5 Continuous random variables

We deviate from the order in the book for this chapter, so the subsections in this chapter do not correspond to those in the text.

5.1 Densities of continuous random variable

Recall that in general a random variable X is a function from the sample space to the real numbers. If the range of X is finite or countable infinite, we say X is a discrete random variable. We now consider random variables whose range is not countably infinite or finite. For example, the range of X could be an interval, or the entire real line.

For discrete random variables the probability mass function is $f_X(x) = \mathbf{P}(X = x)$. If we want to compute the probability that X lies in some set, e.g., an interval $[a, b]$, we sum the pmf:

$$\mathbf{P}(a \leq X \leq b) = \sum_{x: a \leq x \leq b} f_X(x)$$

A special case of this is

$$\mathbf{P}(X \leq b) = \sum_{x: x \leq b} f_X(x)$$

For continuous random variables, we will have integrals instead of sums.

Definition 1. *A random variable X is continuous if there is a non-negative function $f_X(x)$, called the probability density function (pdf) or just density, such that*

$$\mathbf{P}(X \leq t) = \int_{-\infty}^{t} f_X(x)\, dx$$

Proposition 1. *If X is a continuous random variable with density $f(x)$, then*

1. $\mathbf{P}(X = x) = 0$ *for any* $x \in \mathbb{R}$.
2. $\mathbf{P}(a \leq X \leq b) = \int_a^b f(x)\, dx$
3. $\int_{-\infty}^{\infty} f(x)\, dx = 1$

Proof. First we observe that subtracting the two equations

$$\mathbf{P}(X \le b) = \int_{-\infty}^{b} f_X(x)\,dx, \quad \mathbf{P}(X \le a) = \int_{-\infty}^{a} f_X(x)\,dx$$

gives

$$\mathbf{P}(X \le b) - \mathbf{P}(X \le a) = \int_{a}^{b} f_X(x)\,dx$$

and we have $\mathbf{P}(X \le b) - \mathbf{P}(X \le a) = \mathbf{P}(a < X \le b)$, so

$$\mathbf{P}(a < X \le b) = \int_{a}^{b} f_X(x)\,dx \tag{1}$$

Now for any n

$$\mathbf{P}(X = x) \le \mathbf{P}(x - 1/n < X \le x) = \int_{x-1/n}^{x} f_X(t)\,dt$$

As $n \to \infty$, the integral goes to zero, so $\mathbf{P}(X = x) = 0$.

Property 2 now follows from eq. (1) since

$$\mathbf{P}(a \le X \le b) = \mathbf{P}(a < X \le b) + \mathbf{P}(X = a) = \mathbf{P}(a < X \le b)$$

Note that since the probability X equals any single real number is zero, $\mathbf{P}(a \le X \le b)$, $\mathbf{P}(a < X \le b)$, $\mathbf{P}(a \le X < b)$, and $\mathbf{P}(a < X < b)$ are all the same.

Property 3 is just the fact that $P(-\infty < X < \infty) = 1$. $\square$

Caution Often the range of X is not the entire real line. Outside of the range of X the density $f_X(x)$ is zero. So the definition of $f_x(x)$ will typically involves cases: in one region it is given by some formula, elsewhere it is simply 0. So integrals over all of $\mathbb{R}$ which contain $f_X(x)$ will reduce to intervals over a subset of $\mathbb{R}$. If you mistakenly integrate the formula over the entire real line you will get nonsense.

5.2 Expected value

Definition 2. *Let X be a continuous RV with density $f_X(x)$. Then the expected value of X is given by*

$$\mathbf{E}[X] = \int_{-\infty}^{\infty} x\, f_X(x)\, dx$$

provided

$$\int_{-\infty}^{\infty} |x|\, f_X(x)\, dx < \infty$$

(If this last integral is infinite we say the expected value of X is not defined.)

Just as with discrete RV's, if X is a continuous RV and g is a function from $\mathbb{R}$ to $\mathbb{R}$, then we can define a new RV by $Y = g(X)$. How do we compute the mean of Y ? One approach would be to work out the density of Y and then use the definition of expected value. We have not yet seen how to find the density of Y, but for this question there is a shortcut just as there was for discrete RV.

