AWS 2022: RIGIDITY METHOD FOR AUTOMORPHIC FORMS OVER FUNCTION FIELDS

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1. Course outline

The goal of these lectures is to introduce an explicit way of constructing interesting automorphic forms for groups over the function field $F = \mathbb{F}_q(t)$, as well as their Langlands parameters. We call this the *rigidity method*, in analogy with the rigidity method that proved powerful in solving the inverse Galois problem (see [6] for an introduction) and the theory of rigid local system by Katz [4]. In fact, both rigidity methods can be thought of as corresponding to each other under the Langlands correspondence.

Classical modular forms for GL(2) are very concrete: one can think of them as power series using Fourier expansion. For larger groups it is much harder to describe automorphic forms explicitly (or even to give a single concrete example of a cusp form), either over number fields or function fields. In this course, we will see that for the function field $F = \mathbb{F}_q(t)$, by carefully imposing local conditions, one can arrange that the space of automorphic forms satisfying the prescribed local conditions is essentially one-dimensional. The advantage of such a situation is that the unique (up to scalar) automorphic form in this one-dimensional space is guaranteed to be a Hecke eigenform. Such a Hecke eigenform is called rigid. They can be described as an explicit function on double cosets of $G(\mathbb{A}_F)$.

One can then try to compute the Hecke eigenvalues of a rigid Hecke eigenform. Better, we would like to construct the Langlands parameter of this eigenform, which in the function field case should be a local system on a punctured projective line $\mathbb{P}^1$. The construction involves upgrading automorphic forms to automorphic sheaves, and uses sheaf-theoretic versions of the Hecke operators (which relies on the Geometric Satake Equivalence). With these geometric tools, the local systems on the punctured $\mathbb{P}^1$ can be described motivically, namely as the direct image sheaf of a family of algebraic varieties over $\mathbb{P}^1$.

It is expected that all rigid local systems arise as the Langlands parameters of rigid Hecke eigenforms. Many famous local systems are known to arise this way. A first such example is the Kloosterman local system, see [1], closely related to the Kloosterman sum

$$Kl(a) = \sum_{x \in (\mathbb{Z}/p\mathbb{Z})^\times} \exp\left(\frac{2\pi i}{p}(x + \frac{a}{x})\right).$$

Another class of examples is hypergeometric local systems, see the work of Kamgarpour-Yi [3].

The rigidity method for automorphic forms has been applied not only to solve questions about function fields, but also over number fields, see [7] and [9]. It also appears (surprisingly) in the work of Lam-Templier [5] on the mirror conjecture for partial flag varieties.

- Lecture 1: automorphic forms for groups over function fields, interpretation using bundles over a curve, Hecke operators as modification of bundles.
- Lecture 2: parahoric subgroups and local conditions, the notion of rigidity, Kloosterman examples.
- Lecture 3: more examples, sheaf-to-function dictionary, brief mention of the Geometric Satake Equivalence
- Lecture 4: calculating the eigen local system, analogy with the rigidity method in inverse Galois theory, applications.

A general reference for the course is [8].

Date: Sep. 1, 2021.

2. Projects

I can imagine projects surrounding two themes:

(1) Creating rigid situations. This means designing local conditions at two or three places of F to make the corresponding space of automorphic forms essentially one-dimensional. One can start with a known rigid local system on a punctured $\mathbb{P}^1$ and try to guess what the local conditions are. For example, try to find local conditions for a rigid situation for GL(2) such that the corresponding local system has icosahedron monodromy (image in PGL(2) is isomorphic to A_5).

This project requires some familiarity with the representation theory of p-adic groups and the local Langlands correspondence.

(2) Calculating Hecke eigenvalues. We will start with local conditions that are rigid, then try to write the Hecke eigenvalues of the corresponding eigenform into something like a Kloosterman sum. For example, for a lot of rigid situations constructed in [2], the Hecke eigenvalues have not been calculated and it would be interesting to calculate them.

This project requires familiarity with spherical Hecke algebra and the geometric viewpoint of automorphic forms for function fields in terms of vector bundles on algebraic curves.

Konstantin Jakob will be the project assistant.

References

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