INTRODUCTION TO MODEL THEORY WITH APPLICATIONS

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6. Model Theory of fields II

In the last lecture we continue our excursion in the model theory of fields. This time we focus on the (ordered) field of real numbers. We discuss the axiomatization of the theory $Th(\mathbb{R},+,-,\times,0,1)$ as well as the theory $Th(\mathbb{R},+,-,\times,0,1,<)$. We prove a quantifier elimination result in this setting and give some basic applications. Finally we talk briefly about o-minimality, a powerful tool in model theory and which now has many applications in geometry and number theory.

6.1. **Real closed fields.** Recall that $\mathcal{L}_r = \{+,-,\times,0,1\}$ denotes the language of rings while $\mathcal{L}_{or} = \mathcal{L}_r \cup \{<\}$ the language of ordered rings.

Proposition 6.1. *$Th(\mathbb{R},+,-,\times,0,1)$ does not have quantifier elimination in $\mathcal{L}_r$.*

Proof. We work in $\mathcal{R} = (\mathbb{R},+,-,\times,0,1)$. We claim that every quantifier free definable subset of $X \subseteq \mathbb{R}$ is finite or cofinite. Indeed let $\phi(x)$ be a quantifier free $\mathcal{L}_r$-formula in one variable x. Then $\phi(x)$ is a Boolean combination of atomic $\mathcal{L}_r$-formula. But recall that an atomic formula is of the form $p(x) = 0$ for $p \in \mathbb{Z}[x]$ and the set defined by $p(x) = 0$ is finite. Hence the set X defined by $\phi(x)$ is a Boolean combination of finite sets and so is finite or cofinite.

But consider the formula $\phi(x) := \exists y(y^2 = x)$. Then $\phi(\mathcal{R}) = \mathbb{R}_{\geq 0}$ is the set of non-negative real numbers which is not finite or cofinite. Hence $\phi(x)$ is not equivalent to a quantifier free formula. □

As we have seen in Example 2.5, the ordering $<$ on $\mathbb{R}$ is definable in $(\mathbb{R},+,-,\times,0,1)$. So even though we will prove quantifier elimination in $\mathcal{L}_{or}$, it follows that every $\mathcal{L}_{or}$-definable sets in $\mathbb{R}^n$ will be $\mathcal{L}_r$-definable. Let us discuss the axiomatization of $\mathcal{R} = (\mathbb{R},+,-,\times,0,1)$. The following are some of the basic properties of $\mathbb{R}$ we wish to isolate.

Definition 6.2. Let F be a field.

(1) F is said to be **orderable** if there is a linear order $<$ on F making $(F,<)$ an ordered field.
(2) F is said to be **formally real** if -1 is not a sum of squares of elements in F.
(3) F is said to be **real closed** if it is formally real with no proper formally real algebraic extensions.

Recall that the axioms (in a language containing $<$) for linear orders are

- $\forall x\, \neg(x < x)$
- $\forall x \forall y \forall z\, ((x < y \wedge y < z) \rightarrow x < z)$

- $\forall x \forall y\, (x < y \vee x = y \vee y < x)$.

On the hand (in $\mathcal{L}_{or}$) the axioms for ordered field consists of the axioms for field, linear orders and the axioms

- $\forall x \forall y \forall z\, (x < y \rightarrow x + z < y + z)$
- $\forall x \forall y \forall z\, ((x < y \wedge z > 0) \rightarrow x \times z < y \times z)$.

Notice that F is formally real if and only if for $a_1, \ldots, a_n \in F$, if $a_1^2 + \ldots + a_n^2 = 0$ then $a_1 = \cdots = a_n = 0$. If F is an ordered field, we say that $a \in F$ is negative if $a < 0$. It is not hard to see, using the axioms, that the squares in F are nonnegative. It hence follows that any orderable field is formally real. The following result of Artin and Schreier tells us that the converse is also true.

Fact 6.3. *If F is formally real, then F is orderable. Furthermore, if $a \in F$ is not the sum of squares, then there is an ordering of F where a is negative.*

The next result, also due to Artin and Schreier, will provide an axiomatization of $Th(\mathbb{R}, +, -, \times, 0, 1)$ in $\mathcal{L}_{or}$.

