Point-counting and applications

Jonathan Pila

Project Descriptions

Arizona Winter School, 2023

Project 1

Synopsis. To consider some variants and special cases of the result on unlikely intersections of an *asymmetric curve* in $Y(1)^n$ (theorem of Habegger-Pila). This will mainly involve getting good/explicit height bounds for the unlikely intersections in suitable settings.

1. Introduction. Effectivity is often very difficult to achieve in diophantine problems even when finiteness is known. The classical example is the proof by Faltings of the Mordell Conjecture: quanitative but not effective. The André-Oort conjecture is also ineffective outside of some very special cases ([6, 7, 9, 30]), and "unlikely intersection" problems are in general ineffective (though see [25]).

In point-counting approaches, one opposes an upper bound from point-counting to a lower bound coming from arithmetic, generally some lower bound for the size of a Galois orbit. Often both bounds are ineffective. Lower bounds for Galois orbits of special points in the André-Oort conjecture are ineffective due to the famously ineffective class number bounds of Landau-Siegel. These bounds are obtained by considering separately the cases where a Riemann Hypothesis is true or false. So the bound one gets depends on some putative failure of RH. Counting results are hard to make effective in general though they do not depend on that kind of argument, and various quite general effective results are known [8, 10].

This project concerns the "unlikely intersection problems" for a curve in $X = Y(1)^3 = \mathbb{C}^3$. Recall that a *special subvariety* of $Y(1)^3$ is an irreducible component of an algebraic subvariety defined by some number of equations of the form $x_i = c$, where c is "special" (i.e. a singular modulus, that is, the j-invariant of an elliptic curve with complex multiplication), or $\Phi_N(x_j, x_k) = 0$ where Φ_N is a classical modular polynomial. (References [54, 41]).

1.1. Conjecture. (Special case of ZP) Let $V \subset X = Y(1)^3$ be a curve that is not contained in any proper special subvariety of X. Then the intersection of V with the union $X^{[2]}$ of all special subvarieties of codimension ≥ 2 is a finite set.

A number of partial results are known [26], [40], [20]. In general the required counting results in the counting approach are not effective, but the Galois lower bounds can be made effective in the known cases. In [26] these use isogeny estimates to get Galois lower bounds from height upper bounds.

The points $(x_1, x_2, x_3) \in V \cap X^{[2]}$ fall into a few different types, the "generic" one being that the coordinates x_1, x_2, x_3 are non-special with the corresponding elliptic curves being pairwise isogenous. Let's call these "totally isogenous points".

The project is to make the height upper bounds for totally isogenous points in [26] fully effective, at least in some special cases. Such as for curves $X_{a,b,c} = \{(t^a, t^b, t^c) : t \in \mathbb{C}\}$ as $(a, b, c) = (1, 2, 3), \ldots$. Assuming some reasonable form for effective point-counting upper bounds one can consider how the bounds depend on a, b, c.

It may be possible to get a fully effective solution to some very special cases for totally real totally isogenous points on some real arcs; see [1], and [8].

Project 2.

Synopsis. To consider generalisations/variants of some results of Jones-Thomas-Wilkie, Wilkie, Jones-Qiu on "integer-valued definable functions". For entire (instead of definable) functions this is a classic topic going back to Pólya (1915) and many variants have been studied. It involves an interplay of complex analysis in o-minimal structures with arithmetic properties.

1. Introduction. A classical theorem of Pólya asserts that the function 2^z is the "smallest" entire transcendental function that takes integer values at all non-negative integer arguments.

1.1. Theorem. (Pólya [49]) *Let $f(z)$ be an entire function with $f(\mathbb{N}) \subset \mathbb{Z}$. If*

$$\limsup_{r\to\infty} \frac{M(f,r)}{2^r} < 1$$

then $f(z)$ is a polynomial.

