MODEL THEORETIC ORIGINS AND APPROACHES TO UNLIKELY INTERSECTION PROBLEMS
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ABSTRACT. We discuss some points of contact between model theory and unlikely intersection problems. These connections include one strand of the origins of these problems in Zilber's program to axiomatize the theory of the complex exponential function, questions of the (un)decidability of the theory of rational functions, and techniques for solving functional versions of these problems using the model theory of differential fields. Notably absent from these notes and the accompanying lectures is the theory of o-minimality.

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1. INTRODUCTION

What we now know as unlikely intersection problems were first formulated by Bombieri, Masser, and Zannier in their study of intersections of algebraic subvarieties of powers of the multiplicative group with translates of subtori [16, 14, 10, 11, 15, 13, 12]. Independently, Zilber proposed his Conjecture on Intersections with Tori [44] as part of his program to explain the theory of the complex exponential function. Shortly thereafter, Pink proposed a way to combine the Mordell-Lang and André-Oort Conjectures using his formalism of mixed Shimura varieties [34]. These conjectures, which go under the name of the "Zilber-Pink conjecture" or "conjectures on unlikely intersections" have been refined, notably in [5], and extended over the years. See [43, 21] for detailed surveys.

In these notes, we will focus on three aspects of the Zilber-Pink conjectures having close connections to mathematical logic. In Section 2 we recount parts of Zilber's proposed axiomatization of the theory of the complex exponential function. This gives us a context to introduce Schanuel's conjecture and Zilber's initial version of the Zilber-Pink conjecture, the Conjecture on Intersections with Tori. In Section 3 we relate the conjectures of Zilber-Pink-type to the problem of describing the induced structure on sets in algebraically closed fields. We will discuss how this is related to the problem of whether the theory of the field of rational functions $\mathbb{C}(t)$ is decidable. In Section 4 we describe an approach to the Zilber-Pink conjecture using the theory of differentially closed fields.

As we noted in the abstract, one of the most salient connections between mathematical logic and the Zilber-Pink conjecture is the use of o-minimality to prove instances of the conjecture. Please see Pila's notes for his course at this 2023 Arizona Winter School for an account of the use of o-minimality in special point and unlikely intersection problems.

The present notes (i.e. the ones you are reading right now) are still rather rough and certainly contain errors and omissions. Please let me know about corrections I should implement. I would also like to know about parts of the notes for which you would like to see more details. Those may be added to this document over time and may be included in the lectures at the Arizona Winter School.

2. Zilber's program for the complex exponential function

The theory of the real exponential function, by which we mean the first-order theory of the structure $\mathbb{R}_{\exp} := (\mathbb{R}, +, \cdot, \exp, \leq, 0, 1)$ is famously tame, indeed, the theory is model complete and o-minimal [41] and if the Schanuel conjecture (which we will discuss in much more detail below) for the real numbers holds, then this theory is even decidable [26]. On the other hand, one the first observations one makes about $\mathbb{C}_{\exp} = (\mathbb{C}, +, \cdot, \exp, 0, 1)$, and as Marker notes in [27], it is often the last observation, is that $\mathbb{Z}$ is definable in $\mathbb{C}_{\exp}$. Hence, the theory of $\mathbb{C}_{\exp}$ suffers from the Gödelian undecidability of arithmetic. We will leave it to you with the following exercise to find a definition of $\mathbb{Z}$.

Exercise 2.1. If we were to allow π and $i = \sqrt{-1}$ as parameters, then it would be easy to define $\mathbb{Z}$ as by the formula $\phi(x) := \exp(2\pi i x) = 1$. However, neither π nor $\sqrt{-1}$ is part of our language for $\mathbb{C}_{\exp}$. Find a definition of $\mathbb{Z}$ which does not use any new parameters.

2.1. Zilber's proposed infinitary axiomatization of $\mathbb{C}_{\exp}$.

Zilber's insight, or really one of Zilber's many insights, was that the theory of $\mathbb{C}_{\exp}$ may be tame relative to the complexity introduced by the integers. In its strongest form, this suggestion takes the form that his theory of pseudoexponentiation, which we describe below, is actually the theory of $\mathbb{C}_{\exp}$. A formally weaker form is his Quasiminimality Conjecture.

Definition 2.2. A structure $\mathfrak{M}$ is *quasiminimal* if its universe $M = |\mathfrak{M}|$ is uncountable and for every definable (with parameters) set $X \subseteq M$ either X or $M \smallsetminus X$ is countable.

Remark 2.3. From the compactness theorem one sees that quasiminimality is a property of a structure, not of its first-order theory. There are infinitary languages in which quasiminimality may be enforced by the theory.

Remark 2.4. In Definition 2.2 we asked that every definable set be countable or co-countable without specifying the logic. While we usually work in first-order logic, we will observe that Zilber's proposed theory for $\mathbb{C}_{\exp}$ is most naturally expressed using the infinitary logic $\mathcal{L}_{\omega_1,\omega}(Q)$ (which will be described in detail below). We might consider a stronger form of quasiminimality which would allow for more expressive logics. Let us say that the structure $\mathfrak{M}$ is *strongly quasiminimal* if its universe $|\mathfrak{M}|$ is uncountable and for every finite subset $A \subseteq |\mathfrak{M}|$ each $\mathrm{Aut}_A(\mathfrak{M})$-invariant subset of $|\mathfrak{M}|$ is countable or cocountable. Strong quasiminimality corresponds to our syntactic notion of quasiminimality where we take "definable set" to mean a set definable in the logic $\mathcal{L}_{\infty,\infty}$ with finitely many parameters.

Conjecture 2.5. $\mathbb{C}_{\exp}$ *is (strongly) quasiminimal.*

One could read Conjecture 2.5 as asserting that the complexity of $\mathbb{C}_{\exp}$ is encapsulated in its definable countable sets. This conjecture has some strong consequences. One immediate consequence is that $\mathbb{R}$ would not be definable in $\mathbb{C}_{\exp}$. A less obvious consequence is that if Conjecture 2.5 holds, then $|\mathrm{Aut}(\mathbb{C}_{\exp})| = 2^{2^{\aleph_0}}$. In particular, there would be some discontinuous field automorphism $\sigma : \mathbb{C} \to \mathbb{C}$ which commutes with the exponential function.

Zilber suggested that the theory of $\mathbb{C}_{\exp}$ should be described by saying that

- the underlying field is an algebraically closed field of characteristic zero of cardinality $2^{\aleph_0}$,
- exp is a surjective homomorphism from the additive group to the multiplicative group,
- the kernel of exp is an infinite cyclic group,
- not too many exponential-algebraic relations hold as formalized by *Schanuel's Conjecture*,
- all systems of exponential-algebraic equations whose solvability do not explicitly contradict Schanuel's Conjecture have many solutions, and
- the "exponential closure" of each finite set (by which we mean the collection numbers obtained by closing off under the field operations, exponentiation, and certain implicitly defined functions) should be countable.

In order to formalize this specification, we will need to explain Schanuel's Conjecture, the condition of not explicitly contradicting this conjecture, and the details of how to form the exponential closure.

Let us start with Schanuel's Conjecture.

Conjecture 2.6. *If $\alpha_1, \ldots, \alpha_n \in \mathbb{C}$ are linearly independent over $\mathbb{Q}$, then the transcendence degree over $\mathbb{Q}$ of the field $\mathbb{Q}(\alpha_1, \ldots, \alpha_n, e^{\alpha_1}, \ldots, e^{\alpha_n})$ generated by $\alpha_1, \ldots, \alpha_n$ and their exponentials is at least n.*

Remark 2.7. The Schanuel conjecture for the real numbers, mentioned above in connection with the decidability of the theory of $\mathbb{R}_{\exp}$, is the restriction of Conjecture 2.6 to the case that $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$.

Other than some special cases covered by classical results, such as the Lindemann-Weierstrass Theorem which asserts that Conjecture 2.6 is true when $\alpha_1, \ldots, \alpha_n$ are all algebraic, Schanuel's Conjecture remains open. Indeed, applying Schanuel's conjecture to the case of $\alpha_1 = 1$ and $\alpha_2 = 2\pi i$ would yield the algebraic independence of e and π, which is unknown. Indeed, as of this writing it is still unknown whether $e + \pi$ is rational.

The condition that every system of exponential algebraic equations which does not explicitly contradict Schanuel's Conjecture may be expressed via the Converse Schanuel Conjecture, also proposed by Schanuel [42].

Conjecture 2.8. *Suppose that K is a countable field of characteristic zero with with an exponential function, that is, a surjective group homomorphism $E : (K, +) \to (K^\times, \cdot)$, having an infinite cyclic kernel and satisfying Schanuel's Conjecture in the sense that whenever $\alpha_1, \ldots, \alpha_n \in K$ are linearly independent over $\mathbb{Q}$, then the transcendence degree over $\mathbb{Q}$ of $\mathbb{Q}(\alpha_1, \ldots, \alpha_n, E(\alpha_1), \ldots, E(\alpha_n))$ is at least n. Then there is an embedding $\sigma : (K, +, \cdot, E, 0, 1) \hookrightarrow \mathbb{C}_{\exp}$.*

The axiom on the countability of exponential closures is less relevant to our concerns, but for the sake of completeness we explain the details. Morally, for a set $A \subseteq \mathbb{C}$ the exponential-algebraic closure of A, $\mathrm{ecl}(A)$, is the set of numbers which are components of isolated solutions to exponential-algebraic equations with coefficients from A. To make this notion precise we need to define what we mean by an exponential-algebraic equation with coefficients from A and we need to abstract the notion of an isolated solution to such an equation so as not to refer to the Euclidean topology.

An exponential ring (A, E) is a commutative ring A given together with a homomorphism $E : (A, +) \to (A^\times, \cdot)$ from the additive to the multiplicative group of A.

Exercise 2.9. Show that if (K, E) is an exponential *field*, by which we mean that it is an exponential ring whose underlying ring is a field, and E is nontrivial, then K has characteristic zero.

Given an exponential ring (A, E) and a natural number n, $A[x_1, \ldots, x_n]^E$, the exponential ring of exponential polynomials with coefficients in A in the variables $x_1, \ldots, x_n$, is the free exponential ring extension of A generated by the variables $x_1, \ldots, x_n$. This ring of exponential polynomials may be constructed as the term algebra in the variables $x_1, \ldots, x_n$ for the language $\mathcal{L}(E, +, -, \cdot, \{a\}_{a \in A})$ modulo the universal axioms for exponential rings and the atomic diagram of A.

Exercise 2.10. Show that $A[x_1, \ldots, x_n]^E$ is characterized by the following universal property. For any maps of exponential rings $\phi : (A, E) \to (B, E)$ and choice $(b_1, \ldots, b_n) \in B^n$ of an n-tuple from B there is a unique map $\widetilde{\phi} : A[x_1, \ldots, x_n]^E \to B$ of exponential rings with $\widetilde{\phi}(x_i) = b_i$ and $\widetilde{\phi} \upharpoonright A = \phi$.

For each $i \leq n$ there is a unique A-derivation $\frac{\partial}{\partial x_i}$ on $A[x_1, \ldots, x_n]^E$ which satisfies the rules that

$$\frac{\partial}{\partial x_i}(x_j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

and

$$\frac{\partial}{\partial x_i}(E(f)) = \frac{\partial f}{\partial x_i} E(f)$$

for any $f \in A[x_1, \ldots, x_n]^E$.

Exercise 2.11. Verify the assertions of the above paragraph. Prove uniqueness by structural induction on terms t to show that if ∂_1 and ∂_2 are two A-derivations on $A[x_1, \ldots, x_n]^E$ satisfying the above rules and f is represented by the term t, then $\partial_1(f) = \partial_2(f)$. For the existence of $\frac{\partial}{\partial x_i}$, observe that $A[x_1, \ldots, x_n]^E$ admits an increasing filtration by subrings where $F_0 := A[x_1, \ldots, x_n]$ and F_{m+1} is the subring of $A[x_1, \ldots, x_n]^E$ generated by $E(F_m)$ over F_m. Check that the requisite derivation exists on each F_m and that these derivations are compatible with the inclusions $F_m \hookrightarrow F_{m+1}$.