Fact 6.4. *Let F be a formally real field. The following are equivalent.*

1. *F is real closed.*
2. *$F(i)$ is algebraically closed (where $i^2 = -1$).*
3. *For any $p \in F[x]$ and $a, b \in F$ such that $a < b$ and such that $p(a)$ and $p(b)$ have opposite signs[1], there exists $c \in F$ such that $a < c < b$ and $p(c) = 0$.*
4. *For any $a \in F$, either a or $-a$ is a square and every polynomial of odd degree has a root.*

Definition 6.5. The theory of real closed fields, RCF, is the $\mathcal{L}_r$-theory axiomatized by

- Axioms for fields
- $\forall x \exists y\, (y^2 = x \wedge y^2 + x = 0)$
- For each $n \in \mathbb{N}_{>0}$, the sentence
$$\forall x_1, \ldots \forall x_n\, (x_1^2 + \cdots x_n^2 + 1 \neq 0)$$
- For each $n \in \mathbb{N}$, the sentence
$$\forall a_0, \ldots \forall a_{2n+1} \exists y\, (a_0 + a_1 x + \ldots + a_{2n+1} x^{2n+1} = 0)$$

The theory of real closed ordered fields, $RCOF$, is the $\mathcal{L}_{or}$-theory RCF together with the axioms for ordered fields.

The following is the analogue of the algebraic closure in the RCF setting.

Definition 6.6. Let K be a formally real field. A field R is a **real closure** of K if $K \subseteq R$ is an algebraic extension and R is real closed.

Proposition 6.7. *If K is a formally real field, then K has a real closure*

[1] We say that x and y have opposite signs if $x < 0 < y$ or $y < 0 < x$.

Proof. Let K be formally real, and recall that K^{alg} denotes its algebraic closure. Consider

$$C = \{R \subseteq K^{alg} : K \subseteq R \text{ and } R \text{ is formally real.}\}$$

It is not hard to see using Zorn's Lemma that C has a maximal element R. So R is an algebraic extension of K which is formally real. We claim that R is real closed. Indeed, suppose $R \subseteq F$ is an algebraic extension that F is formally real. But we can embed F in K^{alg} over R so that $K \subseteq R \subseteq F \subseteq K^{alg}$. Hence $F \in C$ and since R is maximal we get that $R = F$. □

The real closure of a formally real field need not be unique.

Example 6.8. Since π is transcendental, it follows that both $\mathbb{Q}(\sqrt{\pi})$ and $\mathbb{Q}(\sqrt{-\pi})$ are formally real. Let R_1 and R_2 be their respective real closures. By construction R_1 and R_2 are real closures of $\mathbb{Q}(\pi)$. However they are not isomorphic over $\mathbb{Q}(\pi)$ since π is a square in R_1 but not in R_2.

On the other hand, if $(K, <)$ is an ordered field, then there is a unique real closure R, where the ordering on R extend that on K. We are ready to prove QE.

Theorem 6.9. *RCOF has quantifier elimination.*

Proof. We again use criterion $(*)$ of Proposition 4.12 from Lecture 4. Let $\mathcal{M}, \mathcal{N} \models RCOF$ and assume that $\mathcal{A}$ is a common substructure. As argued in the case of ACF we have that $\mathcal{A}$ is an ordered integral domain and we can extend the order to its field of fractions. Let $\mathcal{F} \models RCOF$ be the unique real closure of the field of fractions of $\mathcal{A}$ which extends the order. We may assume (using uniqueness) that $\mathcal{F}$ is a subfield of both $\mathcal{M}$ and $\mathcal{N}$.

The atomic $\mathcal{L}_A$-formulas in one variable x are given by $p(x) = 0$ or $p(x) > 0$ for $p \in A[x]$. Moreover, notice that we can use the ordering to characterize the negated atomic formula:

$$p(x) \neq 0 \leftrightarrow (p(x) > 0 \vee -p(x) > 0)$$
$$p(x) \not> 0 \leftrightarrow (p(x) = 0 \vee -p(x) > 0)$$

We hence have to consider $\phi(x)$ an $\mathcal{L}_A$-formula which are disjunction of conjunctions of atomic formula

$$\bigvee_{k=1}^{\ell} \left(\bigwedge_{i=1}^{r} (p_{k,i}(x) = 0) \wedge \bigwedge_{j=1}^{s} (q_{k,j}(x) > 0) \right)$$

where $p_{k,i}, q_{k,j} \in A[x]$.

However, notice that if $m \in M$ is such that $\mathcal{M} \models \phi(m)$, then

$$\mathcal{M} \models \bigwedge_{i=1}^{r} (p_{k,i}(m) = 0) \wedge \bigwedge_{j=1}^{s} (q_{k,j}(m) > 0)$$

for some k. So we may assume that $\phi(x)$ is of the form

$$\bigwedge_{i=1}^{r} (p_i(x) = 0) \wedge \bigwedge_{j=1}^{s} (q_j(x) > 0).$$

Assume $m \in M$ is such that $\mathcal{M} \models \phi(m)$. We need to show that there is $n \in N$ is such that $\mathcal{N} \models \phi(n)$. If at least one of the $p_i(x) = 0$ appears in $\phi(x)$, then m is algebraic over $\mathcal{A}$. Hence $m \in \mathcal{F} \subseteq \mathcal{N}$ and we are done. Assume that no $p_i(x) = 0$ appears in $\phi(x)$, that is $\phi(x)$ is of the form

