HEIGHTS PROBLEM SET 2

Below you will find some problems to work on for Week 2! There are three categories: beginner, intermediate and advanced.

Beginner problems

Question 1. Suppose that the minimal polynomial $f \in \mathbb{Z}[x]$ of α factors as

$$f(x) = a_0x^n + \ldots + a_n = a_0(x - \alpha_1) \cdots (x - \alpha_n)$$

over $\mathbb{C}$. Then prove that for every i between 0 and n, we have

$$a_i/a_0 = (-1)^i \sum_{1 \leqslant s_1 < s_2 < \cdots < s_i \leqslant n} \alpha_{s_1}\alpha_{s_2} \cdots \alpha_{s_i}.$$

Question 2. In this problem, you will show that $H(\alpha^{-1}) = H(\alpha)$.

(a) If α is a nonzero algebraic number with minimal polynomial $f(x) := a_0x^n + a_1x^{n-1} + \ldots + a_n$, then verify that $1/\alpha$ is also an algebraic number with minimal polynomial

$$f^{\text{rev}}(x) := x^n f(1/x) = a_0 + a_1x + \ldots + a_nx^n$$

if $a_n > 0$, and minimal polynomial $-f^{\text{rev}}(x)$ if $a_n < 0$.

(b) Describe the roots of $f^{\text{rev}}(x)$ in terms of the roots of $f(x)$.

(b) Show that $H(\alpha^{-1}) = H(\alpha)$. *Hint: use Question 1.*

Question 3. This question will introduce you to splitting fields and get you more comfortable computing with number fields. Recall the table from Padma's notes:

Algebraic number	**Minimal polynomial**	**Number field**	**Degree**
$a/b \in \mathbb{Q}$ $\gcd(a,b) = 1,\ b > 0$	$bx - a$	$\mathbb{Q}$	1
i	$x^2 + 1$	$\mathbb{Q}(i) \cong \mathbb{Q}[x]/(x^2+1)$	2
$\sqrt{2} + 1$	$(x-1)^2 - 2$	$\mathbb{Q}(\sqrt{2}) \cong \mathbb{Q}[x]/(x^2-2)$	2
$\sqrt[3]{2}$	$x^3 - 2$	$\mathbb{Q}(\sqrt[3]{2}) \cong \mathbb{Q}[x]/(x^3-2)$	3
ζ_p, a primitive p-th root of unity for a prime p	$\varphi_p(x) := \frac{x^p-1}{x-1}$ p-th cyclotomic polynomial	$\mathbb{Q}(\zeta_p) \cong \mathbb{Q}[x]/(\varphi_p(x))$ p-th cyclotomic field	$p-1$

(a) For each of the rows of the table, do the following.
- Find all of the roots of the minimal polynomial over the number field. How many roots do you find?
- Factor the minimal polynomial over the number field.

(c) Answer the same questions for the polynomial $f(x) := x^3 - 2$ over $S := \mathbb{Q}[x]/(x^6 - 108)$. You should only get linear factors. We call the number field S the splitting field of $f(x)$: the smallest field extension of the base field over which $f(x)$ *splits* (decomposes into linear factors).

Intermediate problems

Question 4. Prove Gauss' lemma: a polynomial $f := a_0x^n + a_1x^{n-1} + \ldots + a_n$ in $\mathbb{Z}[x]$ is irreducible if and only if it is irreducible in $\mathbb{Q}[x]$ and $\gcd(a_0, \ldots, a_n) = 1$.

Question 5. Prove that any irreducible polynomial of degree n in $\mathbb{Q}[x]$ has n *distinct* roots in $\mathbb{C}$.

Question 6. There is also a third definition of a height function H_3, in terms of the *house* ⌂ and *denominator* den of an algebraic number α (See also [Wal00][§ 3.4]):

$$\text{⌂}(\alpha) := \overline{|\alpha|} = \max_{j=1}^{n} |\alpha_j|$$

$$\operatorname{den}(\alpha) := \min\{D \in \mathbb{Z} : \ D > 0,\ D\alpha \text{ has a monic minimal polynomial in } \mathbb{Z}[x]\}$$

$$H_3(\alpha) := \operatorname{den}(\alpha) \max\left(1, \text{⌂}(\alpha)\right).$$

Prove that $\operatorname{den}(\alpha)$ is well-defined and divides the leading coefficient a_0 of the minimal polynomial $a_0x^n + \ldots + a_n$ of α. Prove explicit inequalities relating $H(\alpha), H_2(\alpha)$ and $H_3(\alpha)$.

Question 7. Fix $m \geqslant 1$. Consider the polynomial g defined by

$$g(x) := a_0^m(x - \alpha_1^m) \cdots (x - \alpha_n^m).$$

Show that $g(x) \in \mathbb{Z}[x]$ and that it is a power of the minimal polynomial of α^m.

Question 8. Consider an algebraic number α with minimal polynomial $f(x) = a_0x^n + \ldots + a_n \in \mathbb{Z}[x]$, and conjuagtes $\alpha_1, \ldots, \alpha_n$. Let

$$\operatorname{Disc}(f) = \alpha_0^{2n-2} \prod_{i>j} (\alpha_i - \alpha_j)^2$$

be the discriminant of f. Show that

$$\frac{1}{n} \log |\operatorname{Disc}(f)| \leqslant \log n + (2n-2)h(\alpha).$$

Advanced problems

Question 9.

(a) Prove Liouville's inequality, namely that if α is an algebraic irrational number of degree $n \geqslant 2$, then there is a constant C (depending on α), such that for any rational number a/b with $b > 0$, we have

$$\left|\alpha - \frac{a}{b}\right| \geqslant C/b^n.$$

(Hint: Let f be the minimal polynomial of α. Combine a lower bound on the nonzero rational number $f(a/b)$ and an upper bound for $|f(\alpha) - f(a/b)|/(\alpha - (a/b))$ using the Mean Value Theorem.)

(b) A Liouville number is a real number x with the property that for any integer n, there is a rational number a/b with $b > 1$ such that

$$0 < |x - (a/b)| < 1/b^n.$$

Prove that Liouville numbers are transcendental and that Liouville's constant $\sum_{k=1}^{\infty} \frac{1}{10^{k!}}$ is a Liouville number.

References

[Wal00] Michel Waldschmidt, *Diophantine approximation on linear algebraic groups*, Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 326, Springer-Verlag, Berlin, 2000. Transcendence properties of the exponential function in several variables. MR1756786 ↑