Given an extension of exponential fields $(K, E) \hookrightarrow (L, E)$ we say that $a \in L$ belongs to the exponential algebraic closure of K, written as $a \in \mathrm{ecl}(K)$, if for some n there are exponential polynomials $f_1, \ldots, f_n \in K[x_1, \ldots, x_n]^E$ and a solution $(b_1, \ldots, b_n) \in L^n$ to the system of equations

$$\begin{array}{rcl} f_1(b_1, \ldots, b_n) & = & 0 \\ f_2(b_1, \ldots, b_n) & = & 0 \\ & \cdots & \\ f_n(b_1, \ldots, b_n) & = & 0 \end{array}$$

with

$$\begin{vmatrix} \frac{\partial}{\partial x_1} f_1(b_1, \ldots, b_n) & \frac{\partial}{\partial x_1} f_2(b_1, \ldots, b_n) & \cdots & \frac{\partial}{\partial x_1} f_n(b_1, \ldots, b_n) \\ \frac{\partial}{\partial x_2} f_1(b_1, \ldots, b_n) & \frac{\partial}{\partial x_2} f_2(b_1, \ldots, b_n) & \cdots & \frac{\partial}{\partial x_2} f_n(b_1, \ldots, b_n) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial}{\partial x_n} f_1(b_1, \ldots, b_n) & \frac{\partial}{\partial x_n} f_2(b_1, \ldots, b_n) & \cdots & \frac{\partial}{\partial x_n} f_n(b_1, \ldots, b_n) \end{vmatrix} \neq 0$$

and $b_1 = a$.

More generally, if (L, E) is an exponential field and $A \subseteq L$ is any subset, then $\mathrm{ecl}(A) = \mathrm{ecl}(K)$ where K is the exponential field generated by A, that is, the smallest exponential subfield of L containing A.

Exercise 2.12. Show that in $\mathbb{C}_{\exp}$ if $A \subseteq \mathbb{C}$, then $\mathrm{ecl}(A)$ is countable.

On the face of it, to express all of these axioms we require a stronger logic than ordinary first-order logic. It is not a problem to say that we have an exponential field in which the exponential map is surjective and the underlying field is algebraically closed of characteristic zero, but each of the remaining axioms seems to require to require more.

Exercise 2.13. Give more details as to how to express that we have an exponential field in which the exponential map is surjective and the underlying field is algebraically closed of characteristic zero with a first-order theory.

Zilber's original (conjectural) axiomatization of the theory of $\mathbb{C}_{\exp}$ is expressed in the infinitary language $\mathcal{L}_{\omega_1,\omega}(Q)$. The Conjecture on Intersections with Tori would permit most instances of this infinitary language to be replaced by formulae in usual first-order logic.

In $\mathcal{L}_{\omega_1,\omega}(Q)$, in addition to the usual first-order formula construction operations we are permitted to form countable conjunctions and to apply a new quantifier Q which is intended to mean "there exist uncountably many". That is, if Φ is a countable set of $\mathcal{L}_{\omega_1,\omega}$-formulae, then $\bigwedge \Phi$ is also an $\mathcal{L}_{\omega_1,\omega}$-formula and if ϕ is an $\mathcal{L}_{\omega_1,\omega}(Q)$-formula and x is a variable, then $Qx\phi$ is also an $\mathcal{L}_{\omega_1,\omega}$-formula. If $\mathfrak{M}$ is a structure and a is an (infinitely long) tuple from M to be substituted for the free variables, then

$$\mathfrak{M} \models \bigwedge \Phi(a) \iff \text{ for all } \phi \in \Phi, \mathfrak{M} \models \phi(a) \ .$$

The quantifier Q is interpreted as saying "there are uncountably many". That is,

$$\mathfrak{M} \models Qx\phi \iff \{a \in M : \mathfrak{M} \models \phi(a)\} \text{ is uncountable.}$$

Exercise 2.14. Show that the condition that the kernel of exp is an infinite cyclic group may be expressed in $\mathcal{L}_{\omega_1,\omega}$. (Note: we have omitted the quantifier Q.). Show that the condition that the exponential algebraic closure of each finite set is countable may be expressed in $\mathcal{L}_{\omega_1,\omega}(Q)$. Show that neither of these axioms may be expressed by a first-order theory.

The remaining axioms, expressing Schanuel's Conjecture and its converse form, may also be expressed most naturally using $\mathcal{L}_{\omega_1,\omega}$. The Conjecture on Intersections with Tori, which is Zilber's first contribution to the Zilber-Pink conjectures, may be used to convert these infinitary axioms to a set of axioms that may be expressed in first-order logic.

Fix a natural number n and $f_1, \ldots f_\ell \in \mathbb{Z}[x_1, \ldots, x_n, y_1, \ldots, y_n]$ a sequence of polynomials over the integers in $2n$ variables for which the dimension of the subvariety $V(f_1, \ldots, f_\ell)$ of affine $2n$-space defined by the vanishing of $f_1, \ldots, f_\ell$ is strictly less than n. Consider the $\mathcal{L}_{\omega_1,\omega}$ sentence

$$\theta_{\vec{f}} := (\forall \boldsymbol{x}) \left(\bigwedge_{i=1}^{\ell} f(\boldsymbol{x}, \exp(\boldsymbol{x})) = 0 \to \bigvee_{(m_1,\ldots,m_n) \in \mathbb{Z}^n \smallsetminus \{(0,\ldots,0)\}} m_1 x_1 + \cdots + m_n x_n = 0 \right)$$

The sentence $\theta_{\vec{f}}$ asserts that for each n-tuple $\boldsymbol{a}$ with $\operatorname{tr.deg}_{\mathbb{Q}} \mathbb{Q}(\boldsymbol{a}, \exp(\boldsymbol{a}) < n$ witnessed by $f_1(\boldsymbol{a}, \exp(\boldsymbol{a})) = \cdots = f_\ell(\boldsymbol{a}, \exp(\boldsymbol{a}) = 0$, then some nontrivial $\mathbb{Z}$-linear combination of the coordinates of $\boldsymbol{a}$ is zero, which is equivalent to saying that some nontrivial $\mathbb{Q}$-linear combination is zero. Thus, Schanuel's Conjecture is the countable conjunction over all such $\vec{f}$ of the sentences $\theta_{\vec{f}}$.

Our formulation of $\theta_{\vec{f}}$ uses an infinite disjunction in its conclusion. To express Schanuel's Conjecture in first-order logic, we would like to replace the disjunction over all possible nonzero linear forms over the integers with a disjunction over a finite subcollection of these. The solution, which is dependent on Zilber's Conjecture on Intersections with Tori essentially does just this after a slight adjustment of the hypothesis of $\theta_{\vec{f}}$ and by shifting from linear relations on the arguments to multiplicative dependencies on the exponentials.

To convert to a multiplicative version of Schanuel's Conjecture we begin with a version in which we do not require the arguments to be linearly independent.

Exercise 2.15. Show that Schanuel's Conjecture is equivalent to the statement that for any finite sequence $\alpha_1, \ldots, \alpha_n$ of complex numbers, the transcendence degree of the field $\mathbb{Q}(\alpha_1, \ldots, \alpha_n, \exp(\alpha_1), \ldots, \exp(\alpha_n))$ is at least the dimension over $\mathbb{Q}$ of the $\mathbb{Q}$-vector space generated by $\alpha_1, \ldots, \alpha_n$.

Considering relations on the exponentials instead of linear relations on the arguments, we might like to replace the formulation of Schanuel's Conjecture from Exercise 2.15 with the conclusion that the transcendence degree of the field $\mathbb{Q}(\alpha_1, \ldots, \alpha_n, \exp(\alpha_1), \ldots, \exp(\alpha_n))$ is at least as large as the rank of the subgroup of the multiplicative group generated by $\exp(\alpha_1), \ldots, \exp(\alpha_n)$. It is easy to deduce this latter statement from Schanuel's Conjecture, but the other implication is not obvious (and, as far as I know, not even known to be true). For example, Schanuel's Conjecture in its original form would imply that $\operatorname{tr.deg} \mathbb{Q}(\pi, e) = \operatorname{tr.deg} \mathbb{Q}(1, 2\pi i, e, 1) = 2$, but applying the multiplicative form directly would give only that $\operatorname{tr.deg} \mathbb{Q}(\pi, e) = \operatorname{tr.deg} \mathbb{Q}(1, 2\pi i, e, 1) \geq 1$, which we know to be true. A refinement comes from working over the kernel.

Definition 2.16. Let (L, E) be an exponential field. Let $K := \{a \in L : E(a) = 1\}$ be the kernel of the exponential in L. By $K_\mathbb{Q}$ we mean $K \otimes \mathbb{Q}$ realized as the $\mathbb{Q}$-vector subspace of L generated by K. We say that Schanuel's Conjecture holds over the kernel if whenever $\alpha_1, \ldots, \alpha_n \in L$, then $\operatorname{tr.deg}_{\mathbb{Q}(K)} \mathbb{Q}(\boldsymbol{\alpha}, E(\boldsymbol{\alpha}), K) \geq \dim_\mathbb{Q}(K_\mathbb{Q} + \sum \mathbb{Q}\alpha_i)/K_\mathbb{Q}$.

Schanuel's Conjecture over the kernel does immediately translate into a multiplicative form.

Exercise 2.17. Show that the exponential field (L, E) satisfies Schanuel's Conjecture over the kernel if and only if whenever $\alpha_1, \ldots, \alpha_n \in L$, then $\operatorname{tr.deg}_{\mathbb{Q}(K)} \mathbb{Q}(\boldsymbol{\alpha}, E(\boldsymbol{\alpha}), K)$ is at least the rank of the multiplicative group generated by $E(\alpha_1), \ldots, E(\alpha_n)$.

Exercise 2.18. Show that Schanuel's Conjecture for the usual complex exponential function is equivalent to Schanuel's Conjecture over the kernel for the complex exponential.

Taking into account Exercises 2.17 and 2.18, in attempting to axiomatize $\mathbb{C}_{\exp}$, we may replace Schanuel's Conjecture with Schanuel's Conjecture over the kernel expressed in the multiplicative form.

There is a downside to moving to Schanuel's Conjecture over the kernel in that we will have to work over parameters. That is, Schanuel's Conjecture in its original form may be expressed by saying that for each algebraic variety $X \subseteq \mathbb{G}_a^n \times \mathbb{G}_m^n$ defined over $\mathbb{Q}$ with $\dim(X) < n$, if $(\boldsymbol{a}, \exp(\boldsymbol{a})) \in X$, then some nontrivial $\mathbb{Z}$-linear form vanishes on $\boldsymbol{a}$. As we have seen, we may convert these assertions into countably many $\mathcal{L}_{\omega_1,\omega}$-sentences. For Schanuel's Conjecture over the kernel we need to quantify over all algebraic varieties defined over $\mathbb{Q}(K)$ and as the interpretation of K may change from one model to another, this will require working with families of algebraic varieties and verifying that the relevant geometric properties are definable in parameters.

Fortunately, the properties that we require are in fact definable, though the proofs are nontrivial. Let us see how to define dimension uniformly, starting with a precise formulation as a proposition.

Proposition 2.19. *Let d, ℓ, m, and n be natural numbers. We write $\boldsymbol{x} = (x_1, \ldots, x_n)$ and $\boldsymbol{y} = (y_1, \ldots, y_m)$ for these tuples of variables. Let $f_1, \ldots, f_\ell \in \mathbb{Z}[\boldsymbol{x}, \boldsymbol{y}]$ be polynomials in the variables $\boldsymbol{x}$ and $\boldsymbol{y}$ over the integers. Then there is a quantifier-free formula $\vartheta_{d,\boldsymbol{f}}(\boldsymbol{y})$ in the language of rings $\mathcal{L}(+, \cdot, -, 0, 1)$ with free variables amongst $\boldsymbol{y}$ so that for any algebraically closed field K*

and tuple $\boldsymbol{b} \in K^m$ the set of solutions to

$$\begin{array}{rcl} f_1(\boldsymbol{x}, \boldsymbol{b}) & = & 0 \\ f_2(\boldsymbol{x}, \boldsymbol{b}) & = & 0 \\ & \cdots & \\ f_\ell(\boldsymbol{x}, \boldsymbol{b}) & = & 0 \end{array}$$

in K^n is the set of K-points of an algebraic variety of dimension d if and only if $K \models \vartheta_d(\boldsymbol{b})$.

