On the Fourier coefficients of modular forms

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The goal of this paper is to discuss a Newton-Hodge inequality for modular forms, i.e., a theorem giving lower bounds for the divisibility of Fourier coefficients of modular eigenforms by primes which divide their level. More precisely, for a prime number p and an integer N prime to p we consider the characteristic polynomial of the Hecke operator U_p on the space $S_{k+2}(\Gamma_1(p^nN))$ of modular forms for the congruence subgroup $\Gamma_1(p^nN)$ of $\mathrm{SL}_2(\mathbf{Z})$. The main theorem bounds the Newton polygon of this polynomial (with respect to the p-adic valuation of $\mathbf{Q}$) from below by an explicit polygon defined in terms of the genus and number of cusps of the modular curve $X_1(N)$. Concretely, this means that only so many of the eigenvalues of U_p can be p-adic units; if the maximum possible number are units, then the rest are divisible by p and only so many of those are exactly divisible by p, etc. The main technique is a motivic variation of theorems of Mazur, Ogus, Illusie and Nygaard on the Katz conjecture (according to which the Newton polygon of Frobenius on crystalline cohomology is bounded in terms of dimensions of Hodge cohomology groups) and a computation of these Hodge groups using logarithmic schemes. We get new information because the relevant Hodge filtration is not of type $(k+1,0)$, $(0,k+1)$ as usual, but rather of type $(k+1,0)$, $(k,1)$, …, $(1,k)$, $(0,k+1)$.

It is worth taking a moment to explain the origin of this theorem. If p is a prime number congruent to $3 \pmod 4$ and K denotes the field $\mathbf{F}_p(j)$, where j is an indetermi-

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nate, then there is an elliptic curve E defined over K so that the Hasse-Weil L-function of E satisfies

$$L(E/K,s) = \prod_{\alpha} (1 - \alpha p^{-s})$$

where the product ranges over all eigenvalues α of the Hecke operator U_p on $S_3(\Gamma_0(p), \left(\frac{-}{p}\right))$, the space of modular forms of weight 3 and character $\left(\frac{-}{p}\right)$ (the Legendre symbol) for the congruence subgroup $\Gamma_0(p)$ [U1]. The values and derivatives of the L-function at $s = 1$ are then powers of $\log p$ times rational numbers whose valuations at p are bounded below in terms of those of the α. On the other hand, if r denotes the order of vanishing of the L-function at $s = 1$, then the Birch and Swinnerton-Dyer conjecture predicts the following formula for the value at 1 of its r-th derivative:

$$\frac{1}{r!} L^{(r)}(E/K,1) = \frac{|\text{Ш}| R \tau}{|E(K)_{tor}|^2}$$

where vertical bars indicate the order of a group. Here Ш is the (conjecturally finite) Tate-Shafarevitch group attached to E, R (the regulator) is a power of $\log p$ times a rational number integral at p and τ (the Tamagawa number) is an explicitly computable rational number. Thus the right hand side is $(\log p)^r$ times a rational number and one can estimate the power of p occuring in this rational number. Comparing this estimate with the estimate above for the denominator of the left hand side, and using that similar results hold with K replaced by its finite extensions $K \otimes \mathbf{F}_q$, one finds a Newton-Hodge-style lower bound on the valuations of the α. In particular, roughly speaking at most 1/4 of them can be units at p. We prove this prediction of the conjecture of Birch and Swinnerton-Dyer and extend it to a wide class of modular forms.

Here are the contents of the paper. In addition, the reader can find a sketch of the proof of the main theorems at the end of Section 1.

Contents

It is a pleasure to thank Arthur Ogus for introducing me to the magic of log structures and Luc Illusie for his interest in and comments on this work.

1. Statement of the main theorems Throughout the paper, we view the field of algebraic numbers $\overline{\mathbf{Q}}$ as a subfield of the complex numbers $\mathbf{C}$. Recall that the *conductor* of a Dirichlet character $\psi : (\mathbf{Z}/M\mathbf{Z})^\times \to \mathbf{C}$ is by definition the smallest positive integer M' such that ψ factors through the natural map $(\mathbf{Z}/M\mathbf{Z})^\times \to (\mathbf{Z}/M'\mathbf{Z})^\times$. If $M = M'$ then ψ is by definition *primitive*; in general, there is a unique primitive character $\psi' : (\mathbf{Z}/M'\mathbf{Z})^\times \to \mathbf{C}$ through which ψ factors.

Now for any positive integer M and non-negative integer k, let $S = S_{k+2}(\Gamma_0(M), \psi)$ be the (complex) vector space of modular forms of weight $k+2$ and character $\psi : (\mathbf{Z}/M\mathbf{Z})^\times \to$

$\mathbf{C}$ for the congruence subgroup $\Gamma_0(M)$ of $\mathrm{SL}_2(\mathbf{Z})$. Recall that a form $f \in S$ is *primitive* if it is new (i.e., is orthogonal under the Petersson inner product to all forms coming from lower levels), is an eigenform for all Hecke operators T_ℓ for primes $\ell \not| M$, and is normalized (i.e., has first Fourier coefficient equal to 1); in this case, f is automatically an eigenform for all U_ℓ with $\ell | M$. For any $f \in S$ which is an eigenform for all T_ℓ with $\ell \not| M$ there is a unique primitive form of some level dividing M with the same Hecke eigenvalues as f; by definition, the level of this form is the *conductor* of f.

We recall a result summarizing what is known about these Hecke eigenvalues. Suppose p is a prime number and f is a normalized eigenform of weight $k+2$ and conductor $p^m M$ (with M prime to p) which may not be primitive, but which is "primitive at p" in the sense that the largest power of p dividing the conductor of f is p^m. Suppose further that the character ψ of f has conductor $p^{m'} M'$ with M' prime to p. If $m = 0$ and $T_p f = af$, write $1 - aT + \psi(p)p^{k+1}T^2 = (1-\alpha T)(1-\beta T)$ with $\alpha, \beta \in \mathbf{C}$; if $m > 0$, write $U_p f = \alpha f$. Then we have the following result which combines work of Deligne (for $m = 0$) and of various authors starting with Hecke (for $m > 0$); see [D1] and [Mi], 4.6.17 for proofs.

Proposition 1.1. *The complex number α is an algebraic integer and*

$$\begin{cases} \alpha\overline{\alpha} = p^{k+1} & \text{if } m = m' \\ \alpha^2 = \psi'(p)p^k & \text{if } m = 1,\ m' = 0 \\ \alpha = 0 & \text{if } m > 1,\ m > m' \end{cases}$$

Here ψ' is the primitive character attached to ψ.

This result says a lot about the possible valuations of α: in particular, the archimedean valuations of α are completely determined and at any finite prime of $\overline{\mathbf{Q}}$ not dividing p, α is a unit. Moreover, for any valuation v of $\overline{\mathbf{Q}}$ dividing p, $v(\alpha)$ is completely determined in the last two cases and in the first case, $0 \leq v(\alpha) \leq k+1$ (where v is normalized so that $v(p) = 1$). Noting that if $U_p f = \alpha f$ then the complex conjugate form $\overline{f}$ satisfies $U_p \overline{f} = \overline{\alpha}\overline{f}$ with $\overline{f} \in S_{k+2}(\Gamma_0(p^m M), \overline{\psi})$, we can say that when $m' > 0$, at most half of the eigenvalues of U_p on $S_{k+2}(\Gamma_0(p^m M), \psi) \oplus S_{k+2}(\Gamma_0(p^m M), \overline{\psi})$ are units.

This trivial observation has a smooth rephrasing in terms of Newton polygons. We define the Newton polygon (for a valuation v) of a polynomial with algebraic integer coefficients $P(T) = 1 + \cdots + a_d T^d = \prod_{i=1}^{d} (1 - \alpha_i T)$ where $v(\alpha_1) \leq \cdots \leq v(\alpha_d)$ as the graph of the function defined on the interval $[0, d]$ whose value on integers is defined by the formula $i \mapsto \sum_{j=1}^{i} v(\alpha_j)$ and which is extended by piecewise linearity. We also define the Hodge polygon of a collection of non-negative integers $(l_0, \ldots, l_s)$ as the graph of the function F defined on the interval $[0, \sum l_j]$ with $F(0) = 0$, $F(\sum_{j=0}^{i} l_j) = \sum_{j=0}^{i} j l_j$, and extended by piecewise linearity. The Hodge polygon takes its name from the fact that the numbers l_j often arise from geometry, as the dimensions of certain Hodge cohomology groups. (See Katz [K1] for a readable introduction to Newton and Hodge polygons.) Then the observation becomes that when p divides the conductor of ψ (i.e., $m' > 0$), then the Newton polygon of the characteristic polynomial of U_p on $S_{k+2}(\Gamma_0(p^m M), \psi) \oplus S_{k+2}(\Gamma_0(p^m M), \overline{\psi})$ lies on or above the Hodge polygon of the collection $(d, 0, \ldots, 0, d)$ where d is the dimension of $S_{k+2}(\Gamma_0(p^m M), \psi)$ and 0 is repeated k times.

We are now going to present our theorems on the valuations of coefficients of modular forms, which are strengthenings of the observation above. For a fixed prime number p, non-negative integers k and m, a positive integer N prime to p, and a Dirichlet character

ψ modulo $p^m N$, introduce the Hecke polynomial

$$E(k,m,N,\psi) = \begin{cases} \det\left(1 - U_p T | S_{k+2}(\Gamma_0(p^m N),\psi)\right) & \text{if } m > 0 \\ \det\left(1 - T_p T + \psi(p)p^{k+1}T^2 | S_{k+2}(\Gamma_0(N),\psi)\right) & \text{if } m = 0 \end{cases}$$

Note that these polynomials will in general have eigenvalues of non-primitive (i.e., old) forms among their inverse roots; however, if p^m divides the conductor of ψ, then all forms contributing to $E(k,n,N,\psi)$ are primitive at p. Now the theory of newforms (i.e., an analysis of the various maps $S_{k+2}(\Gamma_0(p^{m-1}N),\psi) \hookrightarrow S_{k+2}(\Gamma_0(p^m N),\psi)$) reduces questions of valuations of eigenvalues of U_p and T_p on all modular forms to the case of p-primitive forms. Note also that Proposition 1.1 gives a formula for the valuation of the eigenvalue of U_p for any p-primitive form whose conductor is divisible by a higher power of p than the conductor of its character is (i.e., $m > m'$). Thus we will only be concerned with those forms for which $m = m'$ and thus only with those $E(k,n,N,\psi)$ where p^m divides the conductor of ψ.

The first result concerns the case when $N > 4$. In the theorem below, ϕ is Euler's function: $\phi(m) =$ the order of $(\mathbf{Z}/m\mathbf{Z})^\times$.

Theorem 1.2. *Fix an arbitrary prime number p, integers k and n with $0 \le k < p$, $n > 0$ and an integer $N > 4$ prime to p. Let g be the genus of the modular curve $X_1(N)$, c the number of cusps on this curve, and set $w = g - 1 + c/2$ (which is an integer as $N > 4$). Then the Newton polygon, with respect to the p-adic valuation of $\mathbf{Q}$, of the polynomial*

$$\prod_{0 \le m \le n} \prod_{\substack{\psi:(\mathbf{Z}/p^m N\mathbf{Z})^\times \to \mathbf{C} \\ p^m | cond(\psi)}} E(k,m,N,\psi)$$

(where the second product is over characters modulo $p^m N$ whose conductor is divisible by

p^m) lies on or above the Hodge polygon associated to the integers

$$\begin{aligned} l_0 =& \phi(p^n)\left(\frac{w}{2}\right)(p^n+2k) - \frac{\phi(p^n)c}{2} + \begin{cases} 1 & \text{if } k=0 \\ 0 & \text{if } k\neq 0 \end{cases} \\ l_1 = \cdots = l_k =& \phi(p^n)w(p^n-2) \\ l_{k+1} =& \phi(p^n)\left(\frac{w}{2}\right)(p^n+2k) - \frac{\phi(p^n)c}{2} + \begin{cases} 1 & \text{if } k=0 \\ 0 & \text{if } k\neq 0 \end{cases} \end{aligned}$$

Moreover, these two polygons have the same endpoints.

