Non-archimedean Dynamics in Dimension One: Lecture 3

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Revisiting the Quadratic Polynomial Example

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- If $|a| > 1$, then $\mathcal{J}_{\phi,\mathrm{Ber}} = \mathcal{J}_\phi \subseteq \mathbb{P}^1(\mathbb{C}_K)$ is the same Cantor set as before.

  Then $\mathcal{F}_{\phi,\mathrm{Ber}} = \mathbb{P}^1_{\mathrm{Ber}} \smallsetminus \mathcal{J}_\phi$, all points of which are attracted to ∞ under iteration.

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Recall that the classical Julia set $\mathcal{J}_\phi$ was not compact; but of course the Berkovich Julia set $\mathcal{J}_{\phi,\text{Ber}}$ must be compact.

In particular, that sequence $\beta_1, \beta_2, \ldots$ (of preimages of the repelling fixed point α) accumulates at $\zeta(0,1) \in \mathcal{J}_{\phi,\text{Ber}}$.

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Let $U \subseteq \mathcal{F}_{\phi,\mathrm{Ber}}$ *be a connected component of* $\mathcal{F}_{\phi,\mathrm{Ber}}$*, and let* $x \in U$*. Then*

- U *is the union of all connected Berkovich affinoids containing* x *and contained in* $\mathcal{F}_{\phi,\mathrm{Ber}}$*.*
- $\phi(U)$ *is a connected component of* $\mathcal{F}_{\phi,\mathrm{Ber}}$*.*
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  Each V_i maps d_i-to-1 onto U, for some $d_i \geq 1$, and $d_1 + \cdots + d_\ell = d$.

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A connected component of $\mathcal{F}_{\phi,\mathrm{Ber}}$ that is not preperiodic is called a *wandering domain*.

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3. *U is a wandering domain.*

Example: Good Reduction

Recall that ϕ has **good reduction** if when we write $\phi(z) = \dfrac{f(z)}{g(z)}$ where $f, g \in \mathcal{O}[z]$ satisfy

- $(f, g) = 1$,
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An n-periodic residue class $D_{\mathrm{Ber}}(a, 1)$ is attracting if and only if $\overline{\phi}$ has a critical point among $\left\{\overline{a}, \overline{\phi}(\overline{a}), \ldots, \overline{\phi}^{n-1}(\overline{a})\right\}$.

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That is, let $V_0 = \mathbb{P}^1(\mathbb{C}_K) \setminus \overline{D}_{\text{Ber}}(a, r) \subseteq \mathcal{F}_\phi$ be the largest open $\mathbb{P}^1_{\text{Ber}}$-disk containing ∞.

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etc. In the end, $W = \bigcup_{n \geq 0} V_n$.

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   For Cantor type, the boundary is uncountable and contained in the Julia set. The boundary can include points of type I, type II, or type IV.

Rivera-Letelier's Classification: Continued

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Let $\phi \in \mathbb{C}_K(z)$ be a rational function of degree $d \geq 2$ with Fatou set $\mathcal{F}_{\phi,\mathrm{Ber}}$, and let $U \subseteq \mathcal{F}_{\phi,\mathrm{Ber}}$ be a **periodic** *connected component of the Fatou set.*

1. *If U is indifferent, then U is a rational open connected affinoid, and ϕ permutes the (finitely many) boundary points of U. The boundary points are all type II periodic Julia points.*
2. *If U is attracting, then U is either a rational open disk or a domain of Cantor type.*
   For an open disk, the unique boundary point is a type II repelling periodic (Julia) point.
   For Cantor type, the boundary is uncountable and contained in the Julia set. The boundary can include points of type I, type II, or type IV.
   (Maybe also type III? Requires a wandering domain with certain properties.)

Example: A Non-disk Indifferent Component

$$\phi(z) = \frac{1}{1-z} + \pi^2 z = \frac{\pi^2 z^2 - \pi^2 z - 1}{z-1} \in \mathbb{C}_K[z], \text{ with } 0 < |\pi| < 1.$$

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It is not hard to check that

$$\phi \text{ maps } \begin{cases} \zeta(0,|\pi|^{-1}) \mapsto \zeta(0,|\pi|) & \text{with multiplicity 2,} \\ \zeta(0,|\pi|) \mapsto \zeta(1,|\pi|) & \text{with multiplicity 1,} \\ \zeta(1,|\pi|) \mapsto \zeta(0,|\pi|^{-1}) & \text{with multiplicity 1,} \end{cases}$$

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It's also easy to check that ϕ maps the open connected affinoid

$$U := D_{\text{Ber}}(0,|\pi|^{-1}) \smallsetminus \left(\overline{D}_{\text{Ber}}(0,|\pi|) \cup \overline{D}_{\text{Ber}}(1,|\pi|)\right)$$

bijectively onto itself.

No Wandering Domains

Definition

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Then $\mathcal{F}_{\phi,\mathrm{Ber}}$ has no wandering domains.

A Power Series Lemma

Lemma

Let $a \in \mathbb{C}_K^\times$*, set* $r = |a|$*, and let* $0 < s < r$*.*
Let $f(z) = c_0 + c_d z^d + \cdots \in \mathbb{C}_K[[z]]$ *converge on* $\overline{D}(0, r)$*, and assume that* $|c_n| r^n < |d c_d| r^d$ *for all* $n > d \geq 1$*.*

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So $\operatorname{diam}\left(f(\overline{D}(a,s))\right) = |dc_d| r^{d-1} s$, and
$\operatorname{diam}\left(f(\overline{D}(0,r))\right) = |c_d| r^d$.

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Change coordinates so that $\mathcal{J}_\phi \subseteq \overline{D}(0,1)$.
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So U has infinitely many non-overlapping iterates of radius bounded below and intersecting the compact set $\mathcal{O}_K$, a contradiction. QED

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The wandering domains in question are just wandering residue classes of ζ whose iterates avoid "bad" residue classes.

No **Other** Wandering Domains

Theorem (RB, 2005)

Let K be a complete **discretely valued** *non-archimedean field of* **residue characteristic zero**, *let $\mathbb{C}_K$ be the completion of an algebraic closure of K, and let $\phi \in K(z)$ be a rational function of degree $d \geq 2$.*

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Theorem (RB, 2002)

Let $\mathbb{C}_K$ have residue characteristic $p > 0$.
Then there is a parameter $a \in \mathbb{C}_K$ (in fact, a dense set of such parameters in $\mathbb{C}_K \setminus \overline{D}(0,1)$) such that

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(Idea of Proof: see Project #4)

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Theorem (Rivera-Letelier, 2005)

Let K be a complete non-archimedean field of residue characteristic p. Then there are polynomials $\phi \in K[z]$ with wild recurrent Julia critical points.

Proof. See project #4.

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Let K be a complete non-archimedean field of residue characteristic p. Then there are polynomials $\phi \in K[z]$ with wild recurrent Julia critical points.

Proof. See project #4.

In both cases (char $K = p > 0$ with wild Julia critical points, or char $K = 0$ with wild recurrent Julia critical points), we don't know whether there can be wandering domains.