\infty} \left(1 - \theta^{1-q^i}\right)^{-1},$$

  then we see

$$\frac{1}{\Omega(\theta)} = -\pi_q.$$

Summary of $\Omega(t)$

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- Specialization:

$$\Omega(\theta) = -\frac{1}{\pi_q}.$$

The “ABP-criterion”

- Theorem of Anderson, Brownawell, P.
- Proof of Wade’s theorem

Theorem (Anderson, Brownawell, P. 2004)

Let $r \geq 1$*. Fix a matrix* $\Phi = \Phi(t) \in \mathrm{Mat}_{r \times r}(\overline{k}[t])$*, such that* $\det(\Phi) = c(t-\theta)^s$ *for some* $c \in \overline{k}^\times$ *and* $s \geq 0$*.*

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$$\psi^{(-1)} = \Phi\psi.$$

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$$\psi^{(-1)} = \Phi\psi.$$

Now suppose that there is a $\overline{k}$*-linear relation among the entries of* $\psi(\theta)$*; that is, there is a row vector* $\xi \in \mathrm{Mat}_{1\times r}(\overline{k})$ *so that*

$$\xi\psi(\theta) = 0.$$

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Now suppose that there is a $\overline{k}$-linear relation among the entries of $\psi(\theta)$; that is, there is a row vector $\xi \in \mathrm{Mat}_{1\times r}(\overline{k})$ so that

$$\xi\psi(\theta) = 0.$$

Then there is a row vector of polynomials $P(t) \in \mathrm{Mat}_{1\times r}(\overline{k}[t])$ so that

$$P(t)\psi(t) = 0, \quad P(\theta) = \xi.$$

Wade's theorem revisited

Theorem (Wade 1941)

The Carlitz period π_q is transcendental over $\overline{k}$.

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- Consider

$$\Phi = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & t-\theta & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & (t-\theta)^m \end{bmatrix}, \qquad \psi = \begin{bmatrix} 1 \\ \Omega(t) \\ \vdots \\ \Omega(t)^m \end{bmatrix}.$$

- The functional equation $\Omega^{(-1)} = (t-\theta)\Omega$ implies

$$\psi^{(-1)} = \Phi\psi.$$

- Use ABP-criterion with Φ, ψ to show π_q cannot satisfy an algebraic relation over $\overline{k}$.

- Suppose

$$\xi_0 - \frac{\xi_1}{\pi_q} - \cdots + (-1)^m \frac{\xi_m}{\pi_q^m} = 0, \quad \xi_i \in \overline{k},\ \xi_0 \xi_m \neq 0.$$

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- If we let $\xi := [\xi_0, \ldots, \xi_m]$, then

$$\xi \psi(\theta) = 0.$$

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- The ABP-criterion implies there exist polynomials $P_0(t), \ldots, P_m(t) \in \overline{k}[t]$ so that

$$P_0(t) + P_1(t)\Omega(t) + \cdots + P_m(t)\Omega(t)^m = 0, \quad P_i(\theta) = \xi_i.$$

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- Since $P_0(t) \neq 0$ and $P_m(t) \neq 0$, it follows that $P_0(t)$ must vanish at the infinitely many zeros of $\Omega(t)$. Contradiction.

Difference equations

- Definitions of difference equations and their solution spaces
- Example for Carlitz logarithms
- Other examples in brief
  - Carlitz zeta values
  - Periods and quasi-periods of Drinfeld modules

Difference equations

- Fix a matrix $\Phi \in \mathrm{GL}_r(\overline{k}(t))$. We consider the system of equations

$$\psi^{(-1)} = \Phi\psi, \quad (\sigma(\psi) = \Phi\psi),$$

for $\psi \in \mathrm{Mat}_{r\times 1}(\mathbb{L})$. (Recall $\mathbb{L}$ = fraction field of the Tate algebra $\mathbb{T}$.)
- Define the space

$$\mathrm{Sol}(\Phi) = \{\psi \in \mathrm{Mat}_{r\times 1}(\mathbb{L}) \mid \psi^{(-1)} = \Phi\psi\}.$$

It is an $\mathbb{F}_q(t)$-vector space.
- The entries of $\mathrm{Sol}(\Phi)$ are then candidates for the application of the ABP-criterion.

