1. (from Lawler) Consider the Markov chain in problem 1 in the first homework set. (The Smiths and their newspapers.)
(a) After a long time, what is the expected number of papers in the pile?
Solution: From the last homework the stationary distribution is 
\[ \pi = (0.3839, 0.2559, 0.1706, 0.1137, 0.0758). \]
The expected number of papers is the expected value of the number of papers for this distribution:
\[ 0 \cdot 0.3839 + 1 \cdot 0.2559 + 2 \cdot 0.1706 + 3 \cdot 0.1137 + 4 \cdot 0.0758 = 1.2414 \]
(b) Suppose we start in state 0 (no papers). What is the expected value of the time it takes to return to state 0?
Solution: This is a return time whose expected value is \( 1 / \pi(0) = 2.6408. \)

2. (from Lawler) Let \( X_n \) be the successive values from rolling a six-sided die infinitely many times. (So \( X_n \) are independent and each one is uniformly distributed on \( \{1, 2, 3, 4, 5, 6\} \).) Let \( S_n = X_1 + X_2 + \cdots + X_n \) \( \text{mod} \ 8. \) The \( S_n \) are a Markov with state space \( \{0, 1, 2, 3, 4, 5, 6, 7, 8\} \).
(a) Find the transition matrix
Solution:
\[
P = \begin{pmatrix}
0 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 0 \\
1/6 & 0 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 \\
1/6 & 0 & 0 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 \\
1/6 & 1/6 & 0 & 0 & 1/6 & 1/6 & 1/6 & 1/6 \\
1/6 & 1/6 & 1/6 & 0 & 0 & 1/6 & 1/6 & 1/6 \\
1/6 & 1/6 & 1/6 & 1/6 & 0 & 0 & 1/6 & 1/6 \\
1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 0 & 0 & 1/6 \\
1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 0 & 0
\end{pmatrix}
\]
(b) Define
\[
T_1 = \min\{n \geq 1 : S_n = 0\} \\
T_2 = \min\{n \geq 1 : S_n = 1\}
\]
Find \( E[T_1] \) and \( E[T_2] \).
Solution: By raising the transition matrix to a large power you see that the stationary distribution is \( \pi(i) = 1/8 \) for \( i = 1, 2, 3, 4, 5, 6, 7, 8 \), which you might expect based on symmetry. Since we start in 0, \( T_1 \) is a return time and so \( E[T_1 : X_0 = 0] = 1/\pi(0) = 8. \) \( T_2 \) is not a return time. It is the time it takes to get from 0 to 1. So we change the chain by making state
1 an absorbing state. All the other states are transient. The submatrix $Q$ corresponding to the transient states is gotten by removing the row and column corresponding to state 1. Then we compute $\sum_j (I - Q)^{-1}_{ij}$ where $i$ corresponds to state 0 and $j$ is summed over the indices corresponding to all state but 1. Since state 1 was deleted from $Q$, this means we sum the row in $(I - Q)^{-1}$ corresponding to state 0. This gives 6.857.

3. (from Lawler) We flip a fair coin repeatedly until we have gotten four heads in a row. What is the expected number of flips needed? Hint: this can be thought of as a Markov chain with state space $\{0, 1, 2, 3, 4\}$.

**Solution:** There are several different ways to set up the transition matrix, all of which will work. We can make 4 an absorbing state so that being in state 4 means that he have gotten 4 heads in a row at some time and we don’t care what happens anymore with the flips. Or we can say that if we are in state 4 and get a heads than we stay in state 4; if we are in state 4 and get a tails than we move to state 0 since the run is broken and we have to start over. Finally we can say that if we are in state 4 then we transition to state 0 with probability 1 and start the game over. In all these approaches the submatrix $Q$ you get by deleting the row and column for state 4 is the same and you follow the $\sum_j (I - Q)^{-1}_{ij}$ method to get the answer.

The last way of setting up the chain leads to another solution method. In this approach you always go from state 4 immediately to state 0. So starting from 4 the time it takes to get to back to state 4 for the first time is one more than the time it take to reach 4 starting from 0. So the expected value we are trying to compute is the mean return time for state 4 minus 1. The mean return time is $1/\pi(4)$ where $\pi$ is found by raising the matrix to a large power.

In all the approaches you should get a answer of 30.