Remarks: 1) While the theorem can be refined somewhat (see below), in general our method does not allow us to prove a theorem on the Newton polygons of the individual $E(k, m, N, \psi)$. This restriction, as well as the need to take $k < p$, will be explained below when we sketch the proof.

2) Using the remarks immediately preceding the theorem, one can deduce a Newton-Hodge inequality for the characteristic polynomial of U_p on the whole space $S_{k+2}(\Gamma_1(p^n N))$.

3) Our method applies also to the case where $n = 0$, but the result is just the trivial observation above. This is also the case when $k = 0$.

4) As we are taking a product over various non-trivial characters, the theorem really concerns modular forms on $\Gamma_1(p^n N)$. Recall that the $\Gamma_1(p^n N)$ moduli problem is the product of the problems $\Gamma_1(p^n)$ and $\Gamma_1(N)$ (see [KM] 3.5.1). In Theorem 1.2, $\Gamma_1(N)$ can be replaced by any representable moduli problem of finite level prime to p, e.g., $\Gamma(N)$ for $N \geq 3$ and $(p, N) = 1$. The formulae for the lengths of the sides of the Hodge polygon are the same except that g and c are now taken to be the genus and number of cusps of the curve associated to the new moduli problem.

We can obtain finer results, where the product of Hecke polynomials ranges only over certain characters: if $(\mathbf{Z}/p^mN\mathbf{Z})^\times$ is written as a product of cyclic groups of prime power order, then we can take the product only over those characters with fixed restrictions to the various factors $\mathbf{Z}/\ell^e\mathbf{Z}$ (where ℓ is a prime $\neq p$). The next result makes this explicit for the direct factor $(\mathbf{Z}/p\mathbf{Z})^\times$ of $(\mathbf{Z}/p^mN\mathbf{Z})^\times$. To this end, fix a p-adic valuation v of the field of $(p-1)^{st}$ roots of unity $\mathbf{Q}(\mu_{p-1})$, normalized so that $v(p) = 1$. Let $\chi : (\mathbf{Z}/p\mathbf{Z})^\times \to \mathbf{C}$ be the unique character such that $v(\chi(x)-x) > 0$ for all $x \in \mathbf{Z}$ prime to p. (Throughout the paper, χ will be identified with the Teichmüller character via the embedding $\mathbf{Q}(\mu_{p-1}) \to \mathbf{Q}_p$ associated to v.) Then any Dirichlet character $\psi : (\mathbf{Z}/p^mN\mathbf{Z})^\times \to \mathbf{C}$ can be written uniquely as $\chi^a\eta\theta$ with $0 \leq a \leq p-2$, where $\eta : (\mathbf{Z}/N\mathbf{Z})^\times \to \mathbf{C}$ and $\theta : (1 + p\mathbf{Z}_p) \to \mathbf{C}$ are characters of finite order.

Theorem 1.3. *Fix an arbitrary prime number p, integers k, n, N, and a with $0 \leq k < p$, $n > 0$, $N > 4$ prime to p, and $0 \leq a \leq p-2$. Let g be the genus of the modular curve $X_1(N)$, c the number of cusps on this curve, and set $w = g - 1 + c/2$. Let $a' = 0$ if $a = 0$ and let $a' = p - 1 - a$ if $a \neq 0$. Then the Newton polygon, with respect to the valuation v fixed above, of the polynomial*

$$H(k, n, N, a) = \prod_{0 \leq m \leq n} \prod_{\substack{\psi:(\mathbf{Z}/p^mN\mathbf{Z})^\times \to \mathbf{C} \\ \psi = \chi^a\eta\theta \\ p^m | cond(\psi)}} E(k, m, N, \psi)$$

(where the second product is over characters modulo p^mN whose conductor is divisible by p^m and whose restriction to $(\mathbf{Z}/p\mathbf{Z})^\times$ is χ^a) lies on or above the Hodge polygon associated

to the integers

$$l_0 = \begin{cases} \left(p^{2n-1} - p + 2\right) w/2 - p^{n-1}c/2 + 1 & \text{if } k = a = 0 \\ \left(p^{2n-1} - p + 2 + 2kp^{n-1} + 2a\right) w/2 - p^{n-1}c/2 & \text{otherwise,} \end{cases}$$

$$l_1 = \cdots = l_k = \begin{cases} (p^{2n-1} - 2p^{n-1} - p + 2)w & \text{if } a = 0 \\ \\ (p^{2n-1} - 2p^{n-1} + 1)w & \text{if } a \neq 0 \end{cases}$$

$$l_{k+1} = \begin{cases} \left(p^{2n-1} - p + 2\right) w/2 - p^{n-1}c/2 + 1 & \text{if } k = a = 0 \\ \left(p^{2n-1} - p + 2 + 2kp^{n-1} + 2a'\right) w/2 - p^{n-1}c/2 & \text{otherwise,} \end{cases}$$

Moreover, these two polygons have the same endpoints.

Remarks: 1) Remarks 3 and 4 following Theorem 1.2 also apply here.

2) Theorem 1.3 is the basic result: Theorem 1.2 is obtained as a corollary by summing over the various powers χ^a of the Teichmüller character.

3) Of course the theorem depends on the choice of v: changing v changes the valuations of the inverse roots of each E and it changes the value of a in the decomposition $\chi^a\eta\theta$ of a character ψ. This dependence can be made less apparent by working with modular forms with coefficients in $\overline{\mathbf{Q}_p}$.

4) The values of l_0 give upper bounds on the number of forms whose eigenvalues for U_p are p-adic units, i.e., which are *ordinary* in the terminology of Hida. In fact, comparing the theorem with Hida's results, one finds that in many cases the Newton polygon of the Hecke polynomial lies strictly above the Hodge polygon defined here. We will return to more precise versions of this point in another paper.

When $p > 3$ one can also obtain results (although with different formulae) for the cases $N \leq 4$ by the methods used to prove Theorem 1.3. At the suggestion of the referee, we have included a statement of these results, and indications of their proof, in Section 7.

We give an example to illustrate the difference between the trivial observation and the theorem. The most attractive case is $n = N = 1$, $k > 0$ and $a \neq 0$. Then the Hecke polynomial $H(k, 1, 1, a)$ is just $E(k, 1, 1, \chi^a)$ and all of the forms contributing to it are new at p. Below is a diagram of the case $p \equiv 11 \pmod{24}$, $k = 1$ and $a = (p-1)/2$. The numbers above the segments indicate the length of their projections to the x axis; the numbers below indicate their slopes. The diagram illustrates the remark in the introduction that in this situation, roughly speaking at most $1/4$ of the eigenvalues of U_p can be units.

We conclude this section with a sketch of the proof of Theorems 1.3 and 1.4. In Section 2, we construct a smooth projective variety $\tilde{X}$ over $\mathbf{F}_p$, the field of p elements, and a projector $\Pi \in \mathbf{Q}_p[\mathrm{Aut}_{\mathbf{F}_p} X]$ such that the characteristic polynomial of Frobenius on the part of crystalline cohomology of $\tilde{X}$ cut out by Π is the polynomial H appearing in Theorem 1.3. The coefficients of Π lie in $\mathbf{Z}_p$ if and only if $p > 2$ and $k < p$ and in this case we think of $(\tilde{X}, \Pi)$ as a "motive with p-integral coefficients." The argument here is a small variation on that of Scholl [S], combined with previous cohomology calculations of the author [U2]. In Section 3, we prove a variation of the Katz conjecture, saying that the Newton polygon of Frobenius on the part of crystalline cohomology cut out by a projector is bounded below by a Hodge polygon, defined in terms of the dimensions of Hodge cohomology groups cut out by the same projector. Because we are applying our projector to vector spaces over a field of characteristic p, it must have p-integral coefficients and this is the source of the restriction in the theorems that $k < p$; it is also the reason we must take a product of Hecke polynomials rather than working with the individual $E(k, n, N, \psi)$. (Moreover, it forces $p > 2$; we give a different proof for the case $p = 2$, $k = 1$ in Section 6.) The argument in this section is an essentially formal variation of Nygaard's proof of the Katz conjecture.

The real work takes place in the next two sections where we compute the relevant Hodge cohomology groups. Specifically, in Section 4 we construct a logarithmic scheme $X^\times$ (in the sense of Kato [Ko]) which is closely related to $\tilde{X}$ and on whose cohomology Π acts. The main theorem of the section gives the relation between the Π-part of the Hodge cohomology of $\tilde{X}$ and the Π-part of the (log) Hodge cohomology of $X^\times$. The main tools are the theory of log structures as developed in [Ko], some formal use of its predecessor and cousin mixed Hodge theory, and a computation based on the theory of toric varieties. In Section 5 we compute the Π-part of the Hodge cohomology of $X^\times$ in terms of sections of certain sheaves on an Igusa curve and find the dimensions of these groups, yielding Theorem 1.3.

Section 6 contains a somewhat different proof of Theorem 1.3 for weight 3 and any p based on a formula of Milne; this resolves the only relevant case of Theorem 1.3 for $p = 2$. In Section 7 we consider the cases $N \leq 4$, which involves a slight modification of the projector Π and the computation of invariants under the action of the Galois group of a certain covering of modular curves. This group has order divisible by 6, which forces the restriction $p > 3$.

2. Scholl's projector The goal of this section is to find a piece of cohomology on which Frobenius has characteristic polynomial equal to the Hecke polynomial $H(k, n, N, a)$ of Theorem 1.3. We find this cohomology by applying a minor variation of the projector of Scholl [S].

We retain the notations of the introduction: p is any prime, N is an integer prime to p and ≥ 5, n and k are integers; we do not need to assume $k < p$, but to avoid constantly making a special case, we assume $k > 0$; only trivial modifications in the discussion are needed for $k = 0$. Let $X_1(N)$ be the modular curve over $\mathbf{F}_p$ parameterizing generalised elliptic curves with a $\Gamma_1(N)$-structure (i.e., a point of exact order N) and let $I = Ig_1(p^n N)$ be the Igusa covering of level p^n over $X_1(N)$; I parameterizes generalised elliptic curves E with a $\Gamma_1(N)$-structure plus an Igusa structure, i.e., a point of "exact order p^n" on $E^{(p^n)}$, the range of the n-th iterate of the relative Frobenius of E. See [KM] Chapters 3 and 12 for more precise definitions and properties of these curves. We have a universal curve $\mathcal{E} \xrightarrow{\pi} I$ which is the pull-back of the universal curve over $X_1(N)$. We call the points of I representing singular elliptic curves *cusps* and the points representing supersingular elliptic curves the *supersingular points*; other points of I will be refered to as *ordinary*. The map π is smooth away from the cusps and the fibers of π over the cusps are Néron M-gons where $M|N$.