Theorem 1. *Let X be a continuous RV, g a function from $\mathbb{R}$ to $\mathbb{R}$. Let $Y = g(X)$. Then*

$$\mathbf{E}[Y] = \mathbf{E}[g(X)] = \int_{-\infty}^{\infty} g(x)\, f_X(x)\, dx$$

The definition of the variance is analogous to the discrete case. In fact, for any random variable (discrete, continuous, or otherwise) the variance is given by

Definition 3. *The variance of X is*

$$\sigma^2 = \mathbf{E}[(X-\mu)^2], \quad \mu = \mathbf{E}[X]$$

provided the expected value is defined.

Just as in the discrete case, there is an application of the above theorem that gives us a shortcut for computing the variance

Corollary 1. *If X is a continuous random variable with finite variance σ^2 and mean μ, then*

$$\sigma^2 = \mathbf{E}[X^2] - \mu^2 = \int_{-\infty}^{\infty} x^2 \, f_X(x) \, dx - \mu^2$$

Proof. By the theorem

$$\begin{aligned}
\sigma^2 &= \mathbf{E}[(X-\mu)^2] = \int (x-\mu)^2 \, f_X(x) \, dx = \int [x^2 - 2\mu x + \mu)^2] \, f_X(x) \, dx \\
&= \int x^2 \, f_X(x) \, dx - 2\mu \int x \, f_X(x) \, dx + \mu^2 \int f_X(x) \, dx \\
&= \int x^2 \, f_X(x) \, dx - 2\mu^2 + \mu^2 = \int x^2 \, f_X(x) \, dx - \mu^2
\end{aligned}$$

□

5.3 Catalog

As with discrete RV's, two continuous RV's defined on completely different probability spaces can have the same density.

Definition 4. *Two continuous random variables are identically distributed if they have the same pdf.*

There are certain densities that come up a lot. So we start a catalog of them. Note that the mean and variance of the RV only depend on its pdf.

Uniform: (two parameters $a, b \in \mathbb{R}$ with $a < b$) The uniform density on $[a, b]$ is

$$f(x) = \begin{cases} \frac{1}{b-a}, & \text{if } a \leq x \leq b \\ 0, & \text{otherwise} \end{cases}$$

We have seen the uniform distribution before. Previously we said that to compute the probability X is in some subinterval $[c, d]$ of $[a, b]$ you take the length of that subinterval divided by the length of $[a, b]$. This is of course what you get when you compute

$$\int_c^d f_X(x) \, dx = \int_c^d \frac{1}{b-a} \, dx = \frac{d-c}{b-a}$$

Next we find the mean and variance of the uniform distribution on $[a, b]$. The mean is

$$\mu = \int_a^b x\, f(x)\, dx = \int_a^b \frac{x}{b-a}\, dx = \frac{1}{2}\frac{b^2 - a^2}{b-a} = \frac{a+b}{2} \tag{2}$$

For the variance we have to first compute

$$\mathbf{E}[X^2] = \int_a^b x^2\, f(x)\, dx \tag{3}$$

We then subtract the square of the mean and find $\sigma^2 = (b-a)^2/12$.

Exponential: (one real parameter $\lambda > 0$)

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & \text{if } x \geq 0 \\ 0, & \text{if } x < 0 \end{cases}$$

Check that its total integral is 1. Note that the range is $[0, \infty)$.

One of the homework problems is to compute its mean and variance.

Normal: (two real parameters $\sigma > 0$, $\mu \in \mathbb{R}$)

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right)$$

The range of a normal RV is the entire real line. It is anything but obvious that the integral of this function is 1. Try to show it.

End of lecture - Fri, Oct 6

Cauchy:

$$f(x) = \frac{1}{\pi(1+x^2)}$$

Example (skipped): Suppose X has the Cauchy distribution. Find the number c with the property that $\mathbf{P}(X \geq c) = 1/4$.

Example: Suppose X has the density

$$f(x) = \begin{cases} c\,x(2-x) & \text{if } 0 \le x \le 2 \\ 0 & \text{otherwise} \end{cases}$$

where c is a constant. Find the constant c and then compute the mean and variance.