Here $M(f, r)$ denotes the maximum modulus of $f(z)$ at radius r. The theorem is sharp in the sense that 2^z has the stated growth rate. This theorem has been sharpened, strengthened, and generalised in many directions, in work of (just mentioning some early work) Hardy, Gelfond, Selberg; see [11] and the recent survey [50]. It was also influential in transcendental number theory.

A version for "definable" functions is due to Jones–Thomas–Wilkie [28], sharpened by Wilkie [51].

1.2. Theorem. ([28]) *Let $f : [0, \infty)^k$ be a definable analytic function with $f(\mathbb{N}^k) \subset \mathbb{Z}$. Then either $\sum_{|\overline{x}|\le r} |f(\overline{x})|$ grows faster than $\exp(r^\delta)$, for some $\delta > 0$, or f is a polynomial.*

Here the growth exponent is not sharp, but Wilkie [52]) later gave a sharp version. To frame this result, let $\mathcal{F}$ denote the collection of germs at $+\infty$ of univariate function $\mathbb{R}_{\text{an exp}}$-definable functions.

1.3. Theorem. *Let $f \in \mathcal{F}$ and assume that $f(n) \in \mathbb{Z}$ for all sufficiently large n. Suppose that r is a real number satisfying $0 < r < 1$ and that $|f(x)| \le 2^{rx}$ for all sufficiently large x. Then there exists a polynomial P such that $f(x) = P(x)$ for all sufficiently large x.*

The project seeks to give analogues for definable functions of some of the many variants/extensions of Pólya's theorem. Where possible, the aim is to get sharp versions in terms of the exponent, or versions as sharp as their "non-definable" counterparts (sometimes the sharp value of the growth exponent is not known).

In particular for the variants considered in [4, 27, 35, 42, 37, 38] or other variants surveyed e.g. in [50]. A more ambitious goal would be progress towards the following conjecture.

Conjecture. (Wilkie [52]) Let $f : \mathbb{R} \to \mathbb{R}$ be $\mathbb{R}_{\text{an exp}}$-definable and suppose that $f(n) \in \mathbb{Z}$ for all sufficiently large positive integers n. Assume further that for some $r > 0$, we have that $|f(x)| \leq \exp(rx)$ for all sufficiently large $x \in \mathbb{R}$. Then there exist a polynomials $P(x, y_1, \ldots, y_m)$ with rational coefficients, and positive real algebraic integers $a_1, \ldots, a_m$ such that $f(x) = P(x, a_1^x, \ldots, a_m^x)$ for all sufficiently large x.

References

The references for the lecture notes are also included here.

1. J. Armitage, Pfaffian control of some polynomials involving the j-function and Weierstrass elliptic functions, arXiv:2011.09382.
2. Y. André, Finitude des couples d'invariants modulaire singuliers sur une courbe algébrique plane non modulaire, *J. Reine Angew. Math.* **505** (1998), 203–208.
3. J. Ax, On Schanuel's conjectures, *Annals* **93** (1971), 252–268.
4. J.-P. Bézivin, Suites d'entières et fonctions entières arithmétiques, *Ann. Fac. Sci. Toulouse Math.* **3** (1994) 313–334.
5. J.-P. Bézivin, Sur les fonctions entières q-arithmétiques, *Rend. Circulao Math. Palermo (2)* **47** (1998), 447–462.
6. Y. Bilu, D. Masser, and U. Zannier, An effective "theorem of André" for CM points on plane curves, *Math. Proc. Camb. Phil. Soc.* **154** (2013), 145–152.
7. G. Binyamini, Some effective estimates for André-Oort in $Y(1)^n$, with an appendix by E. Kowalski, *Crelle* **767** (2020), 17–35.
8. G. Binyamini, G. O. Jones, H. Schmidt, and M. E. M. Thomas, An effective Pila-Wilkie theorem for sets definable using Pfaffian functions, with some diophantine applications, arXiv:2301.09883.
9. G. Binyamini and D. Masser, Effective André-Oort for non-compact curves in Hilbert modular varieties, arXiv:2101.06412, *C. R. Acad. Sci. Paris, Ser. I*, to appear.
10. G. Binyamini, D. Novikov, and B. Zack, Wilkie's conjecture for Pfaffian structures, arXiv:2202.05305.
11. R. P. Boas, Comments on [49], in George Pólya: Collected Ppers, Volume 1, R. P. Boas, editor, MIT Press, Cambridge, 1974, 771–773.
12. E. Bombieri, P. Habegger, D. Masser, and U. Zannier, A note on Maurin's theorem, *Rend. Lincei. Mat. Appl.* **21** (2010), 251–260.
13. E. Bombieri, D. Masser, and U. Zannier, Intersecting a curve with algebraic subgroups of multiplicative groups, *IMRN* **20** (1999), 1119–1140.
14. E. Bombieri, D. Masser, and U. Zannier, Anomalous subvarieties – structure theorems and applications, *IMRN* **19** (2007), 33 pages.
15. E. Bombieri, D. Masser, and U. Zannier, On unlikely intersections of complex varieties with tori, *Acta Arithmetica* **133** (2008), 309–323.
16. E. Bombieri and J. Pila, The number of integral points on arcs and ovals, *Duke Math. J.* **59** (1989) 337–357.

