6 Systems with continuous symmetry

6.1 The rotator and Heisenberg models

Review the Ising model

Explain symmetry breaking

We now introduce a new class of models. As before we have a lattice, e.g., $\mathbb{Z}^d$. At each lattice site i we have a variable σ_i which now takes values on either the unit circle or the unit sphere. The Hamiltonian is

$$H = -\sum_{<ij>} \sigma_i \cdot \sigma_j$$

where $\sigma_i \cdot \sigma_j$ is the dot product of the two vectors. In the case where σ_i takes values on the unit circle the model is called the (classical) XY model or the rotator model. In the case of the unit sphere it is called the (classical) Heisenberg model. So the patition function is

$$Z = \prod_i \int_{S^p} d\sigma_i \exp(-\beta H) \tag{1}$$

where $d\sigma_i$ denotes Haar measure on S^p. We are mainly interested in $p=1$ and $p=2$ but you can take $p>2$. Note that the Ising model can be thought of as the case of $p=0$. Expectations are defined by

$$< F(\sigma) >= \frac{1}{Z} \prod_i \int_{S^p} d\sigma_i \, F(\sigma) \exp(-\beta H) \tag{2}$$

For the rotator model we can parameterize the vector σ_i by its polar angle θ_i which takes values between 0 and 2π. Then the partition function is

$$Z = \prod_i \int_0^{2\pi} d\theta_i \, \exp[\beta \sum_{<ij>} \cos(\theta_i - \theta_j)]$$

Review Peierls argument for Ising, $d=2$ vs $d=1$

Symmetry breaking for the new models, sphere of gs

Explain why Peierls argument fails for new models

6.2 Absence of symmetry breaking in two dimensions

The theme of this section is that a continuous symmetry can lead to an absence of symmetry breaking in two dimensions. This is usually called the Mermin Wagner theorem. It does not say there is no phase transition in these models. We will prove it for the rotator model. It is true in much great generality (Dobrushin-Shlosman), but one should be careful. At the end of this section we give an example of a model with a continous symmetry that does have symmetry breaking in two dimensions. We should note that the symmetry broken is a discrete one, not the continuous symmetry.

Theorem 1. *Consider the XY (or rotator) model in a finite rectangle with periodic boundary conditions. We include an external field in the z direction:*

$$H = -\sum_{<ij>} \cos(\theta_i - \theta_j) - h \sum_j \cos(\theta_j) \tag{3}$$

Let N denote the number of sites in the rectangle and define

$$m = \frac{1}{N} \sum_j < \cos(\theta_j) > \tag{4}$$

Then

$$\limsup_{h \to 0^+} \lim_{\Lambda \to \infty} m = 0$$

Proof: Define

$$A = \sum_j e^{-ikj} \sin(\theta_j)$$
$$B = -\sum_j e^{-ikj} \frac{\partial H}{\partial \theta_j}$$

Here k is summed of the appropriate set of momenta. For simplicity, take the rectangle to be a square $\Lambda = \{1, 2, \cdots, L\}^2$, $k = (k_1, k_2)$ with $k_i = 2\pi l_i / L$ where $l_i = 0, 1, 2, \cdots L - 1$.

The Cauchy Schwarz inequality implies

$$< \bar{A} B >^2 \leq < \bar{A} A >< \bar{B} B >$$

The trick is to rewrite $< \bar{A}A >$ and $< \bar{B}B >$ using integration by parts.