$$\bigwedge_{j=1}^{s} (q_j(x) > 0).$$

In $\mathcal{F}$, using Fact 6.4 (3), the q_j's can only change sign at their roots. So since $q_j(m) > 0$, we can find $a_j, b_j \in \mathcal{F} \cup \{-\infty, \infty\}$ roots of q_j in F so that $a_j < m < b_j$ and for all $x \in \mathcal{F}$ such that $a_j < x < b_j$ we have that $q_i(x) > 0$. Let $a = max\{a_1, \ldots, a_r\}$ and $b = min\{b_1, \ldots, b_r\}$. Then $a < b$ since $a < m < b$ and for all $x \in \mathcal{F}$ such that $a < x < b$ we have that $\wedge_{j=1}^{s}(q_j(x) > 0)$. So we can find $n \in \mathcal{F} \subseteq \mathcal{N}$ so that $\mathcal{N} \models \phi(n)$. □

As a corollary we obtain that *ROCF* is the theory $Th(\mathbb{R}, +, -, \times, 0, 1, <)$.

Corollary 6.10. *RCOF is complete and model complete.*

Proof. Since *RCOF* has QE, it is model complete. Every model of *RCOF* has characteristic 0 (if not and the characteristic is p, then $-1 = \underbrace{1^2 + \ldots + 1^2}_{(p-1)-\text{times}})$. Hence $\mathbb{Q}$, the prime field, embeds in any model of *RCOF*. Since $\mathbb{Q}$ admits a unique ordering, this is an embedding of ordered fields. Using Proposition 5.11 from Lecture 5, we get that *RCOF* is complete. □

Similarly *RCF* is the theory $Th(\mathbb{R}, +, -, \times, 0, 1)$.

Corollary 6.11. *RCF is complete.*

Proof. Let ϕ be a sentence in $\mathcal{L}_r$. We need to show that either $RCF \models \phi$ or $RCF \models \neg\phi$. We show that if $RCOF \models \phi$ then $RCF \models \phi$. We will be done using completeness of *RCOF*.

Indeed assume $RCOF \models \phi$ and let $\mathcal{M} \models RCF$. Then we can expand $\mathcal{M}$ to a model $\mathcal{M}^*$ of *RCOF* (using Fact 6.3). By assumption $\mathcal{M}^* \models \phi$. However recall that ϕ be a sentence in $\mathcal{L}_r$. Hence its truth does not depend on the interpretation of $<$. It follows that $\mathcal{M} \models \phi$. Since $\mathcal{M}$ was arbitrary $RCF \models \phi$. □

Let $\mathcal{K} \models RCOF$. Recall that a set $X \subseteq K^n$, is *semialgebraic* if it is a finite Boolean combination of sets define by equations $p_i(\overline{x}) = 0$ and inequalities $q_j(\overline{x}) > 0$ where $p_i, q_j \in K[\overline{x}]$. As with *ACF*, from quantifier elimination for *RCOF* we get that definable sets are precisely semialgebraic sets. We leave the proof of the following to the reader.

Corollary 6.12 (Tarski-Seidenberg Theorem)**.** *The projection of a semialgebraic set is semialgebraic.*

6.2. ***o*-minimality.** We now introduce an important tool for understanding the definable sets in *RCOF* (and other important context).

Definition 6.13. Let $\mathcal{L}$ be a language containing $<$. Let T be an $\mathcal{L}$-theory extending the theory of linear orders. We say that T is ***o*-minimal** if for any $\mathcal{M} \models T$, if $X \subseteq M$ is definable, then X is a finite union of points and intervals with endpoints in $M \cup \{-\infty, \infty\}$.

By an (open) interval we mean a set which takes one of the form $(a, b) = \{x \in M : a < x < b\}$, $(-\infty, b) = \{x \in M : x < b\}$, $(a, \infty) = \{x \in M : a < x\}$ or $(-\infty, \infty) = M$ for some $a, b \in M$. These are the basic open sets of the ordered topology on $\mathcal{M}$. So by an open subset of M, will mean the union of some basic open intervals.

Proposition 6.14. *RCOF is o-minimal.*

Proof. Let $\mathcal{M} \models RCOF$. Since $RCOF$ has QE, every definable subset of $\mathcal{M}$ is a finite Boolean combination of sets of the form $\{x \in M : p(x) = 0\}$ and $\{x \in M : q(x) > 0\}$ for polynomial $p, q \in M[X]$. Sets of the first kind are finite, whereas sets of the second form are finite unions of intervals. Thus (recalling as in the proof of Theorem 6.9 how to deal with negation), any definable set is a finite union of points and intervals. □

We use the proposition to show that definable functions in one variable (that is functions in one variable whose graphs are definable) are piecewise continuous. Throughout we let $\mathcal{M} \models RCOF$. We say a definable (or semialgebraic) function $f : X \to Y$ is continuous if the preimage of any open set in Y is open in X where $X, Y \subseteq M$. We say that f is continuous at $x \in X$ if for any open setsubset $V \subseteq Y$ such that $f(x) \in V$ there is an open subset $U \subseteq X$ such that $f(y) \in V$ for any $y \in U$. We need the following lemma.