The gamma function is defined by

$$\Gamma(w) = \int_0^\infty x^{w-1}\, e^{-x}\, dx \tag{4}$$

Integration parts shows that $\Gamma(w+1) = w\Gamma(w)$. It then follows by induction that for positive integers n, $\Gamma(n+1) = n!$.

The gamma distribution has range $[0,\infty)$ and depends on two parameters $\lambda > 0, w > 0$. The density is

$$f(x) = \begin{cases} \frac{\lambda^w}{\Gamma(w)}\, x^{w-1}\, e^{-\lambda x} & \text{if } x \ge 0 \\ 0 & \text{otherwise} \end{cases} \tag{5}$$

In one of the homework problems we compute its mean and variance. You should find that they are

$$\mu = \frac{w}{\lambda}, \quad \sigma^2 = \frac{w}{\lambda^2} \tag{6}$$

5.4 Cumulative distribution function

In this section X is a random variable that can be either discrete or continuous.

Definition 5. *The cumulative distribution function (cdf) of the random variable X is the function*

$$F_X(x) = \mathbf{P}(X \le x)$$

Why introduce this function? It will be a powerful tool when we look at functions of random variables and compute their density.

Example: Let X be uniform on $[-1,1]$. Compute the cdf.

GRAPH !!!!!!!!!!!!!!!!!!!!

x	2	3	4	5	6
$f_X(x)$	1/8	1/8	3/8	2/8	1/8

Example: Let X be a discrete RV whose pmf is given in the table.

GRAPH !!!!!!!!!!!!!!!!!!!!

Example: Compute cdf of exponential distribution.

Theorem 2. *Let X be a continuous RV with pdf $f(x)$ and cdf $F(x)$. Then they are related by*

$$\begin{aligned} F(x) &= \int_{-\infty}^{x} f(t)\,dt, \\ f(x) &= F'(x) \end{aligned}$$

Proof. The first equation is immediate from the def of the cdf. To get the second equation, differentiate the first equation and remember that the fundamental theorem of calculus says

$$\frac{d}{dx}\int_a^x f(t)\,dt = f'(x)$$

□

Theorem 3. *For any random variable the cdf satisfies*

1. *$F(x)$ is non-decreasing, $0 \le F(x) \le 1$.*
2. *$\lim_{x\to-\infty} F(x) = 0$, $\lim_{x\to\infty} F(x) = 1$.*
3. *For a continuous random variable the cdf is continuous.*
4. *For a discrete random variable the cdf is piecewise constant. The set of points where it jumps is the range of X. If x is a point where it has a jump, then the height of the jump is $\mathbf{P}(X = x)$.*

Proof. 1 is obvious ....

To prove 2, let $x_n \to \infty$. Assume that x_n is increasing. Let $E_n = \{X \leq x_n\}$. Then E_n is an increasing sequence of events. By the continuity of the probability measure,

$$\mathbf{P}(\cup_{n=1}^{\infty} E_n) = \lim_{n\to\infty} \mathbf{P}(E_n)$$

Since $x_n \to \infty$, every outcome is in E_n for large enough n. So $\cup_{n=1}^{\infty} E_n = \Omega$. So

$$\lim_{n\to\infty} F(x_n) = \lim_{n\to\infty} \mathbf{P}(E_n) = 1 \tag{7}$$

The proof that the limit as $x \to -\infty$ is 0 is similar.

GAP

□

We will not need the following theorem, but a natural question is whether all functions F with the properties in the previous theorem are the cdf of some random variable.

Theorem 4. *Let $F(x)$ be a function from $\mathbb{R}$ to $[0,1]$ such that*

1. *$F(x)$ is non-decreasing.*
2. *$\lim_{x\to-\infty} F(x) = 0$, $\lim_{x\to\infty} F(x) = 1$.*
3. *$F(x)$ is continuous from the right.*

Then $F(x)$ is the cdf of some random variable, i.e., there is a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ and a random variable X on it such that $F(x) = \mathbf{P}(X \leq x)$.

The proof of this theorem is way beyond the scope of this course. In fact, the resulting random variable need not be a discrete or continuous random variable as we have defined them.