We have

$$< \bar{A}B >= -\sum_{j,l} e^{ik(j-l)} < \sin(\theta_j)\frac{\partial H}{\partial \theta_l} >$$

We will use the following trick several times:

$$e^{-\beta H}\frac{\partial H}{\partial \theta_l} = -\frac{1}{\beta}\frac{\partial}{\partial \theta_l}e^{-\beta H}$$

So

$$\begin{aligned}
< \sin(\theta_j)\frac{\partial H}{\partial \theta_l} > &= \frac{1}{Z}\int d\theta \, \sin(\theta_j)e^{-\beta H}\frac{\partial H}{\partial \theta_l} \\
&= -\frac{1}{\beta Z}\int d\theta \, \sin(\theta_j)\frac{\partial}{\partial \theta_l}e^{-\beta H} \\
&= \frac{1}{\beta Z}\int d\theta \, e^{-\beta H}\frac{\partial}{\partial \theta_l}\sin(\theta_j) \\
&= \frac{1}{\beta}\delta_{j,l} < \cos(\theta_j) >
\end{aligned}$$

So

$$< \bar{A}B > \;=\; -\frac{1}{\beta}\sum_{j,l} e^{ik(j-l)}\delta_{j,l} < \cos(\theta_j) >= -\frac{1}{\beta}Nm$$

For the other term,

$$< \bar{B}B >= \sum_{j,l} e^{ik(j-l)} < \frac{\partial H}{\partial \theta_j}\frac{\partial H}{\partial \theta_l} >$$

We use integration by parts again:

$$\begin{aligned}
< \frac{\partial H}{\partial \theta_j}\frac{\partial H}{\partial \theta_l} > &= \frac{1}{Z}\int d\theta \frac{\partial H}{\partial \theta_l}e^{-\beta H}\frac{\partial H}{\partial \theta_l} \\
&= -\frac{1}{\beta Z}\int d\theta \frac{\partial H}{\partial \theta_l}\frac{\partial}{\partial \theta_l}e^{-\beta H} \\
&= \frac{1}{\beta Z}\int d\theta \frac{\partial^2}{\partial \theta_l \partial \theta_j}e^{-\beta H}
\end{aligned}$$

We have

$$\frac{\partial H}{\partial \theta_l} = \sum_{m:|m-l|=1} \sin(\theta_l - \theta_m) + h \sin(\theta_l)$$

$$\frac{\partial^2 H}{\partial \theta_j \partial \theta_l} = \delta_{j,l} \sum_{m:|m-l|=1} \cos(\theta_l - \theta_m) - \delta_{|j-l|=1} \cos(\theta_l - \theta_j) + \delta_{j,l}\, h \cos(\theta_l)$$

So

$$\begin{aligned}
< \bar{B}B > &= \frac{1}{\beta} \sum_{j,l} e^{ik(j-l)} [\delta_{j,l} \sum_{m:|m-l|=1} < \cos(\theta_l - \theta_m) > \\
&- \delta_{|k-l|=1} < \cos(\theta_l - \theta_j) > + \delta_{j,l} h < \cos(\theta_l) >] \\
&= \frac{1}{\beta} \sum_{l} \sum_{m:|m-l|=1} < \cos(\theta_l - \theta_m) > \\
&- \frac{1}{\beta} \sum_{j,l:|j-l|=1} e^{ik(j-l)} < \cos(\theta_j - \theta_l) > + \frac{h}{\beta} \sum_{l} < \cos(\theta_l) > \\
&= \frac{1}{\beta} \sum_{j,l:|j-l|=1} < \cos(\theta_j - \theta_l) > [1 - e^{ik(j-l)}] + \frac{h}{\beta} Nm
\end{aligned}$$

Now rewrite the CS inequality as

$$< \bar{A}A > \geq \frac{< \bar{A}B >^2}{< \bar{B}B >}$$

Now

$$< \bar{A}A > \geq \sum_{j,l} e^{ik(j-l)} < \sin(\theta_j) \sin(\theta_l) >$$

Sum over k:

$$\sum_{k} < \bar{A}A > = N \sum_{l} < \sin^2(\theta_l) > \leq N^2$$

So

$$\begin{aligned}
N^2 &\geq \sum_{k} \frac{< \bar{A}B >^2}{< \bar{B}B >} && (5) \\
&= \frac{1}{\beta} \sum_{k} \frac{N^2 m^2}{\sum_{j,l:|j-l|=1} < \cos(\theta_j - \theta_l) > [1 - e^{ik(j-l)}] + hNm} && (6)
\end{aligned}$$