Lemma 6.15. *If $f : M \to M$ is semialgebraic, then for any open interval $U \subseteq M$ there is a point $x \in U$ such that f is continuous at x.*

Proof. Using completeness of $RCOF$ it suffices to show the result for $\mathbb{R}$. First assume that there is an open set $V \subseteq U$ such that f the image $f(V)$ is finite. Let $b \in f(V)$ be such that $X = \{x \in V : f(x) = b\}$ is infinite. By assumption, X is a definable subset of R. Hence using o-minimality and the fact that X is infinite, there is an open set $V_0 \subseteq V$ such that f is constantly b on V_0. So f is continuous at any $x \in V_0 \subseteq U$.

Next, let us assume that no such V exists. We build a chain $U = V_0 \supset V_1 \supset V_2 \supset \cdots$ of open subsets of U such that the closure $\overline{V}_{n+1}$ of V_{n+1} is contained in V_n. Assume that we have already build V_n. Consider the image $f(V_n)$. By our assumption, $f(V_n)$ is infinite and so using o-minimality, $f(V_n)$ contains an interval (a, b) which we may assume is of length at most $b - a = \frac{1}{n}$. The set $Y = \{x \in V_n : f(x) \in (a, b)\}$ contains an open interval $V_{n+1} \subset V_n$ which works. Using the fact that $\mathbb{R}$ is locally compact (or more precisely the nested interval property), we get

$$\bigcap_i V_i = \bigcap_i \overline{V}_i \neq \emptyset.$$

Our function f is continuous at any $x \in \bigcap_i V_i$. □

Proposition 6.16. *Let $f : M \to M$ be semialgebraic. Then, we can partition $\mathcal{M}$ into $U_1 \cup \ldots \cup U_m \cup X$, where X is finite and the U_i's are pairwise disjoint open intervals with endpoints in $M \cup \{-\infty, \infty\}$ and such that f is continuous on each U_i.*

Proof. Consider the definable set $X = \{x \in M : \mathcal{M} \models \phi(x)\}$ where $\phi(x)$ is the $\mathcal{L}_{or}$-formula stating thaf f is not continuous at x. Using o-minimality X is a finite union of points and intervals with endpoints in $M \cup \{-\infty, \infty\}$. Using Lemma 6.15 it follows that X must be a finite union of points. Hence $M \setminus X$ is a finite union of pairwise disjoint open intervals on which f is continuous and the result follows. □

Finally we end this lecture by mentioning without proof that there is also a nice characterization of definable subsets of M^n which generalizes the definition of o-minimality. We need the following inductive definition definition.

Definition 6.17. The collection of cells is defined inductively as follows.

- $X \subseteq M^n$ is a 0-cell if it is a single point.
- $X \subseteq M$ is a 1-cell if it is an interval (a, b) with $a, b \in M \cup \{-\infty, \infty\}$.
- If $X \subseteq M^n$ is an n-cell and $f : X \to M$ is a continuous definable function, then
$$Graph(f) = \{(x, f(x)) : x \in X\} \subseteq M^{n+1}$$
is an n-cell.
- Let $X \subseteq M^n$ is an n-cell. Suppose that f is either a continuous definable function from X to M or the constant function $-\infty$. Similarly suppose that g is either a continuous definable function from X to M or the constant function ∞. In any case, assume that $f(\overline{x}) < g(\overline{x})$ for all $\overline{x} \in X$. Then the set
$$\{(\overline{x}, y) : \overline{x} \in X \wedge f(\overline{x}) < y < g(\overline{x})\} \subseteq M^{n+1}$$
is an $n+1$-cell.

Theorem 6.18 (Cell Decomposition)**.** *Let $X \subseteq M^n$ be semialgebraic. There are finitely many pairwise disjoint cells $C_1, \ldots, C_k$ such that $X = C_1 \cup \cdots \cup C_k$.*

The following is the analogue of Proposition 6.16 in higher dimension.

Theorem 6.19 (Piecewise Continuity)**.** *For every semialgebraic function $f : X \to M$, with $X \subseteq M^n$, there is a decomposition $X = X_1 \cup \cdots \cup X_k$ of X into a finite union of definable sets such that $f{\restriction}_{X_i}$ is continuous for all i.*