Let $f : X \to I$ be the k-fold fiber product of $\mathcal{E}$ over I. When $k > 1$, the variety X is not smooth over $\mathbf{F}_p$: there are singularities arising from the product of double points in the fibres of π. For $k > 1$, let $\tilde{X}$ denote the resolution of these singularities defined by Deligne ([D1], lemme 5.4 and lemme 5.5); this resolution is explained in detail in [S], §2 and, from another point of view, in Section 4. For $k = 1$, we set $\tilde{X} = X$; in all cases we have a map $\tilde{f} : \tilde{X} \to I$.

Let $C \subseteq I$ be the reduced subscheme of cusps and set $I^o = I \setminus C$, $X^o = \tilde{f}^{-1}(I^o)$. The fibers of $X^o \to I^o$ are k-fold self-products of elliptic curves. We obtain an action of $G = (\mathbf{Z}/N\mathbf{Z} \rtimes \mu_2)^k \rtimes S_k$ on X^o by letting the $\mathbf{Z}/N\mathbf{Z}$'s act by translation by the canonical points of order N, the μ_2's act by inversion in each copy of the elliptic curve and the symmetric group S_k act by permuting the factors in each fiber. This action extends to X and $\tilde{X}$ and covers the identity action on I. We also have the action of the diamond operators $\langle d \rangle = \langle d \rangle_p$ for $d \in (\mathbf{Z}/p\mathbf{Z})^\times$ on I: $\langle d \rangle$ sends the geometric point representing (E, P, i) (where E is an elliptic curve over $\overline{\mathbf{F}_p}$, $P \in E$ is a point of exact order N and $i : (\mathbf{Z}/p^n\mathbf{Z})^\times \to E^{(p^n)}$ is an Igusa structure of level p^n) to $(E, P, \chi(d)i)$ where χ is the

Teichmüller character; the same recipe gives the action of $\langle d\rangle$ on X and this action lifts to $\tilde{X}$.

Following Scholl, define a character $\epsilon : G \to \{\pm 1\}$ by setting $\epsilon|_{\mu_2} = id$, $\epsilon|_{\mathbf{Z}/N\mathbf{Z}} = 1$ and $\epsilon|_{S_k} = sgn$; let Π be the associated idempotent in the group ring $\mathbf{Z}[1/2Nk!][G]$. We note that Π has p-integral coefficients if and only if $p > 2$ and $k < p$. If V is a $\mathbf{Z}[1/2Nk!][G]$-module, we write $V(\epsilon)$ for ΠV. Recall that $\chi : (\mathbf{Z}/p\mathbf{Z})^\times \to \mathbf{Q}_p$ is the Teichmüller character, which we have identified with a character $(\mathbf{Z}/p\mathbf{Z})^\times \to \mathbf{Q}(\mu_{p-1})$. Fix a prime $\ell \neq p$ and a place λ of $\mathbf{Q}(\mu_{p-1})$ over ℓ; we deduce an embedding $\mathbf{Q}(\mu_{p-1}) \hookrightarrow \mathbf{Q}_\ell(\mu_{p-1})$ and we can identify χ with a character $\chi : (\mathbf{Z}/p\mathbf{Z})^\times \to \mathbf{Q}_\ell(\mu_{p-1})^\times$. If V is a $\mathbf{Q}_\ell(\mu_{p-1})$-vector space with $(\mathbf{Z}/p\mathbf{Z})^\times$ action, we write $V(\chi^a)$ for the χ^a eigenspace.

Proposition 2.1. *a) There is a canonical isomorphism*

$$H^*_{\text{ét}}(\tilde{X} \otimes \overline{\mathbf{F}_p}, \mathbf{Q}_\ell)(\epsilon) \cong H^1_{\text{ét}}(I \otimes \overline{\mathbf{F}_p}, \operatorname{Sym}^k R^1\pi_*\mathbf{Q}_\ell)$$

compatible with the actions of $\langle d\rangle$, $d \in (\mathbf{Z}/p\mathbf{Z})^\times$ *and* $\mathrm{Gal}(\overline{\mathbf{F}_p}/\mathbf{F}_p)$.

b) We have an equality of polynomials in T *with coefficients in* $\mathbf{Q}(\mu_{p-1})$*:*

$$\det\left(1 - Fr\, T | H^1_{\text{ét}}(I \otimes \overline{\mathbf{F}_p}, \operatorname{Sym}^k R^1\pi_*\mathbf{Q}_\ell(\mu_{p-1}))(\chi^a)\right) = H(k, n, N, a).$$

Proof: b) is a reiteration of the main theorem of [U2], broken down into eigenspaces for the $(\mathbf{Z}/p\mathbf{Z})^\times$ action; see p. 706. For a) we need to introduce another variety: let $X^* \to I$ be the (open) variety whose fiber over $x \in I$ is the connected component of the Néron model of $X^o \to I^o$. The fiber of X^* over a cusp is $\mathbf{G}_m^k$, and the inclusions $X^o \to X^* \to \tilde{X}$ are isomorphisms away from the fibers over the cusps. We also introduce the subgroup $G_k = {\mu_2}^k \rtimes S_k$ of G and the character $\epsilon_k : G_k \to \{\pm 1\}$, $\epsilon_k = \epsilon|_{G_k}$.

Consider the long exact (Gysin) sequence ([M2], VI.5.4b, p. 244):

$$\to H^{j-2}(C \times \mathbf{G}_m^k)(-1) \to H^j(X^*) \to H^j(X^o) \to H^{j-1}(C \times \mathbf{G}_m^k)(-1) \to$$

(where we write $H^j(V)(n)$ for $H^j_{\text{ét}}(V \otimes \overline{\mathbf{F}_p}, \mathbf{Q}_\ell(n))$ and $H^j(V)$ for $H^j(V)(0)$). We apply the idempotent associated to ϵ_k to this sequence. According to Scholl ([S],1.3.1),

$$H^{j-1}(C \times \mathbf{G}_m^k)(-1)(\epsilon_k) \cong H^{j-(k+1)}(C)(-k-1).$$

Recall that the fiber of π over any cusp x of I is a Néron M-gon with $M|N$, so the action of G on the fiber of $\tilde{f} : \tilde{X} \to I$ over x factors through the quotient $(\mathbf{Z}/M\mathbf{Z} \rtimes \mu_2)^k \rtimes S_k$. As the character ϵ also factors through this quotient, the proof of Scholl's Theorem 3.1.0 applies verbatim (with M replacing his n) and we have an isomorphism

$$H^j(X^*)(\epsilon_k) \cong H^j(\tilde{X})(\epsilon).$$

On the other hand, a standard computation, using the Leray spectral sequence, the Künneth formula, and [KM] 14.3.4.3, shows that

$$H^j(X^o)(\epsilon_k) \cong \begin{cases} H^1_{\text{ét}}(I^o \otimes \overline{\mathbf{F}_p}, \operatorname{Sym}^k R^1\pi_*\mathbf{Q}_\ell) & \text{if } j = k+1 \\ 0 & \text{otherwise} \end{cases}$$

Collating these results, we have

$$H^j(\tilde{X})(\epsilon) = 0 \qquad \text{unless } j = k+1, k+2$$

and an exact sequence

$$0 \to H^{k+1}(\tilde{X})(\epsilon) \to H^1_{\text{ét}}(I^o \otimes \overline{\mathbf{F}_p}, \operatorname{Sym}^k R^1\pi_*\mathbf{Q}_\ell) \to H^0(C)(-k-1) \to H^{k+2}(\tilde{X})(\epsilon) \to 0.$$

But a consideration of weights shows that the last map is zero, so $H^{k+2}(\tilde{X})(\epsilon)=0$ and

$$H^{k+1}(\tilde{X})(\epsilon)=\mathrm{Ker}\,\left(H^1_{\text{ét}}(I^o\otimes\overline{\mathbf{F}_p},\mathrm{Sym}^k\,R^1\pi_*\mathbf{Q}_\ell)\to H^0(C)(-k-1)\right).$$

Now $\mathrm{Sym}^k\,R^1\pi_*\mathbf{Q}_\ell$ restricted to C is $\mathbf{Q}_\ell(-k)$, so applying the Gysin sequence for the pair (I,C) and $\mathrm{Sym}^k\,R^1\pi_*\mathbf{Q}_\ell$, we find

$$\mathrm{Ker}\,\left(H^1_{\text{ét}}(I^o\otimes\overline{\mathbf{F}_p},\mathrm{Sym}^k\,R^1\pi_*\mathbf{Q}_\ell)\to H^0(C)(-k-1)\right)=H^1_{\text{ét}}(I\otimes\overline{\mathbf{F}_p},\mathrm{Sym}^k\,R^1\pi_*\mathbf{Q}_\ell).$$

This gives the isomorphism of the theorem. As all of the maps used in the proof are equivariant for the actions of Galois and $(\mathbf{Z}/p\mathbf{Z})^\times$, the proposition is proved. □

Now combining the proposition with a result of Katz and Messing ([KMe], Theorem 2), we find a piece of crystalline cohomology on which Frobenius has characteristic polynomial equal to the Hecke polynomial H:

Corollary 2.2.

$$\det\left(1-Fr\,T|\left(H^{k+1}_{cris}(\tilde{X}/\mathbf{Z}_p)\otimes\mathbf{Q}_p\right)(\epsilon,\chi^a)\right)=H(k,n,N,a)$$

3. A variation of a conjecture of Katz Fix a prime p and a smooth and proper variety X over a perfect field k of characteristic p. Let $W=W(k)$ be the ring of Witt vectors over k. Then we have the crystalline cohomology groups $H^r_{cris}(X/W)$, which are endowed with a canonical p-linear endomorphism Φ, induced by the absolute Frobenius of X. In fact, the pair $(H=H^r_{cris}(X/W)/torsion,\Phi)$ is an example of an *F-crystal* on k, i.e., a free, finite-rank W-module together with an injective semi-linear endomorphism.

Associated to an F-crystal we have two polygons: its Newton polygon and its abstract Hodge polygon. To define the Newton polygon we assume for simplicity that k is a finite field with p^f elements. (For the general case see [K1].) In this case, Φ^f is a W-linear endomorphism of H and the Newton polygon for the valuation v of (H, Φ) is by definition the Newton polygon, in the sense of §1, of the characteristic polynomial $\det(1 - \Phi^f T|H)$ computed with respect to a valuation $\frac{1}{f}v$. To define the abstract Hodge polygon, we note that since Φ is an injection, $\Phi(H)$ is a W-submodule of H of maximal rank; use elementary divisors to choose bases $v_1, \ldots, v_s$, $w_1, \ldots, w_s$ of H so that $\Phi v_j = p^{e_j} w_j$. Then set $\tilde{h}^i$ to be the number of times i occurs among the e_j. By definition, the $\tilde{h}^i$ are the abstract Hodge numbers of (H, Φ) and its abstract Hodge polygon is the Hodge polygon, in the sense of §1, of its abstract Hodge numbers.

It is an easy result of linear algebra, due to Mazur (see [K1], 1.4.1), that the Newton polygon of (H, Φ) lies on or above its abstract Hodge polygon, and that they have the same end points. Much more interesting is the result, conjectured by Katz and due in various forms to Mazur, Ogus, Illusie and Nygaard ([Ma], [BO] Chapter 8, [I] II.4, and [N]), that the abstract Hodge polygon of (H, Φ) lies on or above the Hodge polygon associated to the geometric Hodge numbers $h^i = \dim_k H^{r-i}(X, \Omega^i_X)$. The main result of this section is a generalization of this result where H is replaced by a piece of cohomology cut out by a projector in the group ring of the automorphism group of X.