There are several methods available to prove Proposition 2.19. Algebraically, you could compute degree bounds necessary to test whether some given subset of the $\boldsymbol{x}$ variables of size d are algebraically independent modulo $\sqrt{(f_1(\boldsymbol{x}, \boldsymbol{b}), \ldots, f_\ell(\boldsymbol{x}, \boldsymbol{b}))}$. A general model theoretic argument could go through the observation that all completions of the theory of algebraically closed fields are strongly minimal, that Morley rank and algebraic dimension agree in this theory, and that Morley rank is uniformly definable in strongly minimal theories. The following exercises outline another proof.

Exercise 2.20. Let K be an algebraically closed field, d a natural number, and $X \subseteq \mathbb{A}^n_K$ an algebraic subvariety of affine n-space over K for some natural number n. Show that $\dim X \geq d$ if and only if there is some coordinate projection $\pi : \mathbb{A}^n_K \to \mathbb{A}^d_K$, given by $(x_1, \ldots, x_n) \mapsto (x_{i_1}, \ldots, x_{i_d})$ for some $1 \leq i_1 < i_2 < \cdots < i_d \leq n$ so that the Zariski closure of $\pi(X(K))$ is $\mathbb{A}^d_K$.

Exercise 2.21. Let K be an algebraically closed field and $X \subsetneq \mathbb{A}^n_K$ a closed subvariety of affine n-space over K with $X \neq \mathbb{A}^n$. Show that there are $a_1, \ldots, a_{n+1} \in K^n$ so that $\bigcap_{i=1}^{n+1} (a_i + X(K)) = \varnothing$.

Exercise 2.22. Prove Proposition 2.19 using Exercises 2.20 and 2.21.

Using Proposition 2.19 we may express Schanuel's Conjecture over the kernel as countable list of $\mathcal{L}_{\omega_1,\omega}$ sentences.

2.2. **Conjecture on Intersections with Tori and first-order axiomatizations.** Let us consider the Conjecture on Intersections with Tori in form that generalizes naturally to other unlikely intersection problems.

If S is a smooth algebraic variety and X and Y are irreducible subvarieties of S, then each component of the intersection $X \cap Y$ has dimension at least $\dim X + \dim Y - \dim S$. Generally, we expect the dimension to be exactly $\dim X + \dim Y - \dim S$. We call a component of such an intersection *atypical* if its dimension is larger than this lower bound and we say that an intersection between X and Y is *unlikely* if $\dim(X) + \dim(Y) < \dim(S)$.

Let $g \in \mathbb{Z}_+$ be a positive integer and write $S := \mathbb{G}_m^g$ for the g^{th} Cartesian power of the multiplicative group, regarded as an algebraic group. For an irreducible subvariety $X \subseteq S$ we define the atypical locus of X, X^{atyp}, to be the union of all atypical components of intersections $X \cap T$ where $T \leq S$ is an algebraic subgroup. Note that we do not require T to be connected.

Conjecture 2.23 (Zilber's Conjecture on Intersections with Tori)**.** *With the notation as in the previous paragraph, X^{atyp} is a Zariski closed subvariety of X.*

Exercise 2.24. Follow the notation and of Conjecture 2.23. Show that each of the following statements is equivalent to Conjecture 2.23.

- If X is not contained in the a proper algebraic subgroup of S, then X^{atyp} is not Zariski dense in X.
- There is a finite set $\mathcal{T}$ of algebraic subgroups of S so that X^{atyp} is the union of the atypical components of intersections $X \cap T$ as T ranges through $\mathcal{T}$.

In the original formulation of Conjecture 2.23, the subvariety X was taken to be defined over the algebraic numbers. It turns out that the conjecture for X defined over $\mathbb{C}$ is equivalent to the conjecture for X defined over $\mathbb{Q}^{\mathrm{alg}}$, though the proof of this equivalence is not obvious (at least, not to me).

How do we use the Conjecture on Intersections with Tori to show that infinitary sentences $\theta_{\vec{f}}$ may be replaced by first-order sentences? Let us begin by describing how we adjust the hypotheses.

Let $f : X \to Y$ be a regular map of irreducible algebraic varieties. For $b \in Y$ we write X_b for the fiber of f over b. It follows from the theorem on semicontinuity of fiber dimension that the set

$$X^{\mathrm{gfd}} := \{x \in X : \dim X = \dim X_{f(x)} + \dim f(X)\}$$

is a constructible, Zariski dense subset of X.

Let now n be a natural number, $X \subseteq \mathbb{G}_a^n \times \mathbb{G}_m^n$ be an irreducible subvariety of the product of the n^{th} Cartesian power of the additive group with the n^{th} Cartesian power of the multiplicative group defined over $\mathbb{Q}$ having dimension strictly less than n, and let $f : X \to \mathbb{G}_m^n$ be the restriction of the natural projection map to the last n coordinates. Fixing another natural number N, we define a new sentence $\vartheta_{X,N}$ by

$$\vartheta_{X,N} := (\forall \boldsymbol{x}) \left((\boldsymbol{x}, \exp(\boldsymbol{x})) \in X^{gfd} \to \bigvee_{\substack{(m_1,\ldots,m_n) \in \mathbb{Z}^n \\ |m_i| \leq N \\ (m_1,\ldots,m_n) \neq (0,\ldots,0)}} \} \quad m_1 x_1 + \cdots + m_n x_n = 0 \right)$$

We will show with the next proposition that if the Conjecture on Intersections with Tori holds, then for each such irreducible $\mathbb{Q}$-algebraic variety $X \subseteq \mathbb{G}_a^n \times \mathbb{G}_m^n$ with $\dim(X) < n$ we may find a natural number $N(X)$ so that modulo the theory of exponential fields, Schanuel's Conjecture is equivalent to the first-order theory axiomatized by the sentences $\vartheta_{X,N(X)}$. Indeed, we may say what number $N(X)$ to take. If the Conjecture on Intersections with Tori holds, then there this a finite set $\mathcal{T} = \mathcal{T}(X)$ of proper algebraic subgroups of $\mathbb{G}_m^n$ so that $\overline{f(X)}^{\mathrm{atyp}}$ is equal to union of the atypical components of $\overline{f(X)} \cap T$ as T ranges through $\mathcal{T}$. For each $T \in \mathcal{T}$, there is some nonzero vector $(m_1, \ldots, m_n) \in \mathbb{Z}^n$ for which T is contained in the group defined by $\prod_{i=1}^n x_i^{m_i} = 1$. Let $M(T) := \min\{N \in \mathbb{Z}_+ : \prod_{i=1}^n x_i^{m_i} = 1\}$ on T for some nontrivial $(m_1, \ldots, m_n) \in \mathbb{Z}^n$ with $\max m_i \leq N\}$. Set $N = N(X) := \max_{T \in \mathcal{T}} M(T)$.

Proposition 2.25. *Schanuel's Conjecture may be expressed by a countable set of first-order sentences if and only if the Conjecture on Intersections with Tori holds.*

Proof. Let us start by showing that if the Conjecture on Intersections with Tori holds and Schanuel's Conjecture holds, then the axioms $\vartheta_{X,N(X)}$ described in the paragraph before the statement of this proposition hold.

Fix some irreducible $\mathbb{Q}$-algebraic variety $X \subseteq \mathbb{G}_a^n \times \mathbb{G}_m^n$ with $\dim X < n$. Suppose that $(\boldsymbol{a}, \exp(\boldsymbol{a})) \in X^{\text{gfd}}$. Let $T \leq \mathbb{G}_m^n$ be the smallest algebraic subgroup containing $\exp(\boldsymbol{a})$. Using the truth of the Schanuel Conjecture, we see that

$$\operatorname{tr.deg}_{\mathbb{Q}} \mathbb{Q}(\boldsymbol{a}, \exp(\boldsymbol{a})) \geq n - \dim T$$

By additivity of transcendence degrees, we have

$$\operatorname{tr.deg}_{\mathbb{Q}} \mathbb{Q}(\boldsymbol{a}, \exp(\boldsymbol{a})) = \operatorname{tr.deg}_{\mathbb{Q}(\exp(\boldsymbol{a}))} \mathbb{Q}(\boldsymbol{a}, \exp(\boldsymbol{a})) + \operatorname{tr.deg}_{\mathbb{Q}} \mathbb{Q}(\exp(\boldsymbol{a}))$$

Let C be a component of $\overline{f(X)} \cap T$ containing $\exp(\boldsymbol{a})$. We aim to show that C is atypical. Since $(\boldsymbol{a}, \exp(\boldsymbol{a}) \in X^{\text{gfd}}$, we have

$$\operatorname{tr.deg}_{\mathbb{Q}(\exp(\boldsymbol{a})} \mathbb{Q}(\boldsymbol{a}, \exp(\boldsymbol{a})) \leq \dim X_{\exp(\boldsymbol{a})} = \dim X - \dim \overline{f(X)} < n - \dim \overline{f(X)}$$

This yields that

$$\dim C > \dim T + \dim \overline{f(X)} - n$$

That is, C is atypical. Therefore, $\exp(\boldsymbol{a}) \in \overline{f(Y)}^{\text{atyp}}$ implying that $\prod_{i=1}^n \exp(a_i)^{m_i} = 1$ for some non-zero vector $(m_1, \ldots, m_n) \in \mathbb{Z}^n$ with $|m_i| \leq N$ for all $i \leq n$. That is, we have verified that each sentence $\vartheta_{X,N(X)}$ holds.

We leave it as an exercise to verify that Schanuel's Conjecture follows from the $\vartheta_{X,N(X)}$ axioms. $\square$

To complete the proposed axiomatization of $\mathbb{C}_{\exp}$ we should find a geometric way to express Conjecture 2.8. Originally, Zilber did this with a Strong Exponential-Algebraic Closedness axioms which said that systems of exponential-algebraic equations which are not obviously inconsistent have generic solutions. It turns out that it suffices to ask merely that such systems of equations have solutions; that is, without specifying that they have generic solutions.

If $X \subseteq \mathbb{G}_a^n \times \mathbb{G}_m^n$ is a subvariety, defined over $\mathbb{Q}$, of the product of the n^{th} Cartesian power of the additive group by the n^{th} Cartesian power of the multiplicative group, and $\dim X < n$, then Schanuel's Conjecture says that there should be no points of the form $(\boldsymbol{a}, e^{\boldsymbol{a}}) \in X(\mathbb{C})$ (let us call these "exponential points on X") unless $\boldsymbol{a}$ lies on some hyperplane defined over $\mathbb{Q}$. So, while it may be possible for there to be some sporadic exponential points on X if its dimension is small, generally we do not expect many. Thus, in any axiom asserting that there are exponential points on such a variety X we should include the condition that $\dim X \geq n$.

There may be hidden reasons why a variety is too small to have many exponential points. Given a matrix

$$M = \begin{pmatrix} \mu_{1,1} & \mu_{1,2} & \cdots & \mu_{1,n} \\ \mu_{2,1} & \mu_{1,2} & \cdots & \mu_{1,n} \\ \vdots & \vdots & \ddots & \vdots \\ \mu_{m,1} & \mu_{m,2} & \cdots & \mu_{m,n} \end{pmatrix} \in M_{m\times n}(\mathbb{Z})$$

we may define a map algebraic groups $\Psi_M : (\mathbb{G}_a^n \times \mathbb{G}_m^n) \to (\mathbb{G}_a^m \times \mathbb{G}_m^m)$ by

$$(x_1, \ldots, x_n, y_1, \ldots, y_n) \mapsto (\left(\sum_{j=1}^{n} \mu_{i,j} x_j \right)_{i=1}^{m}, \left(\prod_{j=1}^{n} y_j^{\mu_{i,j}} \right)_{i=1}^{n}) \ .$$

If for some $X \subseteq \mathbb{G}_a^n \times \mathbb{G}_m^n$ and a matrix $M \in M_{m\times n}(\mathbb{Z})$ with $\mathrm{rk}(M) = n$ we had $\dim \Psi_M(X) < n$, then by Schanuel's Conjecture we would not expect many exponential points on $\Psi_M(X)$, and because Ψ_M maps exponential points to exponential points, not many exponential points on X. Thus, in formulating a condition under which varieties should have exponential points we should take into account these transformations. The relevant notion goes under different names in the literature and admits some variants allowing for relativization. Let us say that $X \subseteq \mathbb{G}_a^n \times \mathbb{G}_m^n$ is *broad* if for every matrix $M \in M_{m\times n}(\mathbb{Z})$ with $\mathrm{rk}(M) = m$ we have $\dim \Psi_M(X) \geq m$.