4. (from Lawler) Let $X_n$ and $Y_n$ be independent Markov chains. Each has state space $\{0, 1, 2\}$ and transition matrix

$$P = \begin{pmatrix}
0.5 & 0.25 & 0.25 \\
0.25 & 0.25 & 0.5 \\
0 & 0.5 & 0.5
\end{pmatrix}$$

(a) We can think of $(X_n, Y_n)$ as a Markov chain with 9 states. Describe the states and give the transition matrix.

**Solution:** The states are $(0, 0)$, $(0, 1)$, $(0, 2)$, $(1, 0)$, $(1, 1)$, $(1, 2)$, $(2, 0)$, $(2, 1)$, $(2, 2)$, where the first entry is the value of $X_n$ and the second entry the value of
Y_n. We label the states 1 to 9. There is a matlab script called hmwk2.m on the web now which will do all of this problem. Run it and you will see the matrix p and the answers to all parts. Note how the script avoids having to type in a 9 by 9 matrix.

(b) If we start with X_0 = 0, Y_0 = 2, what is the probability that X_{10} = 0, Y_{10} = 2?

Solution: You compute p^{10} and pull out the 3,3 entry to get 0.0827.

(c) In the long run, what fraction of the time is X_n = Y_n?

Solution: There are three states with X_n = Y_n which are (0,0), (1,1), (2,2) which have labels 1,5,9. Compute \( \pi \) and then \( \pi(1) + \pi(5) + \pi(9) = 0.3917 \).

(d) Let \( T = \min\{n \geq 0 : X_n = Y_n = 0\} \). Find \( E[T : X_0 = 0, Y_0 = 2] \).

Solution: Delete first row and first column. Do usual \( \sum_j (I - Q)^{-1} \) method to get the answer of 40.45.

5. A graph can be described by a set of vertices \( V \) (which we take to be finite) and an adjacency matrix \( A(i,j) \). \( A(i,j) = 1 \) is there is an edge in the graph between edges \( i \) and \( j \), and \( A(i,j) = 0 \) is there is no such edge. The degree \( d(i) \) of a vertex \( i \) is the number of edges connected to the vertex. So

\[
d(i) = \sum_j A(i,j)
\]

The random walk on the graph is the Markov chain with state space \( V \) and transition matrix

\[
p(i,j) = \frac{A(i,j)}{d(i)}
\]

(a) Show that \( \pi(i) = cd(i) \) is a stationary distribution for a suitable constant \( c \), and give the value of \( c \).

Solution: We first show that \( d(i) \) is a left eigenvector of \( P \) with eigenvalue 1. So we must show

\[
d(i) = \sum_j d(j)p(j,i)
\]

Using the definitions of \( p \),

\[
\sum_j d(j)p(j,i) = \sum_j d(j) \frac{A(j,i)}{d(j)} = \sum_j A(j,i) = \sum_j A(i,j) = d(i)
\]
where we have used $A(i, j) = A(j, i)$.

The stationary distribution is a left eigenvector of $P$ and it is the only one up to a scalar multiple. So it must be that $\pi(i) = cd(i)$. To find $c$, we use the fact that the sum of $\pi(i)$ must be 1. So

$$c = \frac{1}{\sum_i d_i}$$

(b) A chessboard is an 8 by 8 grid. A knight moves by moving two steps in one direction and one step in a perpendicular direction. (It must stay on the board). Suppose that a knight moves at random with no other pieces on the board. In the long run, what fraction of time is the knight at a corner square?

**Solution:** There are 64 state, but we don’t need to set up the 64 by 64 transition matrix. We only need the stationary distribution. So by (a) we only need the $d(i)$. There are 64 of them, but by symmetry we only need to compute those in the upper left 4 by 4 square of the chessboard. We find the following for them:

$$
\begin{pmatrix}
2 & 3 & 4 & 4 \\
3 & 4 & 6 & 6 \\
4 & 6 & 8 & 8 \\
4 & 6 & 8 & 8
\end{pmatrix}
$$

These add up to 84, so the sum of all the $d(i)$ is $4 \times 84 = 336$. So the probability of a single corner in $\pi$ is $2/336$. So the probability the knight is in some corner is $8/336 = 1/41$.

6. We saw that for an irreducible, aperiodic Markov chain, the long time average of the fraction of time spent in a state $i$ is $\pi(i)$ where $\pi$ is the stationary distribution. It is relatively easy to run a Markov chain for a long time on a computer. The script `run_chain.m` will do this. Enter the transition matrix for problem 1 and run this script. It will compute the fraction of time spent in each state and will also compute the long time average of the random variable $X$, where $X$ is the state, i.e., the number of papers in the pile. Note that your result should agree (approximately) with what you found in problem 1.