Lemma

The space $\mathrm{Sol}(\Phi) = \{\psi \in \mathrm{Mat}_{r\times 1}(\mathbb{L}) \mid \psi^{(-1)} = \Phi\psi\}$ *satisfies*

$$\dim_{\mathbb{F}_q(t)} \mathrm{Sol}(\Phi) \le r.$$

- We will show that if $\psi_1, \ldots, \psi_m \in \mathrm{Sol}(\Phi)$ are linearly independent over $\mathbb{F}_q(t)$, then they are linearly independent over $\mathbb{L}$.

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- Suppose $m \geq 2$ is minimal so that we have $\psi_1, \dots, \psi_m \in \mathrm{Sol}(\Phi)$ linearly independent over $\mathbb{F}_q(t)$ but

$$0 = \sum_{i=1}^{m} f_i \psi_i, \quad f_i \in \mathbb{L}, f_1 = 1.$$

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- Multiply both sides by Φ:

$$0 = \sum_{i=1}^{m} f_i \Phi \psi_i = \sum_{i=1}^{m} f_i \psi_i^{(-1)}.$$

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$$0 = \sum_{i=1}^{m} f_i \Phi \psi_i = \sum_{i=1}^{m} f_i \psi_i^{(-1)}.$$

- Twist and subtract the two equations.

- We obtain

$$0 = \sum_{i=1}^{m} (f_i - f_i^{(1)})\psi_i = \sum_{i=2}^{m} (f_i - f_i^{(1)})\psi_i.$$

- By minimality of m, we have $f_i = f_i^{(-1)}$. Thus each

$$f_i \in \mathbb{L}^\sigma = \mathbb{F}_q(t).$$

Fundamental matrix for Φ

Definition

Given $\Phi \in \mathrm{GL}_r(\overline{k}(t))$, a matrix $\Psi \in \mathrm{GL}_r(\mathbb{L})$ is a *fundamental matrix* for Φ if

$$\Psi^{(-1)} = \Phi\Psi.$$

- In this case,

$$\dim_{\mathbb{F}_q(t)} \mathrm{Sol}(\Phi) = r.$$

- The columns of Ψ form a basis for $\mathrm{Sol}(\Phi)$.

$\Omega(t)$ yet again

- Here $r = 1$. We take

$$\Phi = t - \theta, \quad \Omega(t) = \zeta_\theta^{-q} \prod_{i=1}^{\infty} (1 - t/\theta^{q^i})$$

- Difference equation:

$$\Omega^{(-1)}(t) = (t - \theta)\Omega(t).$$

- Specialization:

$$\Omega(\theta) = -\frac{1}{\pi_q}.$$

Carlitz logarithms

- Recall the Carlitz exponential:

$$\exp_C(z) = z + \sum_{i=1}^{\infty} \frac{z^{q^i}}{(\theta^{q^i} - \theta)(\theta^{q^i} - \theta^q) \cdots (\theta^{q^i} - \theta^{q^{i-1}})}.$$

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- Its formal inverse is the Carlitz logarithm,

$$\log_C(z) = z + \sum_{i=1}^{\infty} \frac{z^{q^i}}{(\theta - \theta^q)(\theta - \theta^{q^2})\cdots(\theta - \theta^{q^i})}.$$

- $\log_C(z)$ converges for $|z|_\infty < |\theta|^{q/(q-1)}$ and satisfies

$$\theta \log_C(z) = \log_C(\theta z) + \log_C(z^q).$$

The function $L_\alpha(t)$

- For $\alpha \in \overline{k}$, $|\alpha|_\infty < |\theta|^{q/(q-1)}$, we define

$$L_\alpha(t) = \alpha + \sum_{i=1}^{\infty} \frac{\alpha^{q^i}}{(t-\theta^q)(t-\theta^{q^2})\cdots(t-\theta^{q^i})} \in \mathbb{T},$$

  which converges up to $|\theta^q|_\infty$.

