Automorphic Forms on Unitary Groups and Algebraicity

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Automorphic forms and L-functions play central roles in modern number theory. Although they arise as analytic objects, they also often have nice algebraic properties that encode arithmetically significant information. (For example, consider the values of the Riemann zeta function at negative odd integers, which occur as constant terms of Eisenstein series and also encode information about unique factorization in cyclotomic fields.) While we are often interested in Galois representations and the L-functions associated to them, associated automorphic forms play key roles. Indeed, the behavior of automorphic forms controls the behavior of various L-functions. This is seen in various settings, including for unitary groups, the focus of this mini-course and project.

1. Course Outline: Introduction to Automorphic Forms on Unitary Groups and Algebraicity

Unitary groups provide a particularly fruitful setting in which to work. Unitary groups have associated Shimura varieties, which provide convenient structure for studying algebraic aspects of automorphic forms (which, in turn, arise as sections of a vector bundle over Shimura varieties). We have substantial results about Galois representations associated to automorphic forms on unitary groups (e.g. [Ski12, Che04, Che09, CH13, Har10]). In addition, we have convenient representations of the L-functions associated automorphic forms on unitary groups, which are useful both for proving analytic properties and for extracting algebraic information (and even p-adic properties, as seen in [EHLS20]). Working with unitary groups has enabled major developments (which go far beyond the scope of these lectures but several of which are mentioned here as motivation for learning about automorphic forms on unitary groups), including a proof of the main conjecture of Iwasawa Theory for GL_2 [SU14] and the rationality of special values of certain automorphic L-functions (including [Shi00, Har97, Har08, Har84, Bou15]), as well as progress toward cases of the Bloch–Kato conjecture (including [SU06, Klo09, Klo15, Wan19]), and the Gan–Gross–Prasad conjecture (many recent developments, including [Xue14, Xue19, Zha14, Liu14, Yun11, JZ20, He17, BP20, BPLZZ21]).

The lectures will provide an introduction to automorphic forms on unitary groups and ingredients for proving results concerning algebraicity.

- Lecture I: Introduction to automorphic forms on unitary groups, including motivation and definitions.
- Lecture II: Automorphic L-functions and the doubling method.
- Lecture III: Algebraicity results for automorphic forms and L-functions, as well as an introduction to tools and approaches for proving algebraicity.
- Lecture IV: Sources of examples of automorphic forms, including recent approaches via liftings and pullbacks

Prerequisites: I will assume students are familiar with modular forms, viewed as functions on GL_2, as functions on the upper half plane, and as sections of a line bundle over a moduli space of elliptic curves. In particular, to help build intuition, I will sometimes mention parallels with that setting. Students who have also already worked with automorphic forms on other groups will be at an advantage, since they will be more familiar with some of the pitfalls of working beyond GL_2.

2. Project: Constructing Automorphic Forms on Unitary Groups

If someone asks you for examples of modular forms (for GL_2), you can probably list some. At least, you are probably confident you could open a textbook on modular forms and find some examples, such as Ramanujan's Delta function

$$\Delta(q) = q \prod_{n \geq 1} (1 - q^n)^{24},$$

which is a holomorphic cusp form of weight 12 and level 1. On the other hand, what if someone asks you for an explicit example of a modular form on a higher rank group? What about a vector-weight automorphic form?

This project is designed to help you gain intuition and familiarity with automorphic forms on unitary groups. In particular, it is designed to help you understand connections with more familiar examples, while also giving you the opportunity to prove new results. Continuing to develop such intuition through examples is important for experts and novices alike.

One possible approach to describing automorphic forms on higher rank groups in terms of forms on smaller groups is through certain liftings, such as (Duke–Imamoglu–)Ikeda lifts, which have recently been generalized in various directions, including to unitary groups [DI96, Ike01, Ike08]. (For example, a lift of Δ to Siegel modular forms of degree 4 is a particular form called the *Schottky form*, and it turns out to generate the space of Siegel cusp forms of degree 4, level 1, and weight 8 [PY96].) Constructions via these lifts is often used to produce forms of scalar weight.

The goal of this Arizona Winter School project, on the other hand, is to **construct vector-valued automorphic forms on unitary groups from scalar-valued forms**. To start, we will work out a strategy for producing vector-valued automorphic forms from scalar-valued ones, and we will construct explicit examples of these forms. We will rely heavily on recent work of Cléry and van der Geer, who described a method for constructing vector-valued Siegel modular forms from scalar-valued ones [CvdG15]. We will explore extensions of their approach to the setting of signature (g, g) over a quadratic imaginary field. Assuming there is sustained interest in the project, we can then continue to automorphic forms of other signatures, as well as other CM fields.

The idea is to start with a scalar-valued form f of degree g, viewed as a function on a hermitian symmetric space $\mathfrak{H}_g$, and then restrict f to a product of hermitian symmetric spaces $\mathfrak{H}_j \times \mathfrak{H}_{g-j} \subset \mathfrak{H}_g$ of lower degree. As explained in [CvdG15] in the case of Siegel modular forms, if this restriction of f vanishes, then one can exploit this to produce a (non-vanishing) vector-weight form. Following Cléry and van der Geer's model from the setting of Siegel modular forms, we will use this strategy to produce explicit examples of vector-weight hermitian forms for low degrees (for example, by working with analogues of Δ). Part of this project will likely involve learning about certain extensions of Ikeda lifts to unitary groups [Ike08, HK06], in particular their interplay with the pullbacks that play a key role in our proposed construction. As a first case, before moving to higher degree, we will work through the case of $g = 2$, where some earlier descriptions in special cases will likely provide helpful insight [DK03, DK04, Wil21]. More generally, producing the starting forms before restriction might involve employing Hermitian analogs of Saito–Kurokawa and Maass lifts, such as [Koj82, Kur78, And79, Maa79a, Maa79b, Maa79c, Zag81, Vu19a, Vu19b].

There are (at least) three main pieces (or "subprojects"), which can be done in parallel by different subgroups of students and then put together at the end:

(1) Explore the pullback procedure, and work out an analogue for unitary groups.
(2) Work out explicit descriptions of forms (at least in low degree), including via adaptations of the lifts described above.
(3) Work out explicit actions of Hecke operators in low degrees (as a step toward identifying forms of low degree).

As a next step, it might be interesting to explore relationships between different ways of producing forms. For example, in general, how does the vector-valued form our procedure produces from a lift of a form f relate to the original form f?

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