Now for $|j-l|=1$ by translation and rotation invariance $< \cos(\theta_j - \theta_l) >$ does not depend on j,l. Denote it by E. Note that $-E$ is the average energy per bond in the system. We leave it to the reader to show that $E>0$, in fact E is close to 1 uniformly in the volume if β is large. So the above becomes

$$1 \geq \frac{1}{\beta} \sum_k \frac{m^2}{E \sum_{j,l:|j-l|=1} [1 - e^{ik(j-l)}] + Nhm}$$

Using $1 - \cos(x) \leq \frac{1}{2}x^2$, we have $\sum_{j,l:|j-l|=1} [1 - e^{ik(j-l)}] \leq cNk^2$. Thus

$$1 \geq \frac{1}{\beta} \frac{1}{N} \sum_k \frac{m^2}{Eck^2 + hm}$$

Up to this point we have been working in a finite volume. So $m = m(h,L)$. Let $m(h) = \limsup_{L \to \infty} m(h,L)$. By letting $L \to \infty$ along a subsequence which attains the limsup, we have

$$1 \geq \frac{1}{\beta} \int d^2k \frac{m(h)^2}{Eck^2 + hm(h)}$$

where the integral is over the usual $[0, 2\pi]^2$ square and the convergence of the sum to an integral is the usual convergence of a Riemann sum to an integral. To finish the proof, let $m_0 = \limsup_{h \to 0^+} m(h)$ and let $h \to 0^+$ along a sequence which attains the limsup. Then we get

$$1 \geq \frac{1}{\beta} \int d^2k \frac{m_0^2}{Eck^2}$$

Since $\int d^2k \frac{1}{k^2} = \infty$, this implies $m_0 = 0$

$\square$

The theorem says that if we try to break the symmetry by imposing a field and then letting the field go to zero, the symmetry remains unbroken in the sense that the average manetization is zero. A stronger result has been proved by Dobrushin and Shlosman for both the rotator and Heisenberg models which says that every infinite volume probability measure you can get is invariant under the symmetry group.

Just because there is a continuous symmetry we should not automatically conclude there is no symmetry breaking in two dimensions. Consider the "cigar" model defined as follows. σ_i now takes values on the cigar:

$$\{(x,y,z) : \epsilon(x^2+y^2) + z^2 = 1\}$$

with $\epsilon < 1$. The dot product $\sigma_i \cdot \sigma_j$ is maximized when $\sigma_i = \sigma_j = \pm\hat{z}$ where $\hat{z}$ is the unit vector in the z direction. There are only two ground states in this model: the states with all spins parallel and pointing either in the $\hat{z}$ or $-\hat{z}$ directions. It is possible to use the Peierls argument to show this model has a phase transition.

Exercise: In the cigar model impose boundary conditions in which the boundary spins are fixed to be $\hat{z}$. Use the Peierls argument to prove that at low temperatures the expected value of σ_0^z is bounded away from 0 uniformly in the volume. ($\sigma_i = (\sigma_i^x, \sigma_i^y, \sigma_i^z)$).

Exercise: Our proof of the Mermin Wagner theorem was for the rotator model with a nearest neighbor interaction. This proof works for a more general class of models. For example, we could take the Hamiltonian to include next nearest neighbors with a different coupling:

$$H = -\sum_{<ij>} \cos(\theta_i - \theta_j) - \lambda \sum_{i,j:|i-j|=\sqrt{2}} \cos(\theta_i - \theta_j) - h \sum_j \cos(\theta_j) \qquad (7)$$

Prove the Mermin Wagner theorem for this model.

6.3 Symmetry breaking in three dimensions

We end this section with a discussion of pure states and their relation to decay of correlation functions.