5.5 Function of a random variable

Let X be a continuous random variable and $g : \mathbb{R} \to \mathbb{R}$. Then $Y = g(X)$ is a new random variable. We want to find its density. This is not as easy as in the discrete case. In particular $f_Y(y)$ is not $\sum_{x:g(x)=y} f_X(x)$.

KEY IDEA: Compute the cdf of Y and then differentiate it to get the pdf of Y.

Example: Let X be uniform on $[0,1]$. Let $Y = X^2$. Find the pdf of Y.
!!!!!!! GAP

End of lecture - Mon, Oct 9 (sort of)

Example: Let X be uniform on $[-1,1]$. Let $Y = X^2$. Find the pdf of Y.
!!!!!!! GAP
Calculus review: The FTC says

$$\frac{d}{dx}\int_c^x f(u)du = f(x) \tag{8}$$

Now let $g(x)$ and $h(x)$ be differentiable functions. What are

$$\frac{d}{dx}\int_c^{g(x)} f(u)du? \tag{9}$$

$$\frac{d}{dx}\int_{h(x)}^{g(x)} f(u)du? \tag{10}$$

Example: Let X be uniform on $[0,1]$. Let $\lambda > 0$. $Y = -\frac{1}{\lambda}\ln(X)$. Show Y has an exponential distribution.
!!!!!!! GAP

Example: The "standard normal" distribution is the normal distribution with $\mu = 0$ and $\sigma = 1$. Let Z have a standard normal distribution. Define a new RV by $X = \mu + \sigma Z$. Find the pdf of X.

Example: Find the mean and variance of the normal distribution.
!!!!!!! GAP

Proposition 2. *(How to write a general random number generator) Let X be a continuous random variable with values in $[a,b]$. Suppose that the cdf $F(x)$ is strictly increasing on $[a,b]$. Let U be uniform on $[0,1]$. Let $Y = F^{-1}(U)$. Then X and Y are identically distributed.*

Proof.

$$\mathbf{P}(Y \le y) = \mathbf{P}(F^{-1}(U) \le y) = \mathbf{P}(U \le F(y)) = F(y) \tag{11}$$

□

Application: My computer has a routine to generate random numbers that are uniformly distributed on $[0, 1]$. We want to write a routine to generate numbers that have an exponential distribution with parameter λ.

How do you simulate normal RV's? Not so easy since the cdf cannot be explicitly computed. More on this later.

When $Y = g(X)$ and we know the pdf of X, then we have seen how to compute the pdf of Y. If g is increasing on the range of X or decreasing on the range of X, then there is a formula.

Theorem 5. *Let g be strictly increasing or strictly decreasing on the range of X. Assume also that g is differentiable. Then*

$$f_Y(y) = f_X(g^{-1}(y)) \left| \frac{d}{dy} g^{-1}(y) \right| \tag{12}$$

where g^{-1} is the inverse function of g, i.e., the function such that $g^{-1}(g(y)) = y$.

Note that g^{-1} is not $1/g$. A formula from calculus relates the derivative of the inverse function to the derivative of the original function. It says

$$\frac{d}{dy} g^{-1}(y) = \frac{1}{g'(g^{-1}(y))}$$

Example X is exponential with $\lambda = 1$. $Y = \exp(-X)$. So $g(y) = \exp(-y)$ and $g^{-1}(y) = -\ln(y)$. **GAP**

Proof of theorem GAP

5.6 Histograms and the meaning of the pdf

For a discrete RV the pmf $f(x)$ has a direct interpretation. It is the probability that $X = x$. For a continuous RV, the pdf $f(x)$ is not the probability that $X = x$ (which is zero), nor is it the probability of anything. If $\delta > 0$ is small, then

$$\int_{x-\delta}^{x+\delta} f(u)\,du \approx 2\delta f(x)$$

This is $P(x - \delta \leq X \leq x + \delta)$. So the probability X is in the small interval $[x - \delta, x + \delta]$ is $f(x)$ times the length of the interval. So $f(x)$ is a *probability density*.

Histogram are closely related to the pdf and can be thought of as "experimental pdf's." Suppose we generate N independent random samples of X where N is large. We divide the range of X into intervals of width Δx (usually called "bins"). The probability X lands in a particular bin is $P(x \leq X \leq x + \Delta x) \approx f(x)\Delta x$. So we expect approximately $Nf(x)\Delta x$ of our N samples to fall in this bin.