Let $G = \mathrm{Aut}_k(X)$ and take e in the $\mathbf{Z}_p$-group ring $\mathbf{Z}_p[G]$ such that $e^2 = e$. The ring $\mathbf{Z}_p[G]$ acts on $H^r_{cris}(X/W)$ and on $H^{r-i}(X, \Omega^i_X)$ for all r and i.

Proposition 3.1. *The group $eH^r_{cris}(X/W)/torsion$ is an F-crystal whose abstract Hodge polygon lies on or above the Hodge polygon associated to the integers*

$$h^i = \dim_k eH^{r-i}(X, \Omega^i_X)$$

with $i = 0, \ldots, r$.

Proof: That $eH^r_{cris}(X/W)/torsion$ is an F-crystal is clear: Φ is $\mathbf{Z}_p$-linear and commutes with any automorphism of X. The proof of the statement on the polygons will be an essentially formal variation of the argument of Nygaard, which in fact proves more. We adopt the notations of [N]; in particular, $W_n\Omega^i_X$ are the de Rham-Witt sheaves of Illusie [I]. First of all, if $\sigma \in G$, we have canonical isomorphisms of quasi-coherent sheaves on $W_n(X)$

$$\sigma^* W_n\Omega^i_X \xrightarrow{\sim} W_n\Omega^i_X$$

for all n, i, which are compatible with the operators d, V and F of the de Rham-Witt complex ([I], I.1.14, I.2.17.5). From this we deduce automorphisms of $H^j(X, W_n\Omega^i_X)$ and $\mathbf{H}^j(X, W_n\Omega^{\cdot}_X)$ which commute with d, V and F. We also deduce from σ isomorphisms

$$\sigma^* W\Omega^{\cdot}_X(r,n) \xrightarrow{\sim} W\Omega^{\cdot}_X(r,n)$$

of the complexes of [N] §1 which are compatible with $\tilde{V}$ and $\tilde{F}$. Thus e acts on all the groups appearing in Nygaard's proof, commuting with all the operators there. Finally, noting that $M \mapsto eM$ is an exact functor on the category of $\mathbf{Z}_p[G]$-modules, we can apply the argument of [N], Lemma 2.2 essentially verbatim, inserting e's as necessary. This yields the crucial estimate

$$nr\tilde{h}^0(n) + (nr-1)\tilde{h}^1(n) + \cdots + \tilde{h}^{nr-1}(n) \leq rh^0(n) + (r-1)h^1(n) + \cdots + h^{r-1}(n)$$

for all n and r, where $\tilde{h}^i(n)$ are the abstract Hodge numbers of (eH, Φ^n) and $h^i(n) = \mathrm{length}_W H^{r-i}(X, W_n\Omega^i_X)$. Applying Lemma 2.4 of [N] for $n = 1$ yields the proposition. □

4. The situation at the cusps We return to the notations of §2: I is the modular curve $Ig_1(p^nN)$, $f : X \to I$ is the k-fold fiber product of the universal curve $\mathcal{E} \to I$ and $\tilde{f} : \tilde{X} \to I$ is a certain resolution of singularities of X, equivariant for the action of $G = ((\mathbf{Z}/N\mathbf{Z})^k \rtimes \mu_2^k) \rtimes S_k$ and $(\mathbf{Z}/p\mathbf{Z})^\times$. This resolution is explained in detail in [S], §2; see also below. We assume that $p > 2$ and $k < p$ so that the projector Π associated to the character $\epsilon : G \to \pm 1$ defined in §2 has p-integral coefficients; we have also fixed a power χ^a of the basic (Teichmüller) character χ. The goal is to compute the dimensions of the $H^j(\tilde{X}, \Omega^i_{\tilde{X}})(\epsilon, \chi^a)$. We begin in this section by handling the problems presented by the bad reduction of $\mathcal{E}$ at the cusps and then finish the computation in the next section. The main results of the section (Propositions 4.1 and 4.2) are obvious in the case $k = 0$ so to avoid special cases, we assume $k > 0$ throughout this section.

We will use the logarithmic structures of Fontaine and Illusie as developed in [Ko]. Let $C \subseteq I$ be the reduced subscheme of cusps and consider the log scheme (I, M) where $M \subseteq \mathcal{O}_I$ is the subsheaf of monoids consisting of functions invertible outside C. Similarly, define log structures on $\mathcal{E}$, X and $\tilde{X}$ by using the subsheaf of functions invertible off the cuspidal fibers. We denote the resulting log schemes with a $\times$: $\mathcal{I}^\times$, $\mathcal{E}^\times$, etc. The map of log schemes $\mathcal{E}^\times \to I^\times$ is smooth ([Ko], 3.7(2)) and $X^\times \to I^\times$ is easily seen to be its k-fold fiber product and is thus also smooth. For $\tilde{X}$, note that the fibers of $\tilde{f} : \tilde{X} \to I$ over C are non-reduced divisors: if $x \in I$ is a cusp the components of $\tilde{f}^{-1}(x)$ coming from proper transforms of components of $f^{-1}(x)$ have multiplicity 1, but the new components introduced in the blow-up $\tilde{X} \to X$ have multiplicity 2. Let D be the reduced divisor underlying $\tilde{f}^{-1}(C)$; then D has normal crossings (i.e., étale locally on $\tilde{X}$, D is a union of coordinate hyperplanes in affine space) and, from its definition above, we see that the log structure on $\tilde{X}^\times$ is the one associated to D as in [Ko], 1.5. Now étale locally on $\tilde{X}^\times$ and $I^\times$ we have evident charts (in the sense of [Ko], 2.9) coming from the divisor with normal crossings structure and also a chart for the morphism $\tilde{X}^\times \to I^\times$; using these charts and applying Theorem 3.5 of [Ko] (and the assumption $p > 2$) we find that $\tilde{X}^\times \to I^\times$ is also a smooth map of log schemes. As $I^\times$ is smooth over Spec $\mathbf{F}_p$ (with its trivial log structure), so are $\tilde{X}^\times$ and $X^\times$.

Thus, we have a diagram of log schemes

$$\begin{array}{ccc} & \tilde{X}^\times & \\ \swarrow & & \searrow \\ \tilde{X} & & X^\times \end{array}$$

smooth over Spec $\mathbf{F}_p$. In this section, we will compare the Hodge cohomology of $\tilde{X}$ and of $X^\times$, finding that they are essentially the same after applying the projector Π. To this end, introduce sheaves of log differentials ([Ko], 1.7)

$$\Omega^i_{\tilde{X}^\times} = \Omega^i_{\tilde{X}^\times/\,\mathrm{Spec}\ \mathbf{F}_p},\ \Omega^i_{X^\times} = \Omega^i_{X^\times/\,\mathrm{Spec}\ \mathbf{F}_p},\ \text{and}\ \Omega^i_{I^\times} = \Omega^i_{I^\times/\,\mathrm{Spec}\ \mathbf{F}_p}$$

which are locally free on $\tilde{X}$, X, and I. On I, $\Omega^1_{I^\times} = \Omega^1(C)$ and on $\tilde{X}$, $\Omega^i_{\tilde{X}^\times} = \Omega^i(\log D)$, the sheaf of meromorphic i-forms η which have at worst simple poles along D and whose differentials $d\eta$ also have at worst simple poles along D. We note that any $\sigma \in G$ induces automorphisms of the log schemes $\tilde{X}^\times$, $X^\times$, and $I^\times$, and we have canonical isomorphisms

$$\sigma^*\Omega^i_{\tilde{X}^\times} \xrightarrow{\sim} \Omega^i_{\tilde{X}^\times};$$

similar remarks apply to the other sheaves of differentials considered above, and to the $\langle d\rangle$, $d \in (\mathbf{Z}/p\mathbf{Z})^\times$.

Proposition 4.1. *The inclusions* $\Omega^i_{\tilde{X}} \hookrightarrow \Omega^i_{\tilde{X}^\times}$ *induce isomorphisms*

$$H^j(\tilde{X}, \Omega^i_{\tilde{X}})(\epsilon) \cong H^j(\tilde{X}, \Omega^i_{\tilde{X}^\times})(\epsilon)$$

for all $(i,j) \neq (k+1,0)$ *or* $(k+1,1)$*. Moreover, there is an exact sequence*

$$0 \to H^0(\tilde{X}, \Omega^{k+1}_{\tilde{X}})(\epsilon) \to H^0(\tilde{X}, \Omega^{k+1}_{\tilde{X}^\times})(\epsilon) \to H^0(C, \mathcal{O}_C) \xrightarrow{\delta}$$
$$H^1(\tilde{X}, \Omega^{k+1}_{\tilde{X}})(\epsilon) \to H^1(\tilde{X}, \Omega^{k+1}_{\tilde{X}^\times})(\epsilon) \to 0$$

Remarks: 1) We will show in the next section that in fact $\delta = 0$, so we have isomorphisms for all $(i,j) \neq (k+1,0)$.

2) When $k=0$, $H^0(C, \mathcal{O}_C)$ must be replaced by the subgroup $\{f \in H^0(C, \mathcal{O}_C) | \sum_{c \in C} f(c) = 0\}$.

Proposition 4.2. *The map* $g : \tilde{X}^\times \to X^\times$ *induces isomorphisms*

$$H^j(X, \Omega^i_{X^\times}) \cong H^j(\tilde{X}, \Omega^i_{\tilde{X}^\times})$$

for all i*,* j*, which are compatible with the actions of* G *and* $(\mathbf{Z}/p\mathbf{Z})^\times$*.*

Before beginning the proofs of these propositions, we make a general observation about sheaves of logarithmic differentials. Let Z be a smooth proper variety, D a divisor with normal crossings on Z, and $Z^\times$ the associated log scheme. Etale locally on Z, D is a union of smooth divisors and for $j \geq 1$ we let ${}_jD$ (resp. ${}_j\tilde{D}$) be the union in Z (resp. the disjoint union) of the j-fold intersections of components of D. These constructions glue to give a subvariety ${}_jD$ of Z and a smooth variety ${}_j\tilde{D}$ which is the normalization of ${}_jD$; let $a_j : {}_j\tilde{D} \to {}_jD$ be the natural map. The local $j+1$-fold intersections glue together to define a normal crossings divisor on ${}_j\tilde{D}$ and we let ${}_j\tilde{D}^\times$ be the associated log scheme. (We call the divisor defined here on $\tilde{D} = {}_1\tilde{D}$ the divisor of self-intersections of D.) By convention, we let ${}_0\tilde{D}^\times$ be $Z^\times$, the log scheme supported on Z associated to D. Then for $j \geq 0$ each irreducible component $C^\times$ of ${}_{j+1}\tilde{D}^\times$ (with the induced log structure) admits $j+1$ maps to ${}_j\tilde{D}^\times$ each of whose images are divisors with normal crossings; for each of these maps the Poincaré residue defines a homomorphism of coherent $\mathcal{O}_Z$-modules $a_{j*}\Omega^i_{{}_j\tilde{D}^\times} \to a_{j+1*}\Omega^{i-1}_{C^\times}$. Summing over all components and all maps as above, we get a homomorphism

$$a_{j*}\Omega^i_{{}_j\tilde{D}^\times} \to a_{j+1*}\Omega^{i-1}_{{}_{j+1}\tilde{D}^\times}.$$

We also have an inclusion

$$\Omega^i_Z \hookrightarrow \Omega^i_Z(\log D) = \Omega^i_{{}_0\tilde{D}^\times}.$$

We will also need the following variant: suppose E, E', and $E+E'$ are normal crossings divisors on Z. Define ${}_jE$, ${}_j\tilde{E}$ and $b_j : {}_j\tilde{E} \to {}_jE$ as above using the j-fold interesections of components of E, and let ${}_j\tilde{E}^\times$ be the log scheme supported on ${}_j\tilde{E}$ corresponding to the normal crossings divisor $E+E'$. Then as above, the Poincaré residue defines a homomorphism

$$b_{j*}\Omega^i_{{}_j\tilde{E}^\times} \to b_{j+1*}\Omega^{i-1}_{{}_{j+1}\tilde{E}^\times}$$

of coherent $\mathcal{O}_Z$-modules and we have an inclusion

$$\Omega^i_Z(\log E') \hookrightarrow \Omega^i_Z(\log E+E') = \Omega^i_{{}_0\tilde{E}^\times}.$$

Lemma 4.3. *The sequences of coherent $\mathcal{O}_Z$-modules*

$$0 \to \Omega^i_Z \to \Omega^i_Z(\log D) \to a_{1*}\Omega^{i-1}_{{}_1\tilde{D}^\times} \to \cdots \to a_{i*}\mathcal{O}_{{}_i\tilde{D}^\times} \to 0$$

and

$$0 \to \Omega^i_Z(\log E) \to \Omega^i_Z(\log E + E') \to b_{1*}\Omega^{i-1}_{{}_1\tilde{E}^\times} \to \cdots \to b_{i*}\mathcal{O}_{{}_i\tilde{E}^\times} \to 0$$

are exact for the étale topology.