To motivate this consider the two dimensional Ising model. We have seen that using $+$ and $-$ boundary conditions we get different infinite volume limit states. Denote them by $< \quad >_+$ and $< \quad >_-$. Let

$$< \quad > = \frac{1}{2}[< \quad >_+ + < \quad >_-]$$

This is the state you get if you use free or periodic boundary conditions. For low temperatures there is a constant $m > 0$ such that

$$< \sigma_i >_\pm = \pm m$$

We have

$$< \sigma_i \sigma_j >_+ \approx < \sigma_i >< \sigma_j >_+ = m^2$$

if i and j are far apart. In fact

$$< \sigma_i \sigma_j >_+ - < \sigma_i >< \sigma_j >_+ \approx ce^{-|i-j|/\xi}$$

EXPLAIN THIS

However this is not true for $< \quad >$. **SHOW THIS**

To explain the difference we introduce the idea of a pure state. There is a way to define infinite volume Gibbs states directly called the DLR equations. We will not go into it, but will only use the fact that there is a way to talk about infinite volume states other than by actually taking limits of finite volume states and it have the property that a convex combination of two infinite volume Gibbs states is an infinite volume Gibbs state.

The set of infinite volume Gibbs states is a convex. The extreme points are called *pure states*. So a Gibbs state is pure if you cannot write it as a non-trivial convex combination of two different Gibbs states.

Truncated correlations decay for pure states but not in general for mixed states.

6.4 The Kosterlitz-Thouless phase in 2 dimensions

Just because there is no symmetry breaking does not mean there is no phase transition. For the rotator model, there is a low temperature phase known as the Kosterlitz-Thouless phase in which correlation functions have power lay decay.

Theorem 2. *("easy") In both the rotator and Heisenberg models, if β is sufficiently small, then the truncated correlation functions decay exponentially. In particular, there are constants C and ξ which depend on β but not on the volume such that*

$$| < \sigma_j \cdot \sigma_l > | \leq C \exp(-|j-l|/\xi)$$

Furthermore ξ goes to 0 as $\beta \to 0$.

At low temperatures the behavior (some proven and some conjectured) is drastically different.

Theorem 3. *(Frohlich, Spencer) In the rotator model, for sufficiently large β, there are constants C and β' which depend on β but not on the volume such that*

$$< \sigma_j \cdot \sigma_l > \geq C[1 + |j-l|^{-1/2\pi\beta'}]$$

Furthermore β' goes to ∞ as $\beta \to \infty$.

The corresponding lower bound was proved earlier:

Theorem 4. *(McBryan, Spencer) In the rotator model, for all $\beta > 0$ and $\epsilon > 0$ there is a constant C' which does not depend on the volume such that*

$$< \sigma_j \cdot \sigma_l > \leq C'[1 + |j - l|^{-1/(2\pi+\epsilon)\beta}]$$

Proof: We sketch the proof. See McBryan, Spencer, Communications in Mathematical Physics **53**, 299-302 (1977) for the details.

We take $j = 0$ and use the representation in terms of polar angles. So we want to bound

$$\begin{aligned} < \cos(\theta_l - \theta_0) > &= \frac{1}{Z} \prod_i \int_0^{2\pi} d\theta_i \, \exp[\beta \sum_{<jk>} \cos(\theta_j - \theta_k)] \cos(\theta_l - \theta_0) \\ &= \frac{1}{Z} \prod_i \int_0^{2\pi} d\theta_i \, \exp[\beta \sum_{<jk>} \cos(\theta_j - \theta_k)] \exp(i(\theta_l - \theta_0)) \end{aligned}$$