To construct a histrogram of our N samples we first count how many fall in each bin. We can represent this graphically by drawing a rectangle for each bin whose base is the bin and whose height is the number of samples in the bin. This is usually called a frequency plot. To make it look like our pdf we should rescale the heights so that the area of a rectangle is equal to the fraction of the samples in that bin. So the height of a rectangle should be

$$\frac{number\, of\, samples\, in\, bin}{N\, \Delta x}$$

With these heights the rectagles give the histogram. As we observed above, the number of our N samples in the bin will be approximately $Nf(x)\Delta x$, so the above is approximately $f(x)$. So if N is large and Δx is small, the histogram will approximate the pdf.

5.7 More on expected value

The material in this section will not be "on the test."

For a continuous RV we defined the expected value $E[X]$ to be $\int x\, f_X(x)\, dx$. This is not really how it should be defined. There is a way to define $E[X]$ for any random variable and then prove that it in the case of a continuous RV, it is given by $\int x\, f_X(x)\, dx$. A proper discussion of how to define $E[X]$ for any RV requires the theory of abstract Lebesgue integration which is way beyond the level of this course. Nonetheless, we can still give a non-rigorous explanation of how to define $E[X]$.

For a discrete RV X, the expected value is

$$\mathbf{E}[X] = \sum_x x f_X(x)$$

We will use this definition to define the expected value for a continuous RV. The idea is to write our continuous RV as the limit of a sequence of discrete RV's.

Let X be a continuous RV. We will assume that it is bounded. So there is a constant M such that the range of X lies in $[-M, M]$, i.e., $-M \leq X \leq M$. Fix a positive integer n and divide the range into subintervals of width $1/n$. In each of these subintervals we "round" the value of X to the left endpoint of the interval and call the resulting RV X_n. So X_n is defined by

$$X_n(\omega) = \frac{k}{n}, \quad \textit{where } k \textit{ is the integer with} \quad \frac{k}{n} \leq X(\omega) < \frac{k+1}{n}$$

Note that for all outcomes ω, $|X(\omega) - X_n(\omega)| \leq 1/n$. So X_n converges to X pointwise on the sample space Ω. In fact it converges uniformly on Ω. The expected value of X should be the limit of $\mathbf{E}[X_n]$ as $n \to \infty$.

Definition 6. *(Heuristic) For any random variable the expectation of* $E[X]$ *is* $\lim_{n\to\infty} E[X_n]$.

The random variable X_n is discrete. Its values are k/n with k running from $-Mn$ to $Mn - 1$ (or possibly a smaller set). So

$$\mathbf{E}[X_n] = \sum_{k=-Mn}^{Mn-1} \frac{k}{n} f_{X_n}(\frac{k}{n})$$

Now

$$f_{X_n}(\frac{k}{n}) = \mathbf{P}(X_n = \frac{k}{n}) = \mathbf{P}(\frac{k}{n} \leq X < \frac{k+1}{n}) = \int_{\frac{k}{n}}^{\frac{k+1}{n}} f_X(x)\, dx$$

So

$$\begin{aligned} \mathbf{E}[X_n] &= \sum_{k=-Mn}^{Mn-1} \frac{k}{n} \int_{\frac{k}{n}}^{\frac{k+1}{n}} f_X(x)\, dx \\ &= \sum_{k=-Mn}^{Mn-1} \int_{\frac{k}{n}}^{\frac{k+1}{n}} \frac{k}{n} f_X(x)\, dx \end{aligned}$$

When n is large, the integrals in the sum are over a very small interval. In this interval, x is very close to k/n. In fact, they differ by at most $1/n$. So the limit as $n \to \infty$ of the above should be

$$\sum_{k=-Mn}^{Mn-1} \int_{\frac{k}{n}}^{\frac{k+1}{n}} x\, f_X(x)\, dx = \int_{-M}^{M} x\, f_X(x)\, dx = \int_{-\infty}^{\infty} x\, f_X(x)\, dx$$

The last equality comes from the fact that $f_X(x)$ is zero outside $[-M, M]$. The above is not a proof, but it should make the following plausible:

Theorem 6. *Let X be a continuous RV. If we define $E[X]$ to be $\lim_{n\to\infty} E[X_n]$, then*

$$\mathbf{E}[X] = \int_{-\infty}^{\infty} x\, f_X(x)\, dx$$

We can now use these ideas to give a non-rigorous derivation of a theorem we stated before:

Theorem 7. *Let X be a continuous RV, g a function from $\mathbb{R}$ to $\mathbb{R}$. Let $Y = g(X)$. Then*

$$\mathbf{E}[Y] = \mathbf{E}[g(X)] = \int_{-\infty}^{\infty} g(x)\, f_X(x)\, dx$$

Proof. Since we do not know how to find the density of Y, we cannot prove this yet. We just give a non-rigorous derivation. Let X_n be the sequence of discrete RV's that approximated X defined above. Then $g(X_n)$ are discrete RV's. They approximate $g(X)$. In fact, if the range of X is bounded and g is continous, then $g(X_n)$ will converge uniformly to $g(X)$. So $\mathbf{E}[g(X_n)]$ should converges to $\mathbf{E}[g(X)]$.

Now $g(X_n)$ is a discrete RV, and by the law of the unconscious statistician

$$\mathbf{E}[g(X_n)] = \sum_x g(x)\, f_{X_n}(x) \tag{13}$$

Looking back at our previous derivation we see this is

$$\begin{aligned}
\mathbf{E}[g(X_n)] &= \sum_{k=-Mn}^{Mn-1} g(\frac{k}{n}) \int_{\frac{k}{n}}^{\frac{k+1}{n}} f_X(x)\, dx \\
&= \sum_{k=-Mn}^{Mn-1} \int_{\frac{k}{n}}^{\frac{k+1}{n}} g(\frac{k}{n})\, f_X(x)\, dx
\end{aligned}$$

which converges to

$$\int g(x)\, f_X(x)\, dx \tag{14}$$

$\square$

Recall that for a discrete random variable that only takes on values in $0, 1, 2, \cdots$, we showed in a homework problem that

$$E[X] = \sum_{k=0}^{\infty} P(X > k) \tag{15}$$

There is a similar result for non-negative continuous random variables.

Theorem 8. *Let X be a non-negative continuous random variable with cdf $F(x)$. Then*

$$\mathbf{E}[X] = \int_0^\infty [1 - F(x)]\, dx \tag{16}$$

provided the integral converges.

Proof. We use integration by parts on the integral. Let $u(x) = 1 - F(x)$ and $dv = dx$. So $du = -f dx$ and $v = x$. So

$$\int_0^\infty [1 - F(x)]\, dx = x(1 - F(x))|_{x=0}^\infty + \int_0^\infty x\, f(x)\, dx = \mathbf{E}[X] \tag{17}$$

Note that the boundary term at ∞ is zero since $F(x) \to 1$ as $x \to \infty$. $\square$

We can use the above to prove the law of the unconscious statistician for a special case. We assume that $X \geq 0$ and that the function g is from $[0, \infty)$ into $[0, \infty)$, is strictly increasing, and $g(0) = 0$. Note that this implies that g has an inverse. Then

$$\begin{aligned}
\mathbf{E}[Y] &= \int_0^\infty [1 - F_Y(x)]\, dx = \int_0^\infty [1 - \mathbf{P}(Y \leq x)]\, dx && (18) \\
&= \int_0^\infty [1 - \mathbf{P}(g(X) \leq x)]\, dx = \int_0^\infty [1 - \mathbf{P}(X \leq g^{-1}(x))]\, dx && (19) \\
&= \int_0^\infty [1 - F_X(g^{-1}(x))]\, dx && (20)
\end{aligned}$$

Now we do a change of variables. Let $s = g^{-1}(x)$. So $x = g(s)$ and $dx = g'(s)ds$. So above becomes

$$\int_0^\infty [1 - F_X(s)]\, g'(s)\, ds \tag{21}$$

Now integrate this by parts to get

$$[1 - F_X(s)]\, g(s)|_{s=0}^\infty + \int_0^\infty g(s)\, f(s)\, ds \tag{22}$$

which proves the theorem in this special case.