ARIZONA WINTER SCHOOL 2015 COURSE OUTLINE: GROWTH IN GROUPS, GROWTH THROUGH ACTIONS

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The course will be a brief introduction to the study of growth in finite groups. Its emphasis will lie on developments in the last decade; the underlying methods involve a mixture of geometry and probability theory with a distinct combinatorial flavor.

The main idea is to show how to study growth by means of actions - whether that of a group on itself or that of the group on a different object. The nature of that object depends on the kind of group we work with: it is affine space if we work with a linear algebraic group, and an unstructured set, small relative to the size of the group, if we work with $Sym(n)$. For simplicity, we will use SL_2 as a paradigmatic example for linear algebraic groups in general.

(The exposition will be based mainly on [Hel15], which in turn is based on [Hel08], [Hel11], [PS16], [BGT11], [HS14]; it will also incorporate some elements of [Tao15].)

Growth: how and why

- Growth, expansion, mixing times: an introduction from multiple perspectives
- Basic tools:
  (a) additive combinatorics (Ruzsa tripling lemma [RT85]),
  (b) actions (orbit-stabilizer theorem for sets)

Growth in linear algebraic groups.

- Extremely basic algebraic geometry (to be omitted if there are no neophytes): varieties, dimension, degree, intersections
- Escape from subvarieties
- Dimensional estimates
- Large sets and large multiplicity
- Induction on a group; rich tori. Main result for SL_2.
- An overview of applications: expanders [BG08], affine sieve [BGS10], spectral gaps in quotients of $\mathbb{H}$ [BGS11].

Growth in permutation groups

- Random walks in small graphs
- Using elements of small support [BBS04]
- Splitting. Long stabilizer chains.
- Induction and descent. Use of the Classification.

If time allows, we will also treat the case of random generators both for SL_2 and $Sym(n)$.

Project: As is usually the case, the main open problems in the field are likely to be quite hard. One possibility is to work on small but non-trivial improvements. For instance, very little is known about the case of $SL_n(\mathbb{F}_q)$, q fixed, n going to infinity, in spite of some interesting recent work [BY]. Can one say anything whatsoever by adapting some of the techniques used for $Sym(n)$?

An important issue that might attract participants with a background in finite group theory would be to attempt to remove the use of the Classification of finite simple groups from proofs of diameter bounds (even at the cost of weakening the result slightly).

Yet another possible problem: the best known diameter bound for random generators of $Sym(n)$ and the second best bound for the same use rather different techniques; in particular, the latter uses character theory much more heavily. Is it possible to combine some ideas to go further?

References

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