One other obstruction to X having exponential points is that there may be some nontrivial integer vector $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_n)$, which we could think of as an $n \times 1$ matrix, for which the projection of $\Psi_{\boldsymbol{\mu}}(X)$ to $\mathbb{G}_a$ is contained in the the rational hull of the kernel of exp or the projection of $\Psi_{\boldsymbol{\mu}}$ to $\mathbb{G}_m$ is contained in the roots of unity. We say that X is free if this does not happen. In practice, we also need to consider relativizations of this this condition.

Each of the conditions of breadth and freeness seem to depend on quantificaton over a countable set. The Conjecture on Intersections with Tori may be used to convert them to conditions expressible in first-order logic.

When working with the proposed axioms for $C_{\exp}$ in $\mathcal{L}_{\omega_1,\omega}$, Zilber showed that for each uncountable cardinal κ there is exactly one model of size κ up to isomorphism. In particular, there is unique model $\mathbb{B}$ of cardinality $2^{\aleph_0}$. Zilber's conjecture is that $\mathbb{B} \cong \mathbb{C}_{\exp}$.

3. Special point conjectures and induced structure

It is a long standing open problem [36] whether the first-order theory of the field $\mathbb{C}(t)$ of rational functions in the single variable t with coefficients from the field $\mathbb{C}$ of complex numbers is decidable. In various public lectures, Pheidas has described a strategy, which he attributes to an anonymous reviewer of a grant proposal, for showing that this theory is undecidable based on interpreting complicated structure on the complex numbers.

Using the fact that there are no nonconstant rational maps from the projective line to curves of positive genus, it is easy to see that $\mathbb{C}$ is definable in $\mathbb{C}(t)$. For example, we have

$$\mathbb{C} = \{a \in \mathbb{C}(t) : (\exists y) a^3 + y^3 = 1\} \ .$$

The harder observation is that the set of j-invariants of elliptic curves with complex multiplication is also definable in $\mathbb{C}(t)$. We pause to recall some some facts about elliptic curves in Subsection 3.1, but you would do well to consult [40] for more details. Some of the constructions and concepts will be familiar to you from Tsimerman's lectures at this Arizona Winter School. We return in Subsection 3.2 to the details of how to define complicated structures on $\mathbb{C}$ in $\mathbb{C}(t)$.

3.1. **Primer on elliptic curves.** Recall that an elliptic curve over the complex numbers is a connected, projective algebraic group of dimension one. For an elliptic curve E over $\mathbb{C}$, the complex Lie group of the complex points on E may be realized as $E(\mathbb{C}) = \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau)$, a quotient of the additive group of the complex numbers by a lattice of the form $\mathbb{Z} + \mathbb{Z}\tau$ with $\tau \in \mathfrak{h} = \{z \in \mathbb{C} : \mathrm{Im}(\tau) > 0\}$.

On general grounds, one sees that if E_1 and E_2 are two complex elliptic curves realized as $E_j(\mathbb{C}) = \mathbb{C}/\Lambda_j$ then we may identify the group of maps of elliptic curves $\mathrm{Hom}(E_1, E_2)$, by which we mean maps of complex algebraic groups $\psi : E_1 \to E_2$, so group homomorphisms given by regular maps of algebraic varieties, with the set

$$\{\lambda \in \mathbb{C} : \lambda\Lambda_1 \leq \Lambda_2\} .$$

Indeed, in one direction, one sees that if $\lambda \in \mathbb{C}$ and $\lambda\Lambda_1 \leq \Lambda_2$, then the linear map $x \mapsto \lambda x$ from $\mathbb{C} \to \mathbb{C}$ descends to a map of complex Lie groups $\psi_\lambda : \mathbb{C}/\Lambda_1 \to \mathbb{C}/\Lambda_2$. By Chow's Theorem, which asserts that every closed, complex analytic subvariety of projective space is necessarily algebraic, the graph of ψ_λ is algebraic, and, hence, ψ_λ is itself map of elliptic curves $E_1 \to E_2$. In the other direction, a map of elliptic curves $\psi : E_1 \to E_2$ must lift to a homomorphism between their universal covering spaces, that is, $\mathbb{C}$ itself, $\widetilde{\psi} : \mathbb{C} \to \mathbb{C}$. Moreover, because ψ is complex analytic, $\widetilde{\psi}$ is also complex analytic. As all complex analytic group endomorphisms of $\mathbb{C}$ are given by scalar multiplication, we may identify $\widetilde{\psi}$ with multiplication by some complex number λ. Since $\widetilde{\psi}$ descends to ψ, necessarily $\lambda\Lambda_1 \leq \Lambda_2$.

Applying this analysis to the case that $E_1 = E_2 =: E$, we determine the only possibilities for $\mathrm{End}(E) = \mathrm{Hom}(E, E)$ (which has a ring structure with multiplication given by composition along with addition given by adding the homomorphisms as functions). Present E as $E(\mathbb{C}) = \mathbb{C}/\Lambda$ where $\Lambda = \mathbb{Z} + \mathbb{Z}\tau$ with $\tau \in \mathfrak{h}$. Certainly, $\mathrm{End}(E) \geq \mathbb{Z}$. For the endomorphism ring to be larger, we would need to have some $\lambda \in \mathbb{C} \smallsetminus \mathbb{Z}$ with $\lambda\Lambda \in \Lambda$. Since $1 \in \Lambda$, we have $\lambda = \lambda \cdot 1 \in \Lambda = \mathbb{Z} + \mathbb{Z}\tau$. Thus, there are $c \in \mathbb{Z}$ and $d \in \mathbb{Z}$ with $\lambda = c + d\tau$. Since $\lambda \notin \mathbb{Z}$, we have $d \neq 0$. We also have that $\tau \in \Lambda$, so there are $a, b \in \mathbb{Z}$ with $\lambda\tau = a + b\tau$. Using the formula that $\lambda = c + d\tau$, we compute that $d\tau^2 + (c - b)\tau - a = 0$. Recalling that $d \neq 0$, we see that τ satisfies a quadratic equation. We say that an elliptic curve E has *complex multiplcation* or is *CM* (or that its moduli point is CM) if $\mathrm{End}(E) \neq \mathbb{Z}$. From this discussion, we see that if E has complex multiplication, then $\mathrm{End}(E) = \mathbb{Z}[\lambda]$ for some quadratic imaginary number λ. In particular, the rank of $\mathrm{End}(E)$ as an abelian group is two.

The projective general linear group GL_2 of invertible two-by-two matrices by acts by algebraic automorphisms on the projective line via linear fractional transformations expressed with matrices acting on points in projective coordinates as

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot [x_0 : x_1] = [ax_0 + bx_1 : cx_0 + dx_1]$$

or in affine coordinates taking $x_1 = 1$, as

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot x = \frac{ax+b}{cx+d}$$

If we restrict the action to the real Lie group $\mathrm{GL}_2(\mathbb{R})^+ := \{g \in \mathrm{GL}_2(\mathbb{R}) : \det(g) > 0\}$, we see that $\mathrm{GL}_2(\mathbb{R})^+$ maps the upper halfplane $\mathfrak{h}$ back to itself.

There is an analytic function $j : \mathfrak{h} \to \mathbb{C}$ characterized as the unique analytic function which is invariant under precomposition with linear fractional transformations coming from $\mathrm{SL}_2(\mathbb{Z})$, has a simple pole at infinity, and takes the values $j(i) = 1728$ and $j(\exp(\frac{2\pi i}{3})) = 0$. More explicitly, j may be computed as a ratio of two modular forms. In what follows we express our functions of the variable τ ranging over $\mathfrak{h}$ as series in the variable $q := \exp(2\pi\tau)$.

For each natural number r we define a function $\sigma_r : \mathbb{Z}_+ \to \mathbb{Z}$ by

$$\sigma_r(n) := \sum_{d|n} d^r$$

For even $r \geq 2$ we have the Eisenstein series

$$E_r := 1 - \frac{2r}{B_r} \sum_{n=1}^{\infty} \sigma_{r-1}(n) q^n$$

where B_r is the r^{th} Bernouli which admits the closed form expresion

$$B_r = \frac{(-1)^{\frac{r}{2}+1} 2(r!)}{(2\pi)^r} \zeta(r)$$

where ζ is the Riemann zeta function.

The discriminant form Δ is defined by

$$\Delta = q \prod_{n=1}^{\infty} (1 - q^n)^{24}$$

We may then express the j-function as

$$j = \frac{E_4^3}{\Delta}$$

The set of complex numbers serves as the coarse moduli space of elliptic curves over $\mathbb{C}$. That is, to each elliptic curve E over $\mathbb{C}$ we may assign a complex number $j(E) \in \mathbb{C}$ so that for two such elliptic curves E_1 and E_2, we have that $j(E_1) = j(E_2)$ if and only if $E_1 \cong E_2$ as complex algebraic groups. The quantity $j(E)$ may be defined analytically using the presentation $E(\mathbb{C}) = \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau)$ of the complex points of E as a quotient of the additive group of $\mathbb{C}$ by a lattice with and setting $j(E) := j(\tau)$ where j is the analytic j-function. This construction of the j-invariant may be familiar to you from study of special point problems through the o-minimal counting method discussed at the Arizona Winter School.

It turns out that $j(E)$ may be computed as a polynomial in the coefficients of the equations giving E. On general grounds, every complex elliptic curve may be realized as a smooth cubic curve in the projective plane described in affine coordinates by an equation of the form $y^2 = x^3 + Ax + B$ where A and B are constants, the identity element of the group is the unique point on this curve at infinity, and the group operation is given by the secant and tangent method. From this equation for E we may compute $j(E)$ as

$$j(E) = \frac{1728(4A)^3}{-16(4A^3 + 27B^2)}$$

The comparison to the analytic formula for the j-invariant comes from recognizing the coefficients A and B as certain scalar multiples of the Eisenstein series E_4 and E_6, respectively, and then computing an expression for Δ in terms of E_4 and E_6.

By an *isogeny* $\psi : E_1 \to E_2$ between elliptic curves we mean a map of algebraic groups with a finite kernel. Analytically, all pairs of elliptic curves for which there is an isogeny from the first to the second may be obtained from the following construction.

If $N \in \mathbb{Z}_+$ is a positive integer, then for any $\tau \in \mathfrak{h}$, the identity map $\mathbb{C} \to \mathbb{C}$ induces a map of complex Lie groups

$$E_{j(N\tau)}(\mathbb{C}) = \mathbb{C}/(\mathbb{Z} + \mathbb{Z}N\tau) \longrightarrow \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau) = E_{j(\tau)}(\mathbb{C})$$

whose kernel is isomorphic to $\mathbb{Z}/N\mathbb{Z}$.

There is polynomial $\Phi_N(x, y) \in \mathbb{Z}[x, y]$ in two variables with integer coefficients, monic as a polynomial in x and as a polynomial in y, so that for any pair of complex numbers $(a, b) \in \mathbb{C}^2$ we have

$$\Phi(a, b) = 0 \Longleftrightarrow (\exists \tau \in \mathfrak{h})\ (a, b) = (j(N\tau), j(\tau)) \ .$$

That is, there is an isogeny $\psi : E_1 \to E_2$ having kernel isomorphic to $\mathbb{Z}/N\mathbb{Z}$ if and only if $\Phi_N(j(E_1), j(E_2)) = 0$.

