Aim/Description of the Course

The aim of this course is to show how arithmetic and birational geometry is encoded in the elementary theory of a function field K over some base field k, where the base k is either a number field, or a finite field, of an algebraically closed field (such as the complex numbers). One of the main questions here is to give *sentences* in the language of fields by which one can "detect" the transcendence degree $\mathrm{td}(K|k)$. Clearly, the usual way we do this, namely: "$\exists (t_1, \dots, t_d) \in K$ which do not satisfy a non-trivial polynomial relation over k, and all $x \in K$ are algebraic over $k(t_1, \dots, t_d)$" is not a sentence in the language of fields. Second, supposing that we have found sentences as above, give further sentences in the language of fields by which one can describe the isomorphism type of K (as a field).

Concerning answers: Detecting the transcendence degree is "difficult" in the case k is a number field. (The proof uses the Milnor Conjecture, as proved by Voevodsky et al, but just as a "black box".) To the contrary, the case where k is finite or algebraically closed is easy... The problem of detecting the isomorphy type of K from its elementary theory is not completely solved yet. We will nevertheless show that this *is possible* in the case K is a function field of "general type". Here we should remark that the "conservation of difficulties" applies: The geometric situation is much more difficult to tackle than the arithmetic one...

Literature:

- F. Pop: *Elementary equivalence versus Isomorphism,* Invent. Math. 150 (2002), 385–408.

  In order to be able to understand the arguments of the proof, thus what we plan to do, the following basic knowledge is necessary/expected:

- The usual model theoretic rudiments (like in any introductory book on model theoretic algebra). See e.g., Chang–Keisler: *Model Theory* (3rd ed.), North Holland, Amsterdam 1990.
- Basic facts about Galois cohomology, in particular the relation between cohom.dim. and the transcendence degree. Much of these facts mostly used as "black boxes". See for instance J.-P. Serre: *Galois Cohomology,* Springer Verlag 1997.
- Basic facts about quadratic forms and the Witt ring (of anisotropic quadratic forms), Pfister forms. In particular, their relation to algebraic K-theory (and the Milnor Conjecture), and the Galois cohomology. Much of these facts mostly used as "black boxes". See e.g., A. Pfister: *On the Milnor Conjecture: History, Influence, applications,* Jahresbericht DMV 102 (2002), no.1, 15–41. Here lots of further literature on the subject can be found.
- Basic facts about varieties (over number fields, finite fields, or algebraically closed fields). See e.g., D. Mumford: *The red book on Varieties and Schemes.* Mostly (but not only) used as a "black box".
- Also, basic facts about: First, schemes of finite type, e.g., the Chebotarev Density Theorem, like in J.-P. Serre: *Zeta and L Functions,* Arithm. Alg. Geometry, pp. 82–92, Harper&Row, New York 1965; and second, varieties of general type, like in Sh. Iitaka: *Algebraic geometry,* Springer Verlag. Facts mostly used as "black boxes".

A possible project: Give the "axiomatic" of the finitely generated fields and/or function fields over algebraically closed fields. (This should follow from a close analysis of the proof of the result under discussion: "elem. equiv. $\Rightarrow$ isom.")