1.1 Sequences

Definition of convergence: We say a sequence a_n converges to L (denoted $a_n \to L$) if for all $\varepsilon > 0$, there exists an N such that for all $n > N$, $|a_n - L| < \varepsilon$.

In what follows, n is approaching infinity, c is any constant, and p is a positive constant.

Convergent sequences:

- $\frac{1}{n} \to 0$
- $\frac{\ln(n)}{n} \to 0$
- $\frac{(-1)^n}{n} \to 0$
- $c - \frac{1}{n} \to c$
- $c^n \to 0$ if $|c| < 1$
- $c^{-n} \to 0$ if $|c| > 1$
- $p^{1/n} \to 1$
- $\sqrt[n]{n} \to 1$
- $\left(1 + \frac{c}{n}\right)^n \to e^c$
- $\frac{c^n}{n!} \to 0$

1.2 Hierarchy of Functions

We consider how fast different types of functions approach infinity.

$$\cdots < \ln(\ln(n)) < \ln(n) < n^{1/k} < n^k < k^n < n! < n^n < n^{n^n} < \cdots$$

1.3 Series

- **Divergence Test**

  If $a_n \not\to 0$, then $\sum_{n=1}^{\infty} a_n$ diverges. In other words, if $\lim_{n\to\infty} a_n$ does not exist or $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{n=1}^{\infty} a_n$ will diverge.

- **Geometric Series**

  $a + ar + ar^2 + ar^3 + \cdots + ar^n + \cdots = \sum_{k=0}^{\infty} ar^k = \dfrac{a}{1-r} \quad |r| < 1, a \neq 0$

- **Harmonic Series**

  The Harmonic Series $\sum_{n=1}^{\infty} \dfrac{1}{n} = 1 + \dfrac{1}{2} + \dfrac{1}{3} + \dfrac{1}{4} + \cdots$ diverges.

- ***p*-Series Test**

  The p-series

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \to \begin{cases} \text{converge} & \text{if } p > 1 \\ \text{diverge} & \text{if } p \leq 1 \end{cases}$$

- **Integral Test**

  Let $\{a_n\}$ be a sequence of positive terms. Suppose f is a continuous, positive, decreasing function, and that $a_n = f(n)$. Then the series $\sum_{n=1}^{\infty} a_n$ and the integral $\int_1^{\infty} f(x)\,dx$ both converge or both diverge.

- **Comparison Test**

  Let $\sum_{n=1}^{\infty} a_n, \sum_{n=1}^{\infty} b_n, \sum_{n=1}^{\infty} c_n$ be series with nonnegative terms. Assume there exists an $M > 0$ such that for all $n > M$, $a_n \leq b_n \leq c_n$. Then

  (i) If $\sum_{n=1}^{\infty} c_n$ converges, then $\sum_{n=1}^{\infty} b_n$ also converges.

  (ii) If $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} b_n$ also diverges.

- **Limit Comparison Test**
  Let $\sum a_n$ and $\sum b_n$ be series with strictly positive terms.

  1. If $\lim_{n\to\infty} \frac{a_n}{b_n} = c > 0$, then $\sum a_n$ and $\sum b_n$ both converge or both diverge.
  2. If $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ and $\sum b_n$ converges, then $\sum a_n$ converges as well.
  3. If $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$ and $\sum b_n$ diverges, then $\sum a_n$ diverges as well.

- **Ratio Test**
  Let $\sum a_n$ be a series with strictly positive terms, and suppose $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \mathcal{R}$.

  (a) If $\mathcal{R} < 1$, then $\sum a_n$ converges.
  (b) If $\mathcal{R} > 1$, then $\sum a_n$ diverges.
  (c) If $\mathcal{R} = 1$, the test is *inconclusive*.

- **Root Test**
  Let $\sum a_n$ be a series with $a_n \geq 0$, and suppose $\lim_{n\to\infty} \sqrt[n]{a_n} = \mathcal{R}$.

  (a) If $\mathcal{R} < 1$, then $\sum a_n$ converges.
  (b) If $\mathcal{R} > 1$, then $\sum a_n$ diverges.
  (c) If $\mathcal{R} = 1$, the test is *inconclusive*.

- **Alternating Series Test**
  The alternating series

$$\sum_{n=1}^{\infty} (-1)^{n-1} b_n = b_1 - b_2 + b_3 - b_4 + \cdots$$

  Converges if all three of the following conditions are satisfied:

  1. All the b_n's are positive
  2. $b_{n+1} \leq b_n$ for all $n > N$
  3. $\lim_{n\to\infty} b_n = 0$

- **Absolute Convergence Test**
  If $\sum_{n=1}^{\infty} |a_n|$ converges, then $\sum_{n=1}^{\infty} a_n$.

1.4 Strategies

1. The first thing we should check is the Divergence Test. Unless $a_n \to 0$, the series diverges.
2. If we can spot a Geometric Series, then we know right away if it converges or diverges since $\sum ar^n$ converges only if $|r| < 1$; otherwise it diverges.
3. p-series are also easy to spot and check convergence. $\sum 1/n^p$ converges if $p > 1$; otherwise it diverges.
4. If the series has nonnegative terms: Try the Integral Test, Ratio Test, or Root Test. Try comparing to a known series with the Comparison Test or the Limit Comparison Test.
5. Series with some negative terms: Does $\sum |a_n|$ converge? If yes, so does $\sum a_n$ since absolute convergence implies convergence.
6. Alternating series: $\sum a_n$ converges if the series satisfies the conditions of the Alternating Series Test.