Proof: The question is étale local, so we can assume Z is Zariski open in affine space and D, E, E', and $E + E'$ are unions of coordinate hyperplanes. The result in this case can be obtained by direct calculation. Alternatively, one can note that the sequence of the lemma is a special case of a more general construction of Ishida and the lemma is a consequence of [O], Theorem 3.6(4). □

As the next step toward proving the propositions, we have to discuss the geometry of the resolution $\tilde{X} \to X$ from a slightly different point of view than that of [S]. Recall that the fiber of $f : X \to I$ over a cusp of I is a k-fold product of a Néron M-gon where M depends on the cusp and $M|N$. The singularities of X arise from products of double points in these fibers. The resolution $\tilde{X}$ is obtained by a sequence of $k-1$ blow-ups: let Y_i, $i = 0, \ldots, k-2$, be the union in X of the subscheme of the cuspidal fibers consisting of products of i copies of $\mathbf{P}^1$ and $k - i$ double points. Set $X\langle 0\rangle = X$, $Y\langle 0\rangle = Y_0$ and define inductively $X\langle i\rangle$ as the blow-up of $X\langle i-1\rangle$ along $Y\langle i-1\rangle$ and $Y\langle i\rangle$ as the proper transform of Y_i in $X\langle i\rangle$. Then Y_{k-2} is the singular locus of X and $\tilde{X} = X\langle k-1\rangle$ is smooth. We define B_i as the exceptional divisor of $X\langle i+1\rangle \to X\langle i\rangle$ and D_i as the proper transform of B_i in $\tilde{X}$. Also let D_k be the proper transform in $\tilde{X}$ of the fibers of $f : X \to I$ over the cusps. The divisor D is the union of the D_i, $i = 0, \ldots, k-2, k$. We call the irreducible components of D_i *type i components* of D; the dimension of the image under $\tilde{X} \to X$ of a type i component is i. Note that each D_i is stabilized by G and $(\mathbf{Z}/p\mathbf{Z})^\times$.

To describe the D_i and their intersections, we introduce some other varieties. For $2 \le r \le k$ consider the projective space $\mathbf{P}^{2r-1}$ over some field $\mathbf{F}$ with coordinates $x_1, y_1, \dots, x_r, y_r$ and let P_r be the complete intersection

$$x_1 y_1 = \cdots = x_r y_r.$$

For $0 \le i \le r-2$ let $P_{r,i}$ be the subvariety of P_r where at least $r-1-i$ pairs x_j, y_j of coordinates vanish. Set $P_r\langle 0\rangle = P_r$ and define inductively $P_r\langle i\rangle$ as the blow-up of $P_r\langle i-1\rangle$ along the proper transform of $P_{r,i-1}$. Then $P_{r,r-3}$ is the singular locus of P_r and $\tilde{P}_r = P_r\langle r-2\rangle$ is smooth. Similarly, for $0 \le r \le k$, set $Q_r = \left(\mathbf{P}^1\right)^r$ and for $0 \le i \le r-1$ let $Q_{r,i}$ be the union of all products with i factors equal to $\mathbf{P}^1$ and $r-i$ factors equal to 0 or $\infty \in \mathbf{P}^1$. Let $Q_r\langle 0\rangle = Q_r$ and define inductively $Q_r\langle i\rangle$ as the blow-up of $Q_r\langle i-1\rangle$ along the proper transform of $Q_{r,i-1}$; finally set $\tilde{Q}_r = Q_r\langle r-1\rangle$.

The following lemma records the geometry of the resolution $\tilde{X}$ that we need. We leave the proof as an exercise; see [S], §2 for hints.

Lemma 4.4.

a) *For $0 \le i \le k-2$, $Y\langle i\rangle$ is a disjoint union of copies of $\tilde{Q}_i$.*

b) *For $0 \le i \le k-2$, $B_i \to Y\langle i\rangle$ is a P_{k-i} bundle, i.e., locally on $Y\langle i\rangle$ $B_i \cong Y\langle i\rangle \times P_{k-i}$ and $B_i \to Y\langle i\rangle$ is the projection.*

c) *For $0 \le i < i' \le k-2$, each component of the proper transform of $Y_{i'}$ to $X\langle i+1\rangle$ meets each fiber of $B_i \to Y\langle i\rangle$ either in $\emptyset$ or in an irreducible component of $P_{k-i,i'-1-i}$ and each such component occurs once. Each irreducible component of D_i is a $\tilde{P}_{k-i}$ bundle over $\tilde{Q}_i$.*

d) For $0 \leq i < i' \leq k-2$, the intersection of a component of $D_{i'}$ and a component of D_i, viewed as a subscheme of $D_{i'}$, is a $\tilde{P}_{k-i'}$ bundle over a smooth divisor of $\tilde{Q}_{i'}$. If $i < i'-1$ then this divisor is a component of the proper transform of the exceptional divisor of $Q_{i'}\langle i+1\rangle \to Q_{i'}\langle i\rangle$ and each such component occurs exactly once; if $i = i'-1$, this divisor is a component of the proper transform of $Q_{i',i}$ to $\tilde{Q}_{i'}$ and each such component occurs exactly once.

e) The divisor D_k has normal crossings and its normalization $\tilde{D}_k$ is a disjoint union of copies of $\tilde{Q}_k$. Over a cusp with $M > 1$, each component of D_k itself is a copy of $\tilde{Q}_k$ and the intersection of one of these components with all the others is the proper transform of the divisor $Q_{k,k-1} \subseteq Q_k$ to $\tilde{Q}_k$. Each component of this intersection is isomorphic to $\tilde{Q}_{k-1}$. Over a cusp with $M = 1$ there is exactly one component of D_k and the divisor of self-intersections of this component (in $\tilde{D}_k$) is the proper transform of the divisor $Q_{k,k-1} \subseteq Q_k$ to $\tilde{Q}_k$; thus again, each component is isomorphic to $\tilde{Q}_{k-1}$. For $0 \leq i \leq k-2$, the divisor defined on each component of $\tilde{D}_k$ by D_i is the proper transform in $\tilde{Q}_k$ of the exceptional divisor of the blow-up $Q_k\langle i+1\rangle \to Q_k\langle i\rangle$.

Now we define a log structure on $\tilde{Q}_k$ as follows: consider the divisor D' which is the union of the proper transforms of the exceptional divisors of the blow-ups $Q_k\langle i\rangle \to Q_k\langle i-1\rangle$ for $i = 1, \ldots, k-1$ and the proper transform of the divisor $Q_{k,k-1}$. The log strucure on $\tilde{Q}_k$ is then the one associated to this divisor with normal crossings and we denote the resulting log scheme by $\tilde{Q}_k^{\times}$. Note that according to part e) of the lemma, the divisor D' is the divisor of self-intersections of D, restricted to a component of $\tilde{D}_k$.

We have an action of the group $G_r = \mu_2^r \rtimes S_r$ on $\tilde{P}_r$ (induced by the action of G_r on

P_r which permutes the coordinates $x_1, \ldots, y_r$ preserving the pairs x_j, y_j) and an action of G_k on $\tilde{Q}_k$ (induced by the action of G_k on Q_k where the μ_2's act by $z \mapsto 1/z$ on the factors $\mathbf{P}^1$ and S_k permutes the factors). Let $\epsilon_r : G_r \to \pm 1$ be the character which is the identity on the μ_2's and the sign character on S_r. We recall that k is assumed to be $< p$ so the projector associated to ϵ_r in the group ring of G_r has p-integral coefficients.

Lemma 4.5. *For all i and j, $H^j(\tilde{P}_r, \Omega^i_{\tilde{P}_r})(\epsilon_r) = 0$. Also, $H^j(\tilde{Q}_k, \Omega^i_{\tilde{Q}_k^\times})(\epsilon_k) = 0$ unless $i = k$ and $j = 0$ and $H^0(\tilde{Q}_k, \Omega^k_{\tilde{Q}_k^\times})(\epsilon_k) = \mathbf{F}$ (where $\mathbf{F}$ is the ground field).*

Proof: We will use the theory of toric varieties (see [O]), beginning with some general remarks. Let Z be a smooth toric variety. Then (because the fan associated to Z is simplicial) the complement of the open orbit in Z is a divisor D with normal crossings. If $Z^\times$ denotes the associated log scheme, then the sheaves $\Omega^i_{Z^\times}$ are globally trivial; more precisely, $\Omega^i_{Z^\times} \cong \mathcal{O}_Z \otimes_{\mathbf{Z}} \wedge^i N$ where N is the lattice of cocharacters of the torus acting on Z. Now well-known results on the cohomology of equivariant invertible sheaves on toric varieties (e.g., [O], 2.2) say that $H^j(Z, \mathcal{O}_Z) = 0$ unless $j = 0$. Thus $H^j(Z, \Omega^i_{Z^\times}) = 0$ if $j \neq 0$ and $= \mathbf{F} \otimes_{\mathbf{Z}} \wedge^i N$ if $j = 0$.

Now apply this remark to $\tilde{Q}_k$: let T be the torus $\mathbf{G}_m^k$ with cocharacter group $N = \mathbf{Z}^k$. We can give $\tilde{Q}_k$ the structure of a toric variety so that the divisor D' defined before the statement of the lemma is the complement of the open orbit. According to our general remarks, $H^j(\tilde{Q}_k, \Omega^i_{\tilde{Q}_k^\times})$ is 0 unless $j = 0$ and is $\mathbf{F} \otimes \wedge^i N$ if $j = 0$. Now the action of G_k on the cohomology of $\tilde{Q}_k$ is via its action on N which is the obvious one (permuting the coordinates and changing their signs). Thus we have a non-trivial ϵ_k eigenspace only when $i = k$ and $j = 0$, in which case it is one-dimensional. This proves the second statement of the lemma.