An elliptic curve E is CM if and only if there is some $N \in \mathbb{Z}$ with $N \geq 2$ (equivalently, infinitely many such N) for which there an isogeny $\psi : E \to E$ with a cyclic kernel of size N. Indeed, for any nonzero $n \in \mathbb{Z}$, the kernel $E[n]$ of the map $[n]_E : E \to E$ given by multiplication by n in E with respect to the group law of E is isomorphic to the (non-cyclic) group $(\mathbb{Z}/|n|\mathbb{Z})^2$. Conversely, if E has CM, then it admits an endomorphism with a nontrivial cyclic kernel.

From these observations, we see that if E has CM, then $j(E)$ is an algebraic integer. Moreover, since there are infinitely many distinct quadratic imaginary fields, we see that the set of j-invariants of CM elliptic curves is an infinite set of algebraic integers. In the next subsection we will explain how to see this set as a definable subset of $\mathbb{C}(t)$.

3.2. **Defining CM-points and isogeny relations in $\mathbb{C}(t)$.** As we noted in the introduction to this section, the set of complex numbers is a definable subset of $\mathbb{C}(t)$.

Remark 3.1. We have given an existential definition of $\mathbb{C}$ in $\mathbb{C}(t)$. Since quantifier-free formulae in one variable in the language of fields always define finite or cofinite sets, there can be no quantifier-free definition of $\mathbb{C}$. Is there a definition given by universal formulae? Compare this problem to that of defining $\mathbb{Z}$ in $\mathbb{Q}$. It has been known since 1948 [37] that $\mathbb{Z}$ is definable in $\mathbb{Q}$ and much more recently [25] that there are universal definitions of $\mathbb{Z}$ in $\mathbb{Q}$. It remains an open question whether there is an existential definition of $\mathbb{Z}$ in $\mathbb{Q}$. If there were, then we could reduce Hilbert's Tenth Problem for $\mathbb{Q}$ to Hilbert's Tenth Problem for $\mathbb{Z}$, and then conclude from the Matiyasevich-Davis-Putnam-Robinson theorem [18, 28] that no algorithm to decide the solvability of Diophantine equations in the rationals exists.

If X and Y are two smooth, projective curves over the complex numbers, then every rational map $X \dashrightarrow Y$ extends to a regular map from X to Y. Equivalently, $Y(\mathbb{C}(X))$, the set of $\mathbb{C}(X)$-rational points on Y may be identified with the set of regular maps of algebraic varieties from X to Y. Using this observation, we see immediately that if we are given a family $C \subseteq \mathbb{P}^n_B$ of projective curves presented through some system of homogeneous equations on projective n-space with coefficients varying through some definable set B, then we may uniformly define the sets of maps $\mathbb{P}^1 \to C_b$ of curves as b varies through B. Of course, once the genus of C_b is positive, then there are no such nonconstant maps. Thus, it may seem that there is not much to be gained from this observation. The key point is that by making use of the restriction of scalars construction, we may upgrade this simple observation to the stronger conclusion that when given two families of smooth projective curves $C_i \subseteq \mathbb{P}^{n_i}_{B_i}$ for $i = 1, 2$, then we may uniformly define the sets of morphisms $\mathrm{Mor}((C_1)_{b_1}, (C_2)_{b_2})$ as b_i varies through B_i for $i = 1, 2$.

In model-theoretic terms, if K is any field and L is a finite extension of K, then L is interpretable in K. In fact, for any fixed positive integer d, the set of field extensions of K of degree d is uniformly interpretable in K. The construction should be familiar to you from the interpretation of $\mathbb{C}$ in $\mathbb{R}$. That is, using real and imaginary parts we may regard $\mathbb{C}$ as $\mathbb{R}^2$ with addition defined by

$$(x_1, x_2) \oplus (y_1, y_2) := (x_1 + y_1, x_2 + y_2)$$

and multiplication given by the rule

$$(x_1, x_2) \odot (y_1, y_2) := (x_1 y_1 - x_2 y_2, x_1 y_2 + x_2 y_1) .$$

With addition and multiplication defined in this way, the map $\rho : (\mathbb{R}^2, \oplus, \odot) \to (\mathbb{C}, +, \cdot)$ given by $(x_1, x_2) \mapsto x_1 + x_2 i$ is an isomorphism of structures. In this way, we may convert the set of complex valued points in any constructible set to the set of real valued points in some corresponding constructible set living in an ambient space of twice the dimension.

For a general finite extension of fields, L/K, we interpret L in K as follows. First, we fix a K-basis $\alpha_1, \ldots, \alpha_d$ of L over K. Second, we compute the matrix describing multiplication $L \otimes L \to L$ relative to this basis. That is, we find $\mu_{i,j,k} \in K$ for $1 \leq i, j, k \leq d$ so that

$$\alpha_i \alpha_j = \sum_{k=1}^{d} \mu_{i,j,k} \alpha_k$$

for each pair (i, j) with $1 \leq i, j \leq d$. We then define operations on K^d by taking $\oplus$ to be coordinatewise addition and $\odot$ to be the bilinear form given by $\boldsymbol{\mu}$. That is, the k^{th} coordinate of $(x_1, \ldots, x_d) \odot (y_1, \ldots, y_d)$ is

$$\sum_{i=1}^{d} \sum_{j=1}^{d} \mu_{i,j,k} x_i y_j \alpha_k .$$

Then as with the interpretation of $\mathbb{C}$ in $\mathbb{R}$, the map

$$\rho : (K^d, \oplus, \odot) \to (L, +, \cdot)$$

given by

$$(x_1, \dots, x_d) \mapsto \sum_{i=1}^{d} x_i \alpha_i$$

is an isomorphism of structures.

More generally, if we start with K and elements $\mu_{i,j,k}$ for $1 \leq i, j, k \leq d$ we may use $\boldsymbol{\mu}$ to define functions $\oplus$ and $\odot$ on K^d via the rules described above. In each case, we obtain a structure $L_{\boldsymbol{\mu}}$ for the language with two binary function symbols having underlying universe K^d. Because the theory of fields is finitely axiomatizable, we can identify definably for which choices of $\boldsymbol{\mu}$ is $L_{\boldsymbol{\mu}}$ a field. If it is, then by mapping $1 \in K$ to the identity element for $\odot_{\boldsymbol{\mu}}$ in $L_{\boldsymbol{\mu}}$ we obtain a map of fields $K \to L_{\boldsymbol{\mu}}$ expressing $L_{\boldsymbol{\mu}}$ as an extension of K of degree d.

Using these interpretations, definable sets in finite extensions of K may be converted to definable sets relative to K. The only case that will matter to us will be the conversion of algebraic varieties over some $L_{\boldsymbol{\mu}}$ to algebraic varieties over K.

To simplify the notation for the remainder of this discussion we fix a positive integer d and a choice of parameters $\boldsymbol{\mu}$ from K for which $L := L_{\boldsymbol{\mu}}$ is a field extension of K of degree d. We will write $\alpha_1, \dots, \alpha_d \in L$ for the standard basis of $L = K^d$ over K in which α_i is the vector $(0, \dots, 0, 1, 0, \dots, 0)$ with a 1 in the i^{th} coordinate and 0s everywhere else.

For each natural number n, from the interpretation of L in K we obtain a map of L-algebras

$$\vartheta : L[x_1, \dots, x_n] \to L[\{x_{i,j} : 1 \leq i \leq n, 1 \leq j \leq d\}]$$

given by

$$x_i \mapsto \sum_{j=1}^{d} x_{i,j} \alpha_j \ .$$

The polynomial ring $L[\{x_{i,j} : 1 \leq i \leq n, 1 \leq j \leq d\}]$ may be realized as

$$\begin{aligned} L[\{x_{i,j} : 1 \leq i \leq n, 1 \leq j \leq d\}] &= (K[\{x_{i,j} : 1 \leq i \leq n, 1 \leq j \leq d\}])_{\boldsymbol{\mu}} \\ &= \bigoplus_{j=1}^{d} L[\{x_{i,j} : 1 \leq i \leq n, 1 \leq j \leq d\}]\alpha_j \ . \end{aligned}$$

For each $j \leq d$ define

$$\pi_j : L[\{x_{i,j} : 1 \leq i \leq n, 1 \leq j \leq d\}] \to K[\{x_{i,j} : 1 \leq i \leq n, 1 \leq j \leq d\}]$$

by projecting to the coefficient of α_j in this presentation.

If $X \subseteq \mathbb{A}^n_L$ is subvariety of affine n-space defined over L, we let $R_{L/K}X$, the Weil restriction of scalars form L to K of X, be the subvariety of affine dn-space over K defined by the ideal $\{\pi_j(f) : f \in I(X), 1 \leq j \leq d\}$ in $K[\{x_{i,j} : 1 \leq i \leq n, 1 \leq j \leq d\}]$. It is easy to check that there is a natural identification between $X(L)$ and $R_{L/K}X(K)$.

Given a map $f : X \to Y$ of affine algebraic varieties over L, where $X \subseteq \mathbb{A}^n_L$ and $Y \subseteq \mathbb{A}^m_L$, if $\Gamma(f) \subseteq X \times Y$ is the graph of f, then $R_{L/K}\Gamma(f)$ is naturally the graph of a morphism $R_{L/K}(f) : R_{L/K}X \to R_{L/K}Y$. In this way, we construct a functor from the category of embedded affine algebraic varieties over L to the category of embedded affine algebraic varieties over K. In fact, this construction localizes correctly, thereby allowing for an extension to general algebraic varieties (or schemes) and may be understood in terms of an adjunction. We leave the verification of these observations to the following exercises.

Exercise 3.2. Verify that $R_{L/K}$ extends to a functor on the category of algebraic varieties over L to the category of algebraic varieties over K. You will need to check, for instance, that if X is an algebraic variety defined over L with a covering $\mathcal{U}$ by affine open subsets $U \subseteq X$ and for each U we are given an embedding $\phi_U : U \hookrightarrow \mathbb{A}_L^{n_U}$ for which the maps $\psi_{U,V} : \phi_U(V) \to \phi_V(U)$ given by $\psi_{U,V} = \phi_V \circ \phi_U^{-1}$ are regular, then the data of $\{R_{L/K}U : U \in \mathcal{U}\}$ and $\{R_{L/K}(\psi_{U,V}) : (U,V) \in \mathcal{U}^2\}$ defined an algebraic variety $R_{L/K}X$ defined over K. While you are at it, you should check that this construction extends to give a functor from the category of schemes over L to the category of schemes over K.

Exercise 3.3. Tensor product defines a functor

$$- \otimes_K L : \mathrm{Alg}_K \to \mathrm{Alg}_L$$

from K-algebras to L-algebras. Each scheme X over K (respectively, over L) defines the functor of points h_X from Alg_K (respectively, Alg_L) to Set by the rule $A \mapsto X(A)$. Show that we have a natural identification for each K-algebra A of $X(A \otimes_K L)$ with $R_{L/K}X(A)$.

Working scheme theoretically, the tensor product $- \otimes_K L$ corresponds to the base change functor $\mathrm{Sch}_K \to \mathrm{Sch}_L$ from the category of schemes over K (or more precisely, over $\operatorname{Spec} K$) to the category over schemes over L, $X \mapsto X \times_{\operatorname{Spec} K} \operatorname{Spec} L$, which we usually write, abusing notation, as $X \otimes_K L$ or even just as X_L. The identification you have just completed may be used to show that $R_{L/K} : \mathrm{Sch}_L \to \mathrm{Sch}_K$ is right adjoint to the base change functor in the sense that for X a scheme over K and Y a scheme over L we have a natural identification $R_{L/K}Y(X) = Y(X_L)$.

Let us return to the field $\mathbb{C}(t)$ of rational functions and specialize the restriction of scalars construction to the case of function fields of elliptic curves.