We do a complex translation :

$$\theta_j \to \theta_j + i a_j$$

where a_j is a real valued function on the lattice that we will define shortly. There are no poles and the contributions from the vertical sides of the rectangle cancel by periodicity. So the integral is not changed. We have

$$\exp(i(\theta_l - \theta_0)) \to \exp(-(a_l - a_0)) \exp(i(\theta_l - \theta_0))$$

and this has absolute value bounded by $\exp(-(a_l - a_0))$. We also use

$$\exp[\cos(\theta_j - \theta_k)] \to \exp[\cosh(a_j - a_k)\cos(\theta_j - \theta_k) + i \sinh(a_j - a_k)\sin(\theta_j - \theta_k)]$$

which has absolute value bounded by

$$\exp[\cosh(a_j - a_k)\cos(\theta_j - \theta_k)]$$

So

$$\begin{aligned} < \cos(\theta_l - \theta_0) > &\leq \exp(-(a_l - a_0)) \frac{1}{Z} \prod_i \int_0^{2\pi} d\theta_i \, \exp[\beta \sum_{<jk>} \cosh(a_j - a_k)\cos(\theta_j - \theta_k)] \\ &= \exp(-(a_l - a_0)) < \exp[\beta \sum_{<jk>} (\cosh(a_j - a_k) - 1)\cos(\theta_j - \theta_k)] > \\ &\leq \exp(-(a_l - a_0)) \exp[\beta \sum_{<jk>} (\cosh(a_j - a_k) - 1)] \end{aligned}$$

The function a_j will have the property that $|a_j - a_k| \leq 4\beta^{-1}$ for nearest neighbor j, k. So we can bound $\cosh(a_j - a_k) - 1$ by $\frac{1}{2}(1+\epsilon)(a_j - a_k)^2$ where we can make ϵ as small as we want by making β suf. large.

$$\exp(-(a_l - a_0)) \exp[\beta \sum_{<jk>} (\cosh(a_j - a_k) - 1)$$

$$\leq \exp(-(a_l - a_0)) \exp[\frac{1}{2}\beta(1+\epsilon) \sum_{<jk>} (a_j - a_k)^2]$$

$$= \exp(-(a_l - a_0)) \exp[\frac{1}{2}\beta(1+\epsilon)(a, -\Delta a)]$$

Now we define a. Let $C(x)$ be the fundamental solution of the discrete Laplace equation:

$$(-\Delta C)(x) = \delta_0(x) \tag{8}$$

with $C(0) = 0$. It grows asymptotically like $C(x) \approx -\frac{1}{2\pi} \log |x|$. We then take $a_j = \beta^{-1}(C(j) - C(j-l))$. So $-\Delta a = \beta^{-1}(\delta_0 - \delta_l)$

Then $(a, -\Delta a) = \beta^{-1}(a, \delta_0 - \delta_l) = \beta^{-1}(a_0 - a_l)$ so our bound is

$$\exp(-(a_l - a_0)) \exp[\beta\frac{1}{2}(1+\epsilon)(a, -\Delta a)] = \exp(-(a_l - a_0)) \exp[\frac{1}{2}(1+\epsilon)(a_0 - a_l)] =$$
$$\exp[-\frac{1}{2}(1-\epsilon)(a_0 - a_l)] = \exp[\frac{1}{2}(1-\epsilon)\beta^{-1}(C(l) + C(-l))]$$

For large l $C(l) \approx -\frac{1}{2\pi} \log |l|$. So the bound is $\exp[-(1-\epsilon)\beta^{-1} \log |l|/2\pi] = l^{-1/2\pi beta'}$ where $\beta' = (1-\epsilon)\beta$.

$\square$

It is conjectured that the correct power is $\frac{1}{2\pi\beta}$. The two theorems say that the correlation length is finite at high temperatures but infinite at low temperatures. So there is some sort of phase transition that is not accompanied by spontaneous symmetry breaking.

For the Heisenberg model the low temperature behavior is believed to be quite different. It is believed that in two dimensions there is exponential decay of the truncated correlation functions at all temperatures! In other words, the model remains in a high temperature phase no matter how low the temperature. This is related to "asymptotic freedom," for which the 2004 Nobel prize was awarded to David Gross, David Politzer, and Frank Wilczek.