We return for a moment to the general considerations of the first paragraph of the proof. In the notation of Lemma 4.3, each component of ${}_j\tilde{D}^\times$ is itself a smooth toric variety with the log structure associated to the complement of the open orbit. Thus by the above, the sequence

$$0 \to \Omega^i_Z \to \Omega^i_Z(\log D) \to a_{1*}\Omega^{i-1}_{{}_1\tilde{D}^\times} \to \cdots \to a_{i*}\mathcal{O}_{{}_i\tilde{D}^\times} \to 0$$

is an acyclic resolution of Ω^i_Z of length i. We obtain that $H^j(Z, \Omega^i_Z) = 0$ for $j > i$ and Serre duality implies that $H^j(Z, \Omega^i_Z) = 0$ for $j < i$. Also, we have a surjection

$$H^0({}_i\tilde{D}^\times, \mathcal{O}_{{}_i\tilde{D}^\times}) \to H^i(Z, \Omega^i_Z).$$

To finish the proof of the lemma, we will apply this to $\tilde{P}_r$.

Consider the group $\mathbf{Z}^r \subseteq \mathbf{R}^r$ with basis $e_1, \ldots, e_r$ viewed as the cocharacter group of the torus $\mathbf{G}_m^r$ in the usual way. The torus $\mathbf{G}_m^r$ acts on P_r via $(x_1, y_1, \ldots, x_r, y_r) \mapsto (a_1 x_1, a_1^{-1} y_1, \ldots, a_r x_r, a_r^{-1} y_r)$ and the kernel is μ_2 embedded diagonally. Let $N = \mathbf{Z}^r + \frac{1}{2}(e_1 + \cdots + e_r)$ be the cocharacter lattice of the torus $T = \mathbf{G}_m^r/\mu_2$. Then P_r is the toric variety associated to a certain fan in $N \otimes \mathbf{R}$. The cones of positive dimension i in this fan are the following: fix a subset $S \subseteq \{1, \ldots, r\}$ of cardinality $r + 1 - i$ and signs $+$ or $-$ for each element of S. To this data associate the cone generated by the vectors $\frac{1}{2}(\pm e_1 \pm \cdots \pm e_r)$ where the signs of the e_j with $j \in S$ agree with the fixed signs. (To see this, note that to each choice of data as above is associated an $(r - i)$-dimensional orbit of T on P_r: take the subset where $x_j = y_j = 0$ if $j \notin S$ and either x_j or $y_j = 0$ if $j \in S$, depending on the sign attached to j. For simplicity assume the signs are all $+$ so $y_j = 0 \neq x_j$ for all $j \in S$. The union of all orbits whose closure contains this orbit is the open set

$$U = \{x_j \neq 0 | j \in S\}.$$

Then the 1-parameter subgroup $\mathbf{G}_m \to T \subseteq P_r$ associated to a cocharacter $n = \sum a_j e_j \in N$ extends to $\mathbf{A}^1 \to U$ if and only if $a_j \geq |a_{j'}|$ for all $j \in S$, $j' \notin S$. The cone spanned by these cocharacters is exactly the cone associated above to the given data.) The resolution $\tilde{P}_r$ is equivariant, so corresponds to a subdivision of this fan which one checks (using Theorem 10, p. 31 of [KKMS]) can be described as follows: the subdivision consists of simplicial cones, so it suffices to list the r-dimensional ones. To each permutation $\sigma \in S_r$ and function $s : \{1, \ldots, r-1\} \to \pm 1$ we associate the cone spanned by the vectors

$$s(1)e_{\sigma(1)}, \quad s(1)e_{\sigma(1)} + s(2)e_{\sigma(2)}, \quad \ldots, \quad s(1)e_{\sigma(1)} + \cdots + s(r-2)e_{\sigma(r-2)},$$
$$\frac{1}{2}(s(1)e_{\sigma(1)} + \cdots + s(r-2)e_{\sigma(r-2)} + s(r-1)e_{\sigma(r-1)} + e_{\sigma(r)}),$$
$$\frac{1}{2}(s(1)e_{\sigma(1)} + \cdots + s(r-2)e_{\sigma(r-2)} + s(r-1)e_{\sigma(r-1)} - e_{\sigma(r)}).$$

In particular, every i-dimensional cone of this fan with $i < r$ is contained in a hyperplane of the form $\{\sum a_j e_j | a_{j_1} = \pm a_{j_2}\}$ for some choice of $\pm$, j_1 and j_2.

Now the action of G_r on $\tilde{P}_r$ corresponds to the linear action of G_r on $N \otimes \mathbf{R}$ where S_r permutes the e_j and the j-th μ_2 sends e_j to $-e_j$ and leaves the other e_i invariant. In particular, the hyperplanes just mentioned are each fixed pointwise by some element of $g \in G_r$ with $\epsilon_r(g) = -1$. Also, the r-cone above is stabilized by the $g \in G$ such that $g(e_{\sigma(r)}) = -e_{\sigma(r)}$ and $g(e_{\sigma(i)}) = e_{\sigma(i)}$ for $i < r$; again $\epsilon_r(g) = -1$.

Now if D is the complement of the open orbit in $\tilde{P}_r$ then the group $H^0({}_i\tilde{D}^\times, \mathcal{O}_{{}_i\tilde{D}^\times})$ of the general remarks can be canonically identified with $\bigoplus_\phi \mathbf{F}$ where the sum extends over all cones ϕ of dimension i of the fan associated to $\tilde{P}_r$; G acts on this group via its permutation action on the cones of the fan. Then the observations of the previous paragraph show that the ϵ_r part of the source of the surjection

$$\bigoplus_{\phi} \mathbf{F} \to H^i(\tilde{P}_r, \Omega^i_{\tilde{P}_r})$$

is zero for all i. This completes the proof of the lemma. □

Remark: The isomorphism of the second part of the lemma is canonical only up to a sign, depending ultimately on the choice of a generator of $\wedge^k \mathbf{Z}^k \cong \mathbf{Z}$.

Proof of 4.1: Recall the divisor D with normal crossings on $\tilde{X}$ which is the union of the D_j. For $0 \leq i \leq k-2$, let E_i be the divisor $\sum_{j=o}^{i} D_j$ and let E_{-1} be the empty divisor. We note that by Lemma 4.4a, each D_j $(0 \leq j \leq k-2)$ is the disjoint union of its irreducible components (or equivalently, in the notation of Lemma 4.3, ${}_2\tilde{D}_j = \emptyset$). While D_k is not the disjoint union of its irreducible components, its divisor of self-intersections is (by Lemma 4.4e), or equivalently, ${}_3\tilde{D}_k = \emptyset$. Applying the second exact sequence of Lemma 4.3 with $E = E_{k-2}$, $E' = D_k$, we have an exact sequence

$$0 \to \Omega^i_{\tilde{X}}(\log E_{k-2}) \to \Omega^i_{\tilde{X}^\times} \to b_{1*}\Omega^{i-1}_{\tilde{D}_k^\times} \to b_{2*}\Omega^{i-2}_{{}_2\tilde{D}_k^\times} \to 0$$

Here, by Lemma 4.4e, the components of $\tilde{D}_k$ are isomorphic to $\tilde{Q}_k$ and the log structure induced from $\tilde{D}_k^\times$ is that of $\tilde{Q}_k^\times$. Also, the components of ${}_2\tilde{D}_k$ are isomorphic to $\tilde{Q}_{k-1}$ and the log structure induced from ${}_2\tilde{D}_k^\times$ is that of $\tilde{Q}_{k-1}^\times$.

Now applying the second exact sequence of 4.3 again, with $E = E_{j-1}$, $E' = D_j$, we get exact sequences

$$0 \to \Omega^i_{\tilde{X}}(\log E_{j-1}) \to \Omega^i_{\tilde{X}}(\log E_j) \to \Omega^{i-1}_{D_j^\times} \to 0$$

for $0 \leq j \leq k-2$. (Here $D_j = \tilde{D}_j$ since D_j is the disjoint union of its irreducible components.) Note that $E_{-1} = \emptyset$ so $\Omega^i_{\tilde{X}}(\log E_{-1}) = \Omega^i_{\tilde{X}}$. Thus to prove the proposition, it suffices to show that the ϵ parts of the Hodge cohomology of the various $D_j^\times$ and of ${}_2\tilde{D}_k^\times$ vanish except for $H^0(\tilde{D}_k, \Omega^k_{\tilde{D}_k^\times})$.

First consider $\tilde{D}_k^\times$. Under the action of $G = \big((\mathbf{Z}/N\mathbf{Z})^k \rtimes \mu_2^k\big) \rtimes S_k$ on the components of $\tilde{D}_k^\times$ there is one orbit for each cusp. The stabilizer of each component is $\big((M\mathbf{Z}/N\mathbf{Z})^k \rtimes \mu_2^k\big) \rtimes S_k$ (where M depends on the cusp) and $(M\mathbf{Z}/N\mathbf{Z})^k$ acts trivially. Using Lemma 4.5, we find that

$$H^j(\tilde{D}_k, \Omega^i_{\tilde{D}_k^\times})(\epsilon) = \begin{cases} H^0(C, \mathcal{O}_C) & \text{if } i = k \text{ and } j = 0 \\ 0 & \text{otherwise.} \end{cases}$$

(We note that this isomorphism is only canonical up to a choice of sign at each cusp, the ambiguity coming from that of Lemma 4.5.)

As for ${}_2\tilde{D}_k^\times$, note that for any component of ${}_2\tilde{D}_k^\times$, there exists an element $g \in G$ fixing it pointwise and such that $\epsilon(g) = -1$. Thus $H^j({}_2\tilde{D}_k^\times, \Omega^i_{{}_2\tilde{D}_k^\times})(\epsilon) = 0$ for all (i, j).

To finish the proof, we will show that

$$H^*(D_j, \Omega^*_{D_j^\times})(\epsilon) = 0$$

for $0 \leq j \leq k-2$. First note that the stabilizer of an irreducible component of D_j is

$$\Big(\Big((M\mathbf{Z}/N\mathbf{Z})^j \rtimes \mu_2^j\Big) \rtimes S_j\Big) \times \Big(\Big((M\mathbf{Z}/N\mathbf{Z})^{k-j} \rtimes \mu_2^{k-j}\Big) \rtimes S_{k-j}\Big)$$

with the $M\mathbf{Z}/N\mathbf{Z}$'s acting trivially. Now in the description of a component of D_j as a $\tilde{P}_{k-j}$ bundle over $\tilde{Q}_j$, the subgroup $\Big(\Big((M\mathbf{Z}/N\mathbf{Z})^{k-j} \rtimes \mu_2^{k-j}\Big) \rtimes S_{k-j}\Big)$ of the stabilizer acts trivially on the base and in the usual way on the fibers.

Let C be one of the components of D_j with its fibration $h : C \to \tilde{Q}_j$ and let $C^\times$ be the log scheme supported on C which is an irreducible component of $D_j^\times$. Using the Leray spectral sequence, it will be enough to show that

$$\left(Rh_* \Omega^i_{C^\times}\right)(\epsilon_{k-j}) = 0$$

where $\epsilon_{k-j} : G_{k-j} \to \pm 1$ is the character defined above. But Lemma 4.4d) says that the log structure on C (which is induced by the divisor E_{j-1}) is the inverse image of the log structure on $\tilde{Q}_j^\times$. Thus $\Omega^\cdot_{C^\times/\tilde{Q}_j^\times} = \Omega^\cdot_{C/\tilde{Q}_j}$ and the sheaves $\Omega^i_{\tilde{X}^\times}$ have a filtration whose graded pieces are

$$h^* \Omega^{i-i'}_{\tilde{Q}_j^\times} \otimes \Omega^{i'}_{C/\tilde{Q}_j}.$$

Now $C \to \tilde{Q}_j$ is a $\tilde{P}_{k-j}$ bundle, so

$$R^q h_* \Omega^{i'}_{C/\tilde{Q}_j} = H^q(\tilde{P}_{k-j}, \Omega^{i'}_{\tilde{P}_{k-j}}) \otimes \mathcal{O}_{\tilde{Q}_j}.$$

Applying ϵ_{k-j} and Lemma 4.5, $Rh_* \Omega^{i'}_{C/\tilde{Q}_j} = 0$, and this completes the proof of Proposition 4.1. □

Remark: One can also prove Proposition 4.1 by analyzing the weight spectral sequence for $H^j(\tilde{X}, \Omega^i_{\tilde{X}^\times})$ (see [D], 3.2.13iii). This is somewhat messier in that one needs geometric information on all possible intersections of components of D.