As we have noted, a complex elliptic curve may be given by an affine equation of the form $y^2 = x^3 + Ax + B$ for a choice of complex numbers A and B by taking the origin of the group to the point at infinity on the closure of this affine curve in the projective plane. In order for this equation to define a smooth curve, we require that $4A^3 + 27B^2$ be invertible, which is clearly a definable condition. In what follows, we write $E_{A,B}$ for the elliptic curve defined by this equations and we assume that A and B have been chosen so that the curve is nonsingular.

By replacing x with t, the function field of $E_{A,B}$ may be realized as $\mathbb{C}(t)[y]/(y^2 - t^3 - At - B) = \mathbb{C}(t)_{(1,1,1,t^3+At+B)}$ if we use the basis $1, y$ for $\mathbb{C}(E_{A,B})$ over $\mathbb{C}(t)$. Thus, for any algebraic variety X defined over $\mathbb{C}$, using the parameters A and B and the parameters appearing in a definition X, we may interpret $X(\mathbb{C}(E_{A,B})$ in $\mathbb{C}(t)$. In particular, we may uniformly interpret the groups $E_{A',B'}(\mathbb{C}(E_{A,B}))$.

For a pair of complex elliptic curves E_1 and E_2, the group of rational points $E_2(\mathbb{C}(E_1))$, is the same as the group of maps of algebraic varieties from E_1 to E_2 and fits into an exact sequence

$$0 \longrightarrow E_2(\mathbb{C}) \longrightarrow E_2(\mathbb{C}(E_1)) \longrightarrow \operatorname{Hom}(E_1, E_2) \longrightarrow 0$$

Specializing to the case that $E_1 = E_2 =: E$, we see that $E(\mathbb{C}(E))/E(\mathbb{C}) \cong \operatorname{End}(E)$, regarded as an additive group. We know that E has CM if and only if $\operatorname{rk}\operatorname{End}(E) > 1$.

Thus, we may definably recognize that $E_{A,B}$ has CM by checking whether

$$|E_{A,B}(\mathbb{C}(E_{A,B}))/2E_{A,B}(\mathbb{C}(E_{A,B}))| > 2 .$$

With this observation, we see that the set $\{j(E_{A,B}) : E_{A,B} \text{ has CM }\}$ is definable in $\mathbb{C}(t)$.

Exercise 3.4. Work out in more detail the formula used to define the set of j-invariants of elliptic curves with CM.

3.3. **Tameness of the set of CM points.** From our observations in the previous section, we have seen that both the field of complex numbers $\mathbb{C}$ and the set CM of j-invariants of elliptic curves with CM are definable in $\mathbb{C}(t)$. Thus, the complicated structure $\mathfrak{CM} := (\mathbb{C}, +, \cdot, 0, 1, \mathsf{CM})$ consisting of the complex numbers regarded as a field with a predicate for the set of CM points is interpretable in $\mathbb{C}(t)$. One might expect that this would imply that the theory of $\mathbb{C}(t)$ is complicated, and, in particular, undecidable. However, $\mathrm{Th}(\mathfrak{CM})$ is itself stable and decidable. Thus, one cannot prove the undecidability of $\mathrm{Th}(\mathbb{C}(t))$ by reducing to the undecidability of the interpreted structure $\mathfrak{CM}$.

The first hint that $\mathrm{Th}(\mathfrak{CM})$ may be tame comes from the André-Oort conjecture for products of the j-line proven by Pila [30]. If $X \subseteq \mathbb{A}^n$ is an algebraic variety, then there is a finite union of special varieties, by which we mean components of varieties defined by modular equations $\Phi_N(x_i, x_j) = 0$, $Y \subseteq \mathbb{A}^n$ for which

$$X(\mathbb{C}) \cap \mathsf{CM} = Y(\mathbb{C}) \cap \mathsf{CM} .$$

Analyzing the structure of the special varieties, one sees that the André-Oort conjecture implies that the structure induced on CM by the field structure on $\mathbb{C}$ is not complicated. More precisely, it is strongly minimal with trivial forking geometry. General theorems on expansions of strongly minimal structures by predicates with stable induced structure imply that $\mathfrak{CM}$ is itself stable [33]. Moreover, these theorems express how to axiomatize $\mathrm{Th}(\mathfrak{CM})$. We need to start with the axioms of $(\mathbb{C}, +, \cdot, 0, 1)$, namely, ACF_0, the theory of algebraically closed fields of characteristic zero. We then need to express the correspondence between a variety $X \subseteq \mathbb{A}^n$ and $\overline{X(\mathbb{C}) \cap \mathsf{CM}^n}$. Recent work of Binyamini [7] shows that this function is recursive. Hence, $\mathrm{Th}(\mathfrak{CM})$ admits a recursive axiomatization and is therefore decidable.

With our projects we consider some other structures which are definable in $\mathbb{C}(t)$ and for which conjectural effective Zilber-Pink conjectures may imply tameness.

4. Differential algebraic approaches to Zilber-Pink

Differential algebra has played an important role in the problems around unlikely intersections, even before they were formulated as such. As we have noted, Schanuel's Conjecture for complex numbers is a stubbornly difficult problem, but a functional analogue was proven by Ax already in the early 1970s using methods from differential algebra [2]. Ax's theorem and its generalizations for other covering maps are crucial ingredients in the proofs of André-Oort conjecture, functional versions of the Zilber-Pink conjecture, and other results on unlikely intersections.

In this section we recount some of the ideas behind Ax's theorem and its generalizations, discuss how these theorems are used to prove Zilber-Pink-type theorems, and close with a discussion of effective finiteness in differential fields.

4.1. **The Ax-Schanuel theorems.** In the following statement of Ax's theorem we refer to a certain rank. We will be given set $\Delta = \{\partial_1, \ldots, \partial_n\}$ of n commuting derivations and another list $\alpha_1, \ldots, \alpha_m$ of elements of some Δ-field. By $\mathrm{rk}(\partial_i \alpha_j)$ we mean the rank of the Jacobian $m \times n$-matrix

$$\begin{pmatrix} \partial_1\alpha_1 & \partial_2\alpha_1 & \ldots & \partial_n\alpha_1 \\ \partial_1\alpha_2 & \partial_2\alpha_2 & \ldots & \partial_n\alpha_2 \\ \vdots & \vdots & \ddots & \vdots \\ \partial_1\alpha_m & \partial_2\alpha_m & \ldots & \partial_n\alpha_m \end{pmatrix} .$$

Theorem 4.1. *[Ax] Let $n, m \in \mathbb{Z}_+$ be positive integers and let K be a field of characteristic zero with $\Delta = \{\partial_1, \ldots, \partial_n\}$ a set of n commuting derivations on K. Let $C := \{x \in K : \delta_i(x) = 0 \text{ for } 1 \le i \le n\}$ be the field of Δ-constants in K. Let $\alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_m \in K$. We write $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_m)$ and $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_m)$. We assume that these elements of K satisfy*

- *$\beta_j \neq 0$ for $1 \le j \le m$,*
- *$\frac{\partial_i \beta_j}{\beta_j} = \partial_i \alpha_j$ for $1 \le j \le m$ and $1 \le i \le n$, and*
- *the images of the α_is are $\mathbb{Q}$-linearly independent in the vector space K/C.*

Then

$$\mathrm{tr.deg}_C\, C(\boldsymbol{\alpha}, \boldsymbol{\beta}) \ge m + \mathrm{rk}(\partial_i \alpha_j) .$$

We have stated Theorem 4.1 in its differential algebraic formulation. It implies results about actual functions. With the following exercise you should work out what it means about complex analytic functions.

Exercise 4.2. Let $n \in \mathbb{Z}_+$ and let $U \subseteq \mathbb{C}^n$ be a simply connected domain in $\mathbb{C}^n$. Let $\alpha_1, \ldots, \alpha_m$ be a sequence of complex analytic maps $\alpha_j : U \to \mathbb{C}$. For each $j \le m$, set $\beta := \exp \circ \alpha_j$. Suppose that for all non-zero integer vectors $\boldsymbol{\ell} = (\ell_1, \ldots, \ell_m) \in \mathbb{Z}^m$ the function $\prod_{j=1}^m \beta_j^{\ell_j}$ is nonconstant. Show that it follows from Theorem 4.1 that $\mathrm{tr.deg}_{\mathbb{C}}\, \mathbb{C}(\boldsymbol{\alpha}, \boldsymbol{\beta}) \ge m + \mathrm{rk}(\frac{\partial \alpha_j}{\partial z_i})$.

Remark 4.3. In fact, Theorem 4.1 follows formally from the conclusion of Exercise 4.2 using the Seidenberg embedding theorem [38, 39, 29] which says that given a finitely generated differential subfield K of the meromorphic functions on some simply connected open set $U \subseteq \mathbb{C}^n$ and a finitely generated extension L of K (as a differential field with n commuting derivations), possibly after shrinking U to some open, simply connected domain $V \subseteq U$, there is an embedding of differential fields from L into $\mathcal{M}(V)$, the differential field of meromorphic functions on V, over K.

Ax's first proof of Theorem 4.1 in [2] is strictly algebraic. We will outline some of the key steps below. In the following year, in [3] Ax studied a complementary problem about analytic subgroups of algebraic groups using methods of differential geometry and gave an alternative proof Theorem 4.1. Some generalizations of Theorem 4.1 (usually called "Ax-Schanuel theorems") have been proven using differential algebraic methods in the style of Ax's original proof, for example, by Kirby [23] for the Lie exponential on a semiabelian variety. In this century, most proofs of Ax-Schanuel theorems have used o-minimality, for example by Pila and Tsimerman for the j-function [32] or Bakker, Klingler, and Tsimerman for period mappings associated to variations of Hodge structure [4], and then the differential algebraic versions have been deduced using the Seidenberg embedding theorem. More recent techniques of Blázquez Sanz, Casale, Freitag, and Nagloo import ideas from differential geometry into differential algebra to give very general Ax-Schanuel theorems [8, 17].

Let us note the highlights in Ax's proof. He studies properties of differential forms on general fields. An initial crucial observation is that if L/K is an extension of fields and $\delta : L \to L$ is a derivation extends a derivation on K, then there is a unique additive map $\delta^1 : \Omega_{L/K} \to \Omega_{L/K}$ which satisfies $\delta^1(adb) = (\delta a)db + ad(\delta(b))$ for $a, b \in L$. Using this map δ^1, Ax can differentiate differential forms thereby showing that certain differential equations in a differential field induce equations involving Kähler differentials. Another key technical, though elementary, lemma is the observation that if L/K is an extension of fields of characteristic zero, then the map $K \otimes_{\mathbb{Z}} dL/L \to \Omega_{L/K}/dL$ is injective where dL/L is the group of logarithmic forms $\{\frac{db}{b} : b \in L\}$. The proof of this lemma involves a reduction to the case that $\operatorname{tr.deg}_K L = 1$ and then a computation with residues.

4.2. **Differential algebraic approaches to Zilber-Pink type problems.** Differential algebraic methods are not so useful for problems involving numbers or zero dimensional intersections, at least when those intersections are themselves defined over the constants. We will discuss a possible counterexample to this principle below in connection with a function field version of Zilber-Pink. If in the Zilber-Pink conjectures we weaken the conclusion to consider only positive dimensional components of anomalous or unlikely intersections, then in many cases it is possible to prove these weakened conjectures.

Some approaches to Zilber-Pink problems from differential algebra make use of the compactness theorem of first-order logic and are thereby ineffective. However, differential algebra is well suited for explicit computation and various general theorems about computing bounds on the number of solutions to algebraic differential equations are available.

We will start with an inductive approach, due to Hrushovski and Pillay [22] which gives doubly exponential bounds. By using a more refined Bézout theorem, Binyamini improved the bounds substantially to a singly exponential form [6].

The theory of ordinary differentially closed fields of characteristic zero, DCF_0, is the model completion of the theory of differential fields of characteristic zero. This means that every differential field (K, δ) of characteristic zero may be embedded as a differential field in a differentially closed field and the theory DCF_0 eliminates quantifiers. Robinson identified this theory as such (or really, as the model companion of the theory of differential fields of characteristic zero; quantifier elimination was not proven) already in the 1950s [35] and then Blum found clean axioms for the theory, proved quantifier elimination, and established ω-stability in her PhD thesis [9].