17. T. D. Browning and D. R. Heath-Brown, Plane curves in boxes and equal sums of two powers, *Math. Z.* **251** (2005), 233–247.
18. L. Capuano, D. Masser, J. Pila, and U. Zannier, Rational points on Grassmannians and unlikely intersections in tori, *Bull. London Math. Soc* **48** (2016), 141–154.
19. P. Corvaja, D. Masser, and U. Zannier, Torsion hypersurfaces on abelian schemes and Betti coordinates, *Math. Ann.* **371** (2018), 1013–1045.
20. C. Daw and M. Orr, Zilber-Pink in a product of modular curves assuming multiplicative degeneration, arXiv:2208.06338.
21. L. van den Dries, Remarks on Tarski's problem concerning $(\mathbb{R}, +, \cdot, \exp)$, in *Logic colloquium '82,* pp. 97–121, Lolli, Longo, and Marcja, editors, *Studies in Logic and the Foundations of Mathematics* **112**, North Holland, 1984.
22. L. van den Dries, *Tame Topology and O-minimal Structures,* LMS Lecture Note Series **248**, CUP, 1998.
23. M. Gromov, Entropy, homology and semi-algebraic geometry [after Y. Yomdin], Séminaire Bourbaki, 1985-86, exposé 663, *Astérisque* **145-146** (1987), 225–240.
24. P. Habegger, Effective height upper bounds on algebraic tori, arXiv:1201.1815, *Autour de la conjecture de Zilber-Pink*, Course notes, CIRM, 2011, 167–242, Panor. Synthèses **52**, Soc. Math. France, Paris, 2017.
25. P. Habegger, G. Jones, and D. Masser, Six unlikely intersection problems in search of effectivity, *Math. Proc. Camb. Phil. Soc.* **162** (2017), 447–477.
26. P. Habegger and J. Pila, Some unlikely intersections beyond André-Oort, *Compositio* **148** (2012), 1–27.
27. G. O. Jones and S. Qiu, Integer-valued definable functions in $\mathbb{R}_{\text{an exp}}$, *IJNT* **17** (2021), 1739–1752.
28. G. O. Jones, M. E. M. Thomas, and A. J. Wilkie, Integer-valued definable functions, *Bull. LMS* **44** (2012), 1285–1291.
29. J. Knight, A. Pillay, and C. Steinhorn, Definable sets in ordered structures. II, *Trans. AMS* **295** (1986), 593–605.
30. L. Kühne, An effective result of André-Oort type, *Annals* **176** (2012), 651–671.
31. D. Masser and U. Zannier, Torsion anomalous points and families of elliptic curves, *C. R. Acad. Sci. Paris, Ser. I* **346** (2008), 491–494, and *Amer. J. Math* **132** (2010), 1677–1691.
32. D. Masser and U. Zannier, Abelian varieties isogenous to no Jacobian, *Annals* **191** (2020), 635–674.
33. G. Maurin, Courbes algébriques et équations multiplicatives, *Math. Annalen* **341** (2008), 789-824.
34. F. Pellarin, Sur une majoration explicite pour un degré disogénie liant deux courbes elliptiques, *Acta Arith.* **100** (2001), 203–243.
35. A. Perelli and U. Zannier, Su un teorema di Pólya, *Boll. Un. Mat. Ital.* A (5) **18** (1981), 305–307.
36. J. Pila, Geometric postulation of a smooth function and the number of rational points, *Duke Math. J.* **63** (1991) 449–463.
37. J. Pila, Concordant sequences and concordant entire functions, *Ren. Circ. Math. Palermo (2)* **51** (2002) 51–82.
38. J. Pila, Entire functions having a concordant value sequence, *Israel J. Math.* **134** (2003) 317–343