Proof of 4.2: This follows from the more precise claim $Rg_* \Omega^i_{\tilde{X}^\times} = \Omega^i_{X^\times}$ and the fact that g is G- and $(\mathbf{Z}/p\mathbf{Z})^\times$-equivariant. The claim follows from the projection formula and the following two lemmas.

Lemma 4.6. $Rg_*\mathcal{O}_{\tilde{X}} = \mathcal{O}_X$

Proof: This is a general fact about toroidal singularities, but we will give a direct proof here. It suffices to prove the claim for each of the blow-ups $h : X\langle i\rangle \to X\langle i-1\rangle$. That $h_*\mathcal{O} = \mathcal{O}$ is Zariski's main theorem. Also, away from the center of h, $R^j h_*\mathcal{O} = 0$ for $i > 0$ and $j > 0$. We will show that the stalks $(R^j h_*\mathcal{O})_y$ are zero for each closed point y in the center of h and each $j > 0$ by using the theorem on formal functions (cf. [H], V.3.4).

Write $Z = h^{-1}(y)$ and let Z_n be the thickened fiber of order n. Etale locally near the cuspidal fibers, h is the blow-up of

$$\mathrm{Spec}\ \mathbf{F}[x_1, y_1, \ldots, x_{k-i}, y_{k-i}, z_{k-i+1}, \ldots, z_k]/(x_1y_1 = \cdots = x_{k-i}y_{k-i})$$

along the subscheme

$$x_1 = y_1 = \cdots = x_{k-i} = y_{k-i} = 0$$

(where $\mathbf{F}$ is the residue field at the corresponding cusp); thus $X\langle i\rangle$ is étale locally

$$\mathrm{Spec}\ \mathbf{F}[A_1, B_1, \ldots, A_{k-i-1}, B_{k-i-1}, x_{k-i}, z_{k-i+1}, \ldots, z_k]/(A_1B_1 = \cdots = A_{k-i-1}B_{k-i-1})$$

with $h^*(x_j) = A_j x_{k-i}$, $h^*(y_j) = B_j x_{k-i}$ $(j < k-i)$ and $h^*(y_{k-i}) = x_{k-i}A_1B_1$. In particular, the fiber Z is isomorphic to the variety P_{k-i} over $\mathbf{F}$. If $\mathcal{I}$ denotes the ideal of Z in $X\langle i\rangle$, we have exact sequences

$$0 \to \mathcal{I}^n/\mathcal{I}^{n+1} \to \mathcal{O}_{Z_{n+1}} \to \mathcal{O}_{Z_n} \to 0.$$

But $X\langle i\rangle$ is Cohen-Macauly and $\mathcal{I}$ is generated at each point by a regular sequence ($(x_{k-i}, z_{k-i+1}, \ldots, z_k)$ in the coordinates above), so

$$\mathcal{I}^n/\mathcal{I}^{n+1} \cong \mathrm{Sym}^n(\mathcal{I}/\mathcal{I}^2);$$

moreover, in the coordinates above, it is visible that $\mathcal{I}/\mathcal{I}^2 \cong (\mathcal{O}_Z^{k-i} \oplus \mathcal{O}_Z(1))$. Finally, P_{k-i} is a complete intersection and an easy Koszul argument shows that $H^j(P_{k-i}, \mathcal{O}(n)) = 0$ for all $j > 0$, $n \geq 0$. By the theorem on formal functions ([H], III.11.1)

$$\left(R^j h_* \mathcal{O}_{\tilde{Y}}\right)_y^{\wedge} = \varprojlim_n H^j(Z, \mathcal{O}_{Z_n})$$

which is zero by induction. Since $R^j h_* \mathcal{O}_{\tilde{Y}}$ is coherent, it must vanish. $\square$

Lemma 4.7. *$g^*\Omega^i_{X^\times} \cong \Omega^i_{\tilde{X}^\times}$ for all i.*

Proof: Pull-back of differentials gives a homomorphism $g^*\Omega^i_{X^\times} \to \Omega^i_{\tilde{X}^\times}$ of locally free sheaves on $\tilde{X}$ of the same rank which is manifestly an isomorphism away from the cuspidal fibers. But near these fibers one checks directly from the explicit description of g that the pull-back of a set of generating sections of $\Omega^i_{X^\times}$ in fact generate $\Omega^i_{\tilde{X}^\times}$. (Indeed, in the coordinates above, the differentials dx_j/x_j $(j \leq k-i)$, dy_{k-i}/y_{k-i} and dz_j/z_j $(k-i+1 \leq j)$ generate $\Omega^1_{X\langle i-1\rangle^\times}$ and pull-back to $dA_j/A_j + dx_{k-i}/x_{k-i}$ $(j \leq k-i-1)$, dx_{k-i}/x_{k-i}, $dx_{k-i}/x_{k-i} + dA_1/A_1 + dB_1/B_1$, and dz_j/z_j $(j \geq k-i+1)$, which generate $\Omega^1_{X\langle i\rangle^\times}$.) $\square$

This completes the proof of Proposition 4.2.

Remark: As pointed out to me by Illusie, g is in fact log étale: this follows from Lemma 4.6 and [Ko] 3.12 or from [Ko] 3.5 and the proof of Lemma 4.7.

5. Hodge Cohomology of $X^\times$

Next we turn to the calculation of the Hodge cohomology of $X^\times$. It will be convenient to calculate the part cut out by the projector associated to ϵ first and then pass to eigenspaces for the χ^a.

Recall the universal curve $\pi : \mathcal{E} \to I$ and consider the sheaf $R^1\pi_*\mathcal{O}_{\mathcal{E}}$ on I. As $\pi : \mathcal{E} \to I$ is a flat family of curves of arithmetic genus 1, this sheaf is locally free of rank 1; define an invertible sheaf ω by $\omega^{-1} = R^1\pi_*\mathcal{O}_{\mathcal{E}}$. We will also need the sheaf of relative log differentials $\Omega^1_{X^\times/I^\times}$ for $f : X^\times \to I^\times$ ([Ko], 1.7) which sits in an exact sequence

$$0 \to f^*\Omega^1_{I^\times} \to \Omega^1_{X^\times} \to \Omega^1_{X^\times/I^\times} \to 0.$$

This sheaf is locally free of rank k on X. Defining $\Omega^i_{X^\times/I^\times} = \bigwedge^i \Omega^1_{X^\times/I^\times}$, we have exact sequences

$$0 \to f^*\Omega^1_{I^\times} \otimes \Omega^{i-1}_{X^\times/I^\times} \to \Omega^i_{X^\times} \to \Omega^i_{X^\times/I^\times} \to 0. \tag{5.1}$$

Noting that the sheaf $\Omega^1_{\mathcal{E}^\times/I^\times}$ (i.e., $\Omega^1_{X^\times/I^\times}$ for $k = 1$) is the relative dualizing sheaf for π, we find that $\pi_*\Omega^1_{\mathcal{E}^\times/I^\times}$ is invertible and isomorphic to ω and that $R^1\pi_*\Omega^1_{\mathcal{E}^\times/I^\times} \cong \mathcal{O}_I$. There is a canonical section ω_{can} of $\omega = \pi_*\Omega^1_{\mathcal{E}^\times/I^\times}$, defined by the following property: if (E, P, i) represents an ordinary point of I (so E is an ordinary elliptic curve together with a $\Gamma_1(N)$-structure P and an Igusa structure $i : \mathbf{Z}/p^n\mathbf{Z} \hookrightarrow E^{(p^n)}$), then

$$j^*\omega_{can}(E, P, i) = dt/t$$

where $j : \mu_{p^n} \hookrightarrow E$ is the inverse of the Cartier dual of i and dt/t is the standard invariant differential of $\mathbf{G}_m$, restricted to μ_{p^n}. The section ω_{can} generates ω away from the supersingular points and vanishes to order p^{n-1} at each supersingular point. (This is essentially Igusa's theorem that the Hasse invariant has simple zeros; cf. [KM] 12.8.2.)

Now $\Omega^1_{X^\times/I^\times}$ is locally free of rank k on X and isomorphic to a direct sum of k copies of pull-backs of $\Omega^1_{\mathcal{E}^\times/I^\times}$ under the k projections $X \to \mathcal{E}$. Note that since any automorphism $\sigma \in G$ gives a map of log schemes $\sigma : X^\times \to X^\times$ which covers the identity map of $I^\times$, it makes sense to apply the projector associated to ϵ to the sheaves $R^jf_*\Omega^i_{X^\times}$ and $R^jf_*\Omega^i_{X^\times/I^\times}$. The Künneth formula and linear algebra yield the following result.

Lemma 5.2. *There are canonical isomorphisms of sheaves on I:*

$$\left(R^j f_* \Omega^i_{X^\times/I^\times}\right)(\epsilon) \cong \begin{cases} \omega^{i-j} & \text{if } i+j=k \\ 0 & \text{otherwise} \end{cases}$$

Inserting this result into the long exact sequence of Rf_* coming from 5.1 and using the projection formula, we find isomorphisms

$$\left(R^j f_* \Omega^i_{X^\times}\right)(\epsilon) = 0 \qquad \text{if } i+j \neq k \text{ or } k+1$$

and

$$\left(R^k f_* \mathcal{O}_X\right)(\epsilon) \cong \omega^{-k};$$

exact sequences

$$0 \to \left(R^{k-i} f_* \Omega^i_{X^\times}\right)(\epsilon) \to \omega^{2i-k} \xrightarrow{\delta} \Omega^1_I(C) \otimes \omega^{2i-2-k} \to \left(R^{k+1-i} f_* \Omega^i_{X^\times}\right)(\epsilon) \to 0$$

for $1 \leq i \leq k$; and an isomorphism

$$\left(f_* \Omega^{k+1}_{X^\times}\right)(\epsilon) \cong \Omega^1_I(C) \otimes \omega^k.$$

Lemma 5.3. *The (Kodaira-Spencer) map of sheaves*

$$\omega^{2i-k} \xrightarrow{\delta} \Omega^1_I(C) \otimes \omega^{2i-2-k},$$

considered as a global section of $\underline{Hom}(\omega^{2i-k}, \Omega^1_I(C) \otimes \omega^{2i-2-k}) \cong \Omega^1_I(C) \otimes \omega^{-2}$, *is an isomorphism away from the supersingular points and vanishes to order* $p^{n-1}(p^n-2)$ *at each supersingular point.*

Proof: We have a universal curve $\pi' : \mathcal{E}' \to X_1(N)$ and a variety $X' \to X_1(N)$ whose fiber products with $I \to X_1(N)$ are the $\mathcal{E}$ and X under consideration. Moreover, the invertible sheaf ω on I is the pull-back of $R^1\pi'_*\mathcal{O}_{\mathcal{E}'}$ on $X_1(N)$ and the map δ is the pull-back of a similarly defined map δ' on $X_1(N)$. But $X_1(N)$ is a fine moduli space for elliptic curves and so the Kodaira-Spencer map δ' is an isomorphism. As $I \to X_1(N)$ is étale away from the supersingular points and totally ramified of degree $p^n - p^{n-1}$ at the supersingular points, the conductor calculations of [KM], 12.9.3 show that δ vanishes to order $p^{n-1}(p^n - 2)$ at each supersingular point and is an isomorphism elsewhere. $\square$

Remarks: 1) In fact, one can show more precisely that δ is multiplication by a non-zero multiple of the global section $(dq/q)\,\omega_{can}^{-2}$ of $\Omega^1_I(C) \otimes \omega^{-2}$, where ω_{can} the canonical global section of ω defined above and dq/q is a certain log differential (whose q-expansion is dq/q) related to the p-divisible group of $\mathcal{E}$ over I. (For dq/q see, for example, [K2], A.1.3.18.)