The theory DCF_0 is presented in the language of rings augmented by a unary function symbol δ which is intended to name a distinguished derivation. The axioms start with those of algebraically closed fields of characteristic zero and two universal axioms expressing that δ is additive and satisfies the product rule. The differential closure axioms say that for each irreducible polynomial $f(x_0, \dots, x_d)$ in $d+1$ variables (with $d \geq 1$) and every polynomial $g(x_0, \dots, x_{d-1})$ in d variables, there is some solution a to $f(a, \delta a, \dots, \delta^d a) = 0$ and $g(a, \delta a, \dots, \delta^{d-1} a) \neq 0$.

Exercise 4.4. Flesh out how to express the axioms for DCF_0 with a first-order theory. For example, explain how to deal with the quantification over the set of irreducible polynomials in $d+1$ variables.

Rather than working with differential equations and inequations in a single variable, we may express the axioms for DCF_0 by using algebraic equations involving several variables and only one application of the derivation. By analyzing these alternative axioms we will obtain some uniform degree bounds.

To make express these geometric axioms we need to discuss prolongation spaces. Let K be a differential field of characteristic zero with a distinguished derivation δ. For any algebraic variety X defined over K, we define a projective system

$$X = \tau_0 X \longleftarrow \tau_1 X \longleftarrow \tau_2 X \longleftarrow \cdots \longleftarrow \tau_n X \longleftarrow \cdots$$

of algebraic varieties. This system has a consistent system of differentially defined sections $\nabla_n : X \to \tau_n X$. When X is an embedded affine variety and $a \in X(L)$ is a an L-valued point for some differential field extension L of K, then $\nabla(a) = (a, \delta a, \frac{1}{2}\delta^2 a, \ldots, \frac{1}{n!}\delta^n a)$. Morally, $\tau_n X$ should be the Zariski closure of $\{(a, \delta a, \frac{1}{2}\delta^2 a, \ldots, \frac{1}{n!}\delta^n a) : a \in X(L)\}$ as L ranges through the differential field extensions of K. Still working with embedded affine varieties, the equations for $\tau_n X$ are obtained by differentiating the equations for X. Formally the way this is done is by embedding the polynomial ring $K[x_1, \ldots, x_n]$ into the differential polynomial ring $K\{x_1, \ldots, x_n\}_\delta$, which is the free differential ring on the generators $x_1, \ldots, x_n$ extending K. As a ring, $K\{x_1, \ldots, x_n\}_\delta$ is the ordinary polynomial ring on the infinitely many variables $\{\delta^j x_i : j \in \mathbb{N}, 1 \leq i \leq n\}$. If $f \in I(X)$ is a polynomial vanishing on X and $a \in X(L)$ is a point on X valued in the differential field extension L, then from $f(a) = 0$ we compute $0 = \delta(f(a)) = (\delta f)(a, \delta a)$.

Generically, for embedded affine varieties, in particular when they are smooth, these indications of how to construct the prolongations are correct, though there are some technical issues to contend with coming from singularities and patching in defining $\tau_n X$ for more general varieties. An honest construction of the prolongations makes use of the Weil restriction of scalars operation and requires us to consider possible nonreduced schemes. We will leave these issues aside.

In constructing the prolongation space $\tau_n X$, we might have simply iterated the first prolongation construction n-times to get $\tau_1 \cdots \tau_1 X$ rather than building a new variety $\tau_n X$. The points obtained by applying ∇ repeatedly to differential points on X would not be Zariski dense in such iterated prolongation spaces as some coordinates would always be equal to others. We can fix embeddings $\tau_{n+m} X \hookrightarrow \tau_n \tau_m X$ expressing these identifications; and in the differential Bézout theorems they will be important.

If X is an irreducible subvariety of an algebraic variety over a differential field K and

$$X = X_0 \longleftarrow X_1 \longleftarrow X_2 \longleftarrow \cdots \longleftarrow X_n \longleftarrow \cdots$$

is a projective system of dominant maps of algebraic varieties coming from $X_n \subseteq \tau_n X$ for which $X_{n+1} \subseteq \tau_1 X_n$ via the embedding $\tau_{n+1} X \hookrightarrow \tau_1 \tau_n X$, then there is a differential field L extending K and a point $a \in X(K)$ so that for all n, $\nabla_n(a) \in X_n(L)$ is generic in X_n over K. Thus, consistent systems of algebraic differential equations may be recognized by such systems of algebraic equations. This observation together with some computations of higher derivatives allows for the so-called geometric axioms for DCF_0 and gives our effective bounds.

The geometric axioms for DCF_0 also start with ACF_0, the theory of algebraically closed fields of characteristic zero, together with the basic universal axioms expressing that δ is a derivation, and then replace Blum's axioms about one variable ODEs with the system of axioms saying that for every irreducible embedded affine algebraic variety X and irreducible subvariety $Y \subseteq \tau_1 X$ of its first prolongation space for which the natural projection $Y \to X$ is dominant, there is a point a on X with $(a, \delta a) \in Y$.

Exercise 4.5. Explain how to convert these geometric axioms to first-order sentences. Show that DCF_0 is axiomatized by these geometric axioms by showing that every such differential field is existentially closed within the class of differential fields. Explain why this is enough to prove that we have the correct axiomatization.

Our observation about consistent systems of ODEs corresponding to certain projective systems of algebraic varieties can be used to produce an algorithm for computing the Zariski closure of the set of solutions to a system of differential equations in a differentially closed field.

Let us fix a differentially closed field $\mathbb{K}$. for the remainder of this section we will identify algebraic varieties X over $\mathbb{K}$ with their sets $X(\mathbb{K})$ of $\mathbb{K}$-rational points.

Let X be an irreducible algebraic variety. A system of order n differential equations on X is given by an algebraic subvariety $Y \subseteq \tau_n X$ by asking that $\nabla_n(a) \in Y$. What is the Zariski closure of $\{a \in X : \nabla_n(a) \in Y\}$?

Let us call a projective system of varieties $(X_n)_{n=0}^{\infty}$ which satisfies the conditions that

- $X_0 = X$
- each X_n is irreducible
- $X_n \subseteq \tau_n X$
- $X_n \to X_m$ is dominant and is induced by the map $\tau_n X \to \tau_m X$ for $n \geq m$
- $X_{n+1} \subseteq \tau_1 X_n$ via the embedding $\tau_{n+1} X \hookrightarrow \tau_1 \tau_n X$

a coherent δ-sequence.

From our observations above, if $Z = X_n$ for some irreducible component Z of Y in some coherent δ-sequence $(X_n)_{n=0}^{\infty}$, then $\nabla_n^{-1} Y$ is Zariski dense in X. Let us try to construct such a sequence. First, break Y into its irreducible components, $Y = \bigcup_{j=1}^{\ell} Y_j$. Since $\nabla_n^{-1} Y = \bigcup_{j=1}^{\ell} \nabla_n^{-1} Y_j$, we may compute the Zariski closure of $\nabla_n^{-1} Y$ as the finite union of the Zariski closure of the sets $\nabla_n^{-1} Y_j$. So, for the remainder of this computation we may as well assume that Y itself is irreducible.

Starting with this (irreducible) Y, we build a sequence $(X_m)_{m=0}^{\infty}$ of subvarieties of $\tau_n X$ by taking X_m to be the Zariski closure of the image of Y in $\tau_m X$ for $m \leq n$ and X_m to be the (generic part of the) preimage in $\tau_m X$ of $\tau_{m-n} Y$ under the embedding $\tau_m X \to \tau_{m-n} \tau_n X$ for $m \geq n$. If this sequence is a coherent δ-system, then we are done and have computed the Zariski closure of $\nabla^{-1} Y$ as X. If it is not, then we have to see what may have gone wrong. Irreducibility is also not a problem (the case of $m > n$ is somewhat annoying in that we have to be careful about which part of the prolongation we take). It may have happened that $X_0 \subsetneq X$ in which case the Zariski closure of $\nabla^{-1} Y$ is contained in X_0 and we should replace this situation with the equation $Y \cap \tau_n X_0$, and then conclude by

Noetherian induction that we know how to compute the Zariski closure of $\nabla^{-1}(Y \cap \tau_n X_0)$ in X_0. It may have happened that the image of X_{m+1} is not contained in $\tau_1 X_m$ for some m. The way we have constructed X_ℓ for $\ell > n$, this can only cause a problem for $\ell < n$. If it does, then take m minimal, and let X'_{m+1} be the intersection of X_{m+1} with the pullback of X_m. We now have to refine the given sequence $X_0, X_1, \ldots, X_m, X'_{m+1}, X_{m+2}, \ldots$ into a potential coherent δ-sequence. As before, we need to break X'_{m+1} into its irreducible components, work with each of these separately, and replace the X_j for $j \leq m$ with the Zariski closure of the image of each of these components.

We will continue this process, noting that at each stage either we have produced a coherent δ-sequence, from which we can read off the Zariski closure of the set of solutions to our differential equations, or we obtain a sequence of varieties which are coordinatewise contained in the varieties from the previous sequence, at least one properly.

If we start with X a projective or embedded affine variety, then we have a good notion of the degree of a subvariety of X or even of $\tau_n X$. The steps involved in our algorithm involve intersection and projection. We can compute bounds on the the degrees of the varieties we obtain by using Bézout's theorem.

By following the steps in our algorithm, on obtains the following proposition.

Proposition 4.6 (Proposition 3.1 of [22]). *If $X \subseteq \mathbb{A}^n_{\mathbb{K}}$ is a closed subvariety of affine space and $Y \subseteq \tau_\ell X$ is closed subvariety of the ℓ^{th} prolongation space, then the degree of the Zariski closure of $\nabla_\ell^{-1} Y$ is at most* $\deg(X)^{\ell 2^{n\ell}} \deg(Y)^{2^{n\ell}-1}$. *In particular, if the set of solutions to the differential equation is finite, then this number is a bound on its size.*

Binyamini finds much better bounds by using a refined Bézout Theorem of Kušnirenko which invokes computations of volumes of Newton polytopes. Since this discussion may take us too far afield, we direct the reader to [6] for details.

Proposition 4.6 and Binyamini's refinement are used in [31] to prove effective versions of the Zilber-Pink conjecture over function fields. The key idea there is that even though the union of special varieties of a given dimension is a countably infinite union of algebraic varieties, and, hence, not an algebraic variety itself, it does satisfy a nontrivial differential equation. When working over function fields, unlikely or more generally anomalous intersections with this set may be replaced with solutions to some differential equations and the effective finiteness theorems may be used to bound the dimension and degree of the Zariski closure of the anomalous sets. Similar reasoning was used in [19] to bound intersections of algebraic curves with transcendental isogeny orbits on the moduli space of elliptic curves. In each of these cases, while the bounds are finite and explicit, they are still much larger than one might expect the true value to be. In complete generality it may be difficult to improve the bounds much, but for any particular case, it is plausible to expect that one could show that the differential equations themselves have fewer solutions than the worst case bounds allow. One of our projects will be to implement this idea.

Let us illustrate how differential algebra might be used to study a Zilber-Pink problem by considering intersections of one dimensional subgroups of $\mathbb{G}_m^3$ with an algebraic curve $X \subseteq \mathbb{G}_m^3$. We assume about X that it is not contained in any proper algebraic subgroup. If X is defined over some field K, then because each algebraic subgroup T of $\mathbb{G}_m^3$ is defined over $\mathbb{Q}$, if $\dim T = 1$, then $\dim(T \cap X) = 0$ (or the intersection is empty, as expected), and every point in $T \cap X$ is defined over K^{alg}. Thus, if $K \subseteq \mathbb{Q}^{\mathrm{alg}}$, then differential algebraic techniques cannot tell us much about the intersection of X with one dimensional groups because all such intersections will consist of constant points on X. From now on we assume that X is not defined over the algebraic numbers. Let us equip $\mathbb{C}$ with a derivative δ making $\mathbb{C}$ into a differentially closed field with field of constants $\mathbb{Q}^{\mathrm{alg}}$. For the remainder of this argument we will identify algebraic varieties with their sets of $\mathbb{C}$-rational points.