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  which converges up to $|\theta^q|_\infty$.
- Connection with Carlitz logarithms:

$$L_\alpha(\theta) = \log_C(\alpha).$$

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- Connection with Carlitz logarithms:

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- Functional equation:

$$L_\alpha^{(-1)} = \alpha^{(-1)} + \frac{L_\alpha}{t-\theta}.$$

Difference equations for $L_\alpha(t)$

- If we set

$$\Phi = \begin{bmatrix} t-\theta & 0 \\ \alpha^{1/q}(t-\theta) & 1 \end{bmatrix} \in \mathrm{Mat}_2(\bar{k}[t]), \quad \Psi = \begin{bmatrix} \Omega & 0 \\ \Omega L_\alpha & 1 \end{bmatrix} \in \mathrm{Mat}_2(\mathbb{E}),$$

  then

$$\Psi^{(-1)} = \Phi\Psi.$$

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then

$$\Psi^{(-1)} = \Phi\Psi.$$

- Specialization at $t = \theta$:

$$\Psi(\theta)^{-1} = \begin{bmatrix} -\pi_q & 0 \\ -\log_C(\alpha) & 1 \end{bmatrix}.$$

Carlitz zeta values

- For a positive integer n,

$$\zeta_C(n) = \sum_{\substack{a \in \mathbb{F}_q[\theta] \\ a \text{ monic}}} \frac{1}{a^n} \in k_\infty.$$

- **Euler-Carlitz relations:** If $(q-1) \mid n$, then

$$\zeta_C(n) = r_n \pi_q^n, \quad r_n \in \mathbb{F}_q(\theta).$$

For example,

$$\zeta_C(q-1) = \frac{\pi_q^{q-1}}{\theta - \theta^q}.$$

Anderson, Thakur, and $\zeta_C(n)$

Theorem (Anderson-Thakur 1990)

There exist (explicit) $h_0, \ldots, h_\ell \in \mathbb{F}_q[\theta]$ so that

$$\zeta_C(n) = \frac{1}{\Gamma_n} \sum_{i=0}^{\ell} h_i \log_C^{[n]}(\theta^i).$$

Carlitz polylogarithm:

$$\log_C^{[n]}(z) = z + \sum_{i=1}^{\infty} \frac{z^{q^i}}{\left[(\theta - \theta^q)(\theta - \theta^{q^2}) \cdots (\theta - \theta^{q^i})\right]^n}$$

Carlitz factorial: $\Gamma_n \in \mathbb{F}_q[\theta]$

Difference equations for $\zeta_C(n)$

If we let

$$L_{\alpha,n}(t) = \alpha + \sum_{i=1}^{\infty} \frac{\alpha^{q^i}}{\left[(t-\theta^q)(t-\theta^{q^2})\cdots(t-\theta^{q^i})\right]^n},$$

and take

$$\Phi = \begin{bmatrix} (t-\theta)^n & 0 & \cdots & 0 \\ (\theta^0)^{(-1)}(t-\theta)^n & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ (\theta^\ell)^{(-1)}(t-\theta)^n & 0 & \cdots & 1 \end{bmatrix}, \quad \Psi = \begin{bmatrix} \Omega^n & 0 & \cdots & 0 \\ \Omega^n L_{\theta^0,n} & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \Omega^n L_{\theta^\ell,n} & 0 & \cdots & 1 \end{bmatrix},$$

then

$$\Psi^{(-1)} = \Phi\Psi.$$

Furthermore, $\zeta_C(n)$ is essentially an $\mathbb{F}_q(\theta)$-linear combination of the first column of $\Psi(\theta)$.

Periods and quasi-periods of rank 2 Drinfeld modules

Let $\rho : \mathbb{F}_q[t] \to \overline{k}[F]$ be a rank 2 Drinfeld module such that

$$\rho(t) = \theta + \kappa F + F^2.$$

Suppose

$$\ker(\exp_\rho(z)) = \mathbb{F}_q[\theta]\omega_1 + \mathbb{F}_q[\theta]\omega_2 \subseteq \mathbb{C}_\infty.$$

For $i = 1, 2$, set

$$s_j(t) = -\sum_{i=0}^{\infty} \exp_\rho\left(\frac{\omega_j}{\theta^{i+1}}\right) t^i \in \mathbb{T}.$$

Difference equations for rank 2 Drinfeld modules

- We let

$$\Phi = \begin{bmatrix} 0 & 1 \\ t-\theta & -\kappa^{1/q} \end{bmatrix}, \quad \Psi = \begin{bmatrix} 0 & 1 \\ 1 & -\kappa \end{bmatrix} \begin{bmatrix} s_1^{(1)} & s_1^{(2)} \\ s_2^{(1)} & s_2^{(2)} \end{bmatrix}^{-1}.$$

- Then

$$\Psi^{(-1)} = \Phi\Psi,$$

and

$$\Psi(\theta)^{-1} = \begin{bmatrix} \omega_1 & \eta_1 \\ \omega_2 & \eta_2 \end{bmatrix}.$$