39. J. Pila, On the algebraic points of a definable set, *Selecta Math. N. S.* **15** (2009), 151–170.
40. J. Pila, On a modular Fermat equation, *Commentarii Mathematici Helvetici* **92** (2017), 85–103.
41. J. Pila, *Point-Counting and the Zilber–Pink Conjecture,* Cambridge Tracts in Mathematics **228**, CUP, 2022.
42. J. Pila and F. R. Villegas, Concordant sequences and integral-valued entire functions, *Acta Arithmetica* **88** (1999) 239–268.
43. J. Pila and A. J. Wilkie, The rational points of a definable set, *DMJ* **133** (2006), 591–616.
44. J. Pila and U. Zannier, Rational points in periodic analytic sets and the Manin-Mumford conjecture, *Rend. Mat. Acc. Lincei (9)* **19** (2008) 149–162.
45. A. Pillay and C. Steinhorn, Definable sets in ordered structures, *Bulletin AMS* **11** (1984), 159–162.
46. A. Pillay and C. Steinhorn, Definable sets in ordered structures I, *Trans. AMS* **295** (1986), 565–592.
47. A. Pillay and C. Steinhorn, Definable sets in ordered structures III, *Trans. AMS* **309** (1988),469–476.
48. R. Pink, A common generalization of the conjectures of André-Oort, Manin-Mumford, and Mordell-Lang, manuscript dated 17 April 2005 available from the author's webpage.
49. G. Pólya, Über ganzwertige ganze Funktionen, *Rend. Circ.Mat.Palermo* **40** (1915), 1–16. Also collected papers.
50. M. Waldschmidt, Integer–valued functions, Hurwitz functions and related topics: a survey, *31st meeting of the Journées Arithmétiques,* arXiv:2002.01223
51. A. J. Wilkie, Model completeness results for expansions of the ordered field of real numbers by restricted Pfaffian functions and the exponential function, *J. Amer. M. Soc.* **9** (1996), 1051–1094.
52. A. J. Wilkie, Complex continuations of $\mathbb{R}_{\mathrm{an\ exp}}$-definable unary functions with a diophantine application, *J. LMS* **93** (2016), 547–566.
53. Y. Yomdin, C^k-resolution of semi-algebraic mappings. Addendum to "Volume growth and entropy", *Israel J. Math.* **57** (1987), 301–317.
54. D. Zagier, Elliptic modular forms and their applications, *The 1-2-3 of modular forms,* 1–103, J.H. Brunier, G. van der Geer, G. Harder, and D. Zagier, Springer, Berlin, 2008.
55. U. Zannier, *Some problems of unlikely intersections in arithmetic and geometry,* with appendices by D. Masser, Annals of Mathematics Studies **181**, PUP, 2012.
56. B. Zilber, Exponential sums equations and the Schanuel conjecture, *J. London Math. Soc. (2)* **65** (2002), 27–44.

20230131