Exercise 4.7. Show that there are derivations δ on $\mathbb{C}$ with the properties described above. The difficult step is to show that given a differential field K with algebraically closed field of constants C there is a differentially closed extension field L of K also having field of constants C.

Suppose that $(x_1, x_2, x_3) \in T \leq \mathbb{G}_m$ for some one-dimensional group T. Then for two linearly independent triples of integers $(\ell_{1,1}, \ell_{1,2}, \ell_{1,3})$ and $(\ell_{2,1}, \ell_{2,2}, \ell_{2,3})$ we have $\prod_{j=1}^{3} x_j^{\ell_{i,j}} = 1$ for $i = 1, 2$. Applying the logarithmic derivative, we have $\sum_{j=1}^{3} \ell_{i,j} \frac{\delta x_j}{x_j} = 0$. Let us set $y_j = \frac{\delta x_j}{x_j}$. These two linear equations show that the $\mathbb{Q}^{\mathrm{alg}}$-vector space generated by y_1, y_2, and y_3 has dimension one. From the theory of Wronskians, this is the same as saying that the matrix

$$\begin{pmatrix} y_1 & \delta y_1 & \delta^2 y_1 \\ y_2 & \delta y_2 & \delta^2 y_2 \\ y_3 & \delta y_3 & \delta^2 y_3 \end{pmatrix}$$

has rank one. Taking the determinants of the minors of this matrix, we obtain a system of order two equations satisfied by (x_1, x_2, x_3). Using Ax-Schanuel one can show that there only fintely many solutions to these differential equations lying on the curve X. We may then bound the number of such solutions.

Exercise 4.8. Find the subvariety $Y \subseteq \tau_2 \mathbb{G}_m^3$ described by these determinantal conditions and compute its degree.

5. Projects

The three projects associated to this course are grouped by the three main sections of these notes.

5.1. **Exponential-algebraic closedness.**

Establishing that $\mathbb{C}_{\exp}$ is isomorphic to $\mathbb{B}$ appears to be much too difficult given our existing techniques, especially as this would involve proving Schanuel's Conjecture. It would be far from easy to prove the complementary Exponential-Algebraic Closedness axioms, but various recent results, especially those in the doctoral thesis of Gallinaro [20], show that many instances of the exponential-algebraic closedness axioms are within reach.

For this project we will look at two analogous questions and then some low dimensional instances of the the exponential-algebraic closedness conjecture.

5.1.1. *Differential existential closedness.* As we have seen, the special functions we have beeb considering satisfy nontrivial algebraic differential equations. For example, if $g =$

$\exp(f)$, then

$$\frac{g'}{g} = f' \, .$$

The j function satisfies a nonlinear order three. The Schwarzian derivative is given by the formula

$$S(f) := \left(\frac{f''}{f'}\right)' - \frac{1}{2}\left(\frac{f''}{f'}\right)^2 \, .$$

The Schwarzian is characterized by the property that $S(f) = S(g)$ if and only if g is a fractional linear transformation of f. That is, there is some invertible matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with constant entries for which $g = \frac{af+b}{cf+d}$. The Schwarzian satisfies a twisted chain rule:

$$S(f \circ g) = (g')^2 S(f) \circ g + S(g) \, .$$

The j-function satisfies the differential equation

$$S(j) = (j')^2 \frac{j^2 - 1968j + 2\,654\,208}{-2j^2(j-1728)^2} \, .$$

From this equation, and the Schwarzian chain rule, one may deduce a differential relation between a function and the composition of that function with the j-function. We leave it as an exercise for you to work out that formula. Let us write $E_j(x,y)$ for the resulting minimal differential polynomial for which $E_J(f, j(f)) \equiv 0$.

In the paper [1], a variant of the exponential-algebraic closedness axiom for the j-function is considered in which the exact relation $y = j(x)$ is replaced by the differential algebraic relation $E_j(x,y) = 0$. Let us make this more precise. First they define the notion of j-broadness. If $X \subseteq \mathbb{A}^n \times \mathbb{A}^n$ is an irreducible algbraic variety then it is "j-broad" if whenever we take some subset $S \subseteq \{1, \ldots, n\}$ and define $\pi_S : \mathbb{A}^n \times \mathbb{A}^n \to \mathbb{A}^S \times \mathbb{A}^S$ by

$$((x_1, \ldots, x_n), (y_1, \ldots, y_n)) \mapsto ((x_i)_{i \in S}, (y_i)_{i \in S}) \, ,$$

then $\dim \pi_S(X) \geq \#S$. They then show that if we work in a differentially closed field K, then for every j-broad variety $X \subseteq \mathbb{A}^n \times \mathbb{A}^n$, there is a point $((a_1, \ldots, a_n), (b_1, \ldots, b_n)) \in X(K)$ for which $E_j(a_i, b_i) = 0$ for all $i \leq n$. Similar results are known for the equation satisfied by exponentials.

For this subproject we will extend the theorem of [1] to other analytic covering maps. To start, we will work with both j and exp. That is, our basic function $E : \mathbb{C} \times \mathfrak{h} \to \mathbb{C}^\times \times \mathbb{C}$ is given by $(x, \tau) \mapsto (\exp(x), j(\tau))$. You will need to formulate a condition for an algebraic variety $X \subseteq (\mathbb{A}^2)^n \times (\mathbb{G}_m \times \mathbb{A}^1)^n$ to be $(\exp, j)$-broad and then you should should that in a differentially closed field there are always points in such broad varieties for which the coordinates satisfy the corresponding differential equations for exp and j. Once this result is established, you should move onto other covering maps, such as the covering $f : \mathbb{C} \times \mathfrak{h} \to \mathcal{E}$ of the universal elliptic curve given in Legendre form by $y^2 = x(x-1)(x-\lambda)$.

General methods are available to find the relevant differential equations. Proving solvability in a differentially closed field often comes down to an algebraic computation followed by the use of existential closedness of differentially closed fields.

5.1.2. *Blurred exponential-algebraic closedness.* Rather than replacing a covering map by the differential algebraic relation satisfied by the function, we might work another relation called "blurring". For example, from the exponential function $\exp : \mathbb{C} \to \mathbb{C}^\times$ we might consider instead the blurred exponentiation

$$\Gamma_{\exp} := \{(x, y) \in \mathbb{C} \times \mathbb{C}^\times : (\exists q \in \mathbb{Q}) \exp(x + 2\pi i q) = y\} .$$

In [24], Kirby shows that the complex numbers do satisfy the exponential-algebraic closedness axioms provided that one looks for solutions in the blurred $\Gamma_{\exp}$ rather than exactly in the graph of the exponential function.

Analogues of the blurring construction make sense for other covering maps. For example, the blurred graph of the j-function would be given by

$$\Gamma_j := \{(\tau, y) \in \mathfrak{h} \times \mathbb{C} : (\exists \gamma \in \mathrm{GL}_2(\mathbb{Q})) j(\gamma\tau) = y\} .$$

The aim of this subproject is to extend the blurred exponential-algebraic closedness results to other covering maps.

5.1.3. *Exponential-algebraic closedness in low dimensions.* Ultimately, we would like to prove exponential-algebraic closedness in general, but we expect that this would be too much to attempt in a single week. If we restrict the dimension of the variety in which we seek solutions, then in some cases the exponential-algebraic closedness is known. For this subproject, we will consider the first low dimensional case for which the problem remains open. Specifically, we will consider $X \subseteq \mathbb{A}^2 \times \mathbb{G}_m \times \mathbb{A}^1$ an irreducible surface which is $(\exp, j)$-broad (we leave it as an exercise to express precisely what this means) and will attempt to show that there is a point $(a, b, \exp(a), j(b)) \in X(\mathbb{C})$. If we succeed with this problem, then we can move on to higher dimensions or other covering maps.

5.2. **Induced structure on $\mathbb{C}$ in $\mathbb{C}(t)$.** We have observed that the set CM of j-invariants of elliptic curves with complex multiplication is definable in $\mathbb{C}(t)$. We have also noted that as a consequence of an effective form of the the André-Oort Conjecture, the theory of the structure $(\mathbb{C}, +, \cdot, 0, 1, \mathsf{CM})$ is stable and decidable.

Our method of defining CM may be adapted to define the isogeny relation:

$$\mathcal{I} := \{(j(E_1), j(E_2)) : \text{There is an isogeny } \psi : E_1 \twoheadrightarrow E_2\} .$$

Indeed,

$$(j(E_1), j(E_2)) \in \mathcal{I} \Longleftrightarrow E_2(\mathbb{C}(E_1)) \neq E_2(\mathbb{C}) .$$

For this project we will work out the theory of the structure $(\mathbb{C}, +, \cdot, 0, 1, \mathcal{I})$ obtained by naming $\mathcal{I}$ by a basic predicate and then the potentially more complicated structure $(\mathbb{C}, +, \cdot, 0, 1, \mathcal{I}, \mathsf{CM})$ obtained by naming both $\mathcal{I}$ and the set of moduli points of CM elliptic curves by basic predicates.

In order to work out this theory, we may need to work conditionally on the truth of the Zilber-Pink conjecture, and even an effective version of the Zilber-Pink conjecture, for products of the j-line. For any two algebraic varieties $X \subseteq (\mathbb{A}^2)^n$ and $Y \subseteq (\mathbb{A}^2)^n$ defined over $\mathbb{Q}$, this theory will have to decide whether $X \cap \mathcal{I}^n \subseteq Y$ or not. More generally, if $\tau \in \{\pm\}^n$ and we define $\mathcal{I}^+ := \mathcal{I}$, $\mathcal{I}^- := \mathbb{A}^2 \smallsetminus \mathcal{I}$, and $\mathcal{I}^\tau := \prod_{i=1}^n \mathcal{I}^{\tau_i}$, then our theory has to decide whether $X \cap \mathcal{I}^\tau \subseteq Y$ or not. The first of these (i.e. where all components of τ are $+$) should be answerable using an effective version of Zilber-Pink. The general question may follow from some simple manipulations, though it may also implicate new Diophantine geometric issues.

As a separate subproject, we will investigate what complicated structures are definable in $\mathbb{C}(t)$ on moduli spaces of higher dimensional abelian varieties. For example, is the set of moduli points of g-dimensional principally polarized abelian varieties (with sufficient level structure fixed to guarantee the existence of a moduli space) definable in $\mathbb{C}(t)$? A difficulty with using our technique for defining this set in the case of the moduli space of elliptic curves is that we can uniformly interpret function fields of the form $\mathbb{C}(X)$ for X a smooth, projective curve over $\mathbb{C}$, but there is not a natural way to access function fields of higher dimensional algebraic varieites. For a pair of abelian varieties A_1 and A_2, we would like to recognize the morphisms from A_1 to A_2 as $A_2(\mathbb{C}(A_1))$. If A_1 happens to be the Jacobian of a curve X_1, then we have $A_2(\mathbb{C}(A_1)) = A_2(\mathbb{C}(X_1))$, which we can access. Part of this project will involve determining how far we can go with this trick using Jacobians.

5.3. **Better bounds for differential Zilber-Pink.** The effective bounds we have computed in Zilber-Pink problems depend upon general Bézout-style results on the number of solutions to differential equations, but because we are dealing with very special equations of geometric origin, we would expect that the actual number of solutions is much smaller. With this project we will consider some low dimensional cases of the weak Zilber-Pink conjecture, analyzing the differential equations involved to find better bounds.

To start, let us take $X \subseteq \mathbb{G}_m^5$ to be an irreducible surface not contained in any proper algebraic subgroup. To make this even more explicit, let us take X to have low degree, even to be affine to start.

As an exercise, you should write the differential equations describing the Kolchin closure Ξ of the union of the special subvarieties of dimension two. We must now compute $\Xi \cap X$, and more importantly, the union of the positive dimensional components of this differential variety.

As an exercise, compute a bound on the degree using the differential Bézout theorem. I expect that a closer analysis of the differential equations will yield a much lower bound.

After completing this computation, we should move to more complicated situations, by raising the degree of X and/or moving to higher dimensional ambient spaces.

We should also consider the analogous problem with the exponential function replaced by the j-function.

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