2) The fact that this Kodaira-Spencer map is not an isomorphism, i.e., that there is ramification in the map from the modular curve I to $X_1(N)$, accounts for the fact that the Hodge filtration we are considering has the unusual type $((k+1,0), (k,1), \ldots, (0,k+1))$ that it does.

Corollary 5.4. *If $1 \le i \le k$ and $i + j = k + 1$ then*

$$\left(R^j f_* \Omega^i_{X^\times}\right)(\epsilon)$$

is a skyscraper sheaf with stalks of dimension $p^{n-1}(p^n - 2)$ supported at each supersingular point of I. If $i = 0$ and $j = k$, then

$$\left(R^j f_* \Omega^i_{X^\times}\right)(\epsilon) = \omega^{-k}$$

and if $i = k+1$ and $j = 0$, then

$$\left(R^j f_* \Omega^i_{X^\times}\right)(\epsilon) = \Omega^1_{I^\times} \otimes \omega^k.$$

For all other (i,j) we have

$$\left(R^j f_* \Omega^i_{X^\times}\right)(\epsilon) = 0.$$

We write $\underline{C}^i_k$ for $(R^{k+1-i} f_* \Omega^i_{X^\times})(\epsilon)$ and C^i_k for $H^0(I, \underline{C}^i_k)$. It is now easy to compute the ϵ-part of the Hodge cohomology of $X^\times$: the Leray spectral sequence of f splits into short exact sequences

$$0 \to H^1(I, R^{j-1} f_* \Omega^i_{X^\times}(\epsilon)) \to H^j(X, \Omega^i_{X^\times})(\epsilon) \to H^0(I, R^j f_* \Omega^i_{X^\times}(\epsilon)) \to 0.$$

Using the corollary, and Propositions 4.1 and 4.2, we find isomorphisms

$$H^j(\tilde{X}, \Omega^i_{\tilde{X}})(\epsilon) \cong \begin{cases} 0 & \text{if } i+j \neq k+1 \text{ and } (i,j) \neq (k+1,1) \\ C^i_k & \text{if } 1 \leq i \leq k \text{ and } j = k+1-i \\ H^1(I, \omega^{-k}) & \text{if } i = 0 \text{ and } j = k+1 \end{cases}$$

and an exact sequence

$$0 \to H^0(\tilde{X}, \Omega^{k+1}_{\tilde{X}})(\epsilon) \to H^0(I, \Omega^1_I(C) \otimes \omega^k) \xrightarrow{\ r\ } H^0(C, \mathcal{O}_C)$$
$$\to H^1(\tilde{X}, \Omega^{k+1}_{\tilde{X}})(\epsilon) \to 0.$$

The map r can be identified (up to a sign at each cusp which comes from Lemma 4.5 via its use in Proposition 4.1) with H^0 of the residue $\Omega^1_I(C) \otimes \omega^k \to \mathcal{O}_C \otimes \omega^k$ followed by the isomorphism $\mathcal{O}_C \otimes \omega^k \cong \mathcal{O}_C$ deduced from the canonical section ω_{can} of ω. In particular, since ω has positive degree, r is surjective and we find

$$H^0(\tilde{X}, \Omega^{k+1}_{\tilde{X}})(\epsilon) \cong H^0(I, \Omega^1_I \otimes \omega^k)$$

and

$$H^1(\tilde{X}, \Omega^{k+1}_{\tilde{X}})(\epsilon) = 0.$$

(Here we have assumed $k > 0$. If k were 0, the group $H^0(C, \mathcal{O}_C)$ would be modified as in the remark after 4.1 and again r would be surjective.) Collating our results, we have the following.

Theorem 5.5. *We have isomorphisms*

$$H^j(\tilde{X}, \Omega^i_{\tilde{X}})(\epsilon) \cong \begin{cases} 0 & \text{if } i+j \neq k+1 \\ H^0(I, \Omega^1_I \otimes \omega^k) & \text{if } i = k+1 \text{ and } j = 0 \\ C^i_k & \text{if } 1 \leq i \leq k \text{ and } j = k+1-i \\ H^1(I, \omega^{-k}) & \text{if } i = 0 \text{ and } j = k+1 \end{cases}$$

compatible with the action of $(\mathbf{Z}/p\mathbf{Z})^\times$.

Corollary 5.6. *The Hodge to de Rham spectral sequence*

$$E_1^{p,q} = H^q(\tilde{X}, \Omega^p_{\tilde{X}})(\epsilon) \Longrightarrow H^{p+q}_{dR}(\tilde{X})(\epsilon)$$

degenerates at E_1 *and the crystalline cohomology groups* $H^d_{cris}(\tilde{X}/W)(\epsilon)$ *are torsion free for all* d.

Proof: We have inequalities

$$\text{rank}_W\, H^d_{cris}(\tilde{X}/W)(\epsilon) \leq \dim\, H^d_{dR}(\tilde{X})(\epsilon) \leq \sum_{i+j=d} \dim\, H^j(\tilde{X}, \Omega^i_{\tilde{X}})(\epsilon)$$

coming from the universal coefficients exact sequence in crystalline cohomology and the Hodge to de Rham spectral sequence respectively. Both inequalities are equalities for all d if and only if H^d_{cris} is torsion free for all d and the Hodge to de Rham spectral sequence degenerates at E_1. But the numerology just below computes the sum of the dimensions of the $H^j(\tilde{X}, \Omega^i_{\tilde{X}})(\epsilon)$ and Corollary 2.2, together with standard formulae for dimensions of spaces of modular forms (e.g., [CO], Thm. 1), computes the ranks of the $H^d_{cris}(\tilde{X}/W)(\epsilon)$. These numbers turn out to be equal. (Their common value is 0 if $d \neq k+1$ and is the sum of the l_i in the statement of Theorem 1.2 if $d = k+1$.) □

Remark: The proof of Corollary 5.6 also shows that the Newton and Hodge polygons associated to $(\tilde{X}, \epsilon)$ have the same endpoints.

It remains to decompose the groups $H^j(\tilde{X}, \Omega^i_{\tilde{X}})(\epsilon)$ for the action of the diamond operators $\langle d \rangle$, $d \in (\mathbf{Z}/p\mathbf{Z})^\times$ and to compute the dimensions of the pieces. Fix a power χ^a of the basic character $\chi : (\mathbf{Z}/p\mathbf{Z})^\times \to \mathbf{C}$, written with $0 \leq a \leq p-2$.

Recall that the basic character can be identified with the Teichmüller character $\chi : (\mathbf{Z}/p\mathbf{Z})^\times \to \mathbf{Z}_p$ using the valuation v. We compute the action of $\langle d \rangle^*$ on ω_{can} using its characterizing property above: at an ordinary point (E, P, i) where $i : \mathbf{Z}/p^n\mathbf{Z} \hookrightarrow E^{(p^n)}$ has inverse Cartier dual $j : \mu_{p^n} \hookrightarrow E$, we have

$$j^*(\langle d \rangle^* \omega_{can})(E, P, i) = j^* \omega_{can}(E, P, \chi(d) i) = \chi(d)\, dt/t$$

so $\langle d \rangle^* \omega_{can} = \chi(d) \omega_{can}$. On the other hand, as ω_{can} vanishes to order p^{n-1} at each supersingular point as a section of ω, and since ω is the pull-back of an invertible sheaf on $X_1(N)$, it has a generating section which is invariant for the $\langle d \rangle$. We find then that the $\langle d \rangle$ act on the cotangent space to I at each supersingular point via the character χ. Also, at each supersingular point and for each l, $\Omega^1_I(C) \otimes \omega^l$ has a local section invariant for the $\langle d \rangle$ which vanishes to order $p^{n-1}(p^n - 2)$; thus the $\langle d \rangle$ act on a generating section via χ.

To compute dimensions, let g be the genus of $X_1(N)$ and c the number of cusps on this modular curve. The Kodaira-Spencer isomorphism $\Omega^1_{X_1(N)}(\mathit{log\ cusps}) \cong \omega^2$ on $X_1(N)$ gives that $w = g - 1 + c/2$ is the degree of ω on $X_1(N)$. By [KM] 12.9.4, we have that the number of supersingular points on I is $(p-1)w$,

$$\deg_I \omega = p^{n-1}(p-1)w,$$

and

$$2g_I - 2 = p^{2n-1}(p-1)w - p^{n-1}(p-1)c.$$

Applying the observations of the previous paragraph, we see that

$$\dim C_k^i(\chi^a) = \begin{cases} \left(p^{n-1}(p^n-2)+1\right) w & \text{if } a \neq 0 \\ \left(p^{n-1}(p^n-2)-(p-2)\right) w & \text{if } a = 0 \end{cases} \tag{5.7}$$

To find the dimension of $H^0(I, \Omega^1 \otimes \omega^k)(\chi^a)$, let Y be the quotient of I by $(\mathbf{Z}/p\mathbf{Z})^\times$; the curve Y sits in the tower

$$I \to Y \to X_1(N)$$

and carries an invertible sheaf (also called ω) whose pull-back to I is ω. The map $I \to Y$ has degree $p-1$ and is totally ramified at the supersingular points and unramified elsewhere. Let $S \subseteq Y$ be the reduced divisor of supersingular points and let $a' = 0$ if $a = 0$ and $a' = p-1-a$ if $a \neq 0$. We have an isomorphism

$$H^0(I, \Omega^1 \otimes \omega^k)(\chi^a) \cong H^0(I, \Omega^1 \otimes \omega^{k+a'}(-a'p^{n-1}S))(\chi^0)$$

$$s \mapsto s\omega_{can}^{a'}$$

and by the above remarks,

$$H^0(I, \Omega^1 \otimes \omega^{k+a'}(-a'p^{n-1}S))(\chi^0) \cong H^0(Y, \Omega^1_Y \otimes \omega^{k+a'}(bS))$$

where $b = \lfloor (-p^{n-1}a' + p - 2)/(p-1) \rfloor = -a'(p^{n-1}-1)/(p-1)$. (Here $\lfloor x \rfloor$ means the greatest integer $\leq x$.) Now using Riemann-Roch and Riemann-Hurwitz, the dimension of this last group is

$$\begin{cases} \left(p^{2n-1} - p + 2\right) w/2 - p^{n-1}c/2 + 1 & \text{if } k = a = 0 \\ \left(p^{2n-1} - p + 2 + 2kp^{n-1} + 2a'\right) w/2 - p^{n-1}c/2 & \text{otherwise} \end{cases}.$$

Finally, $H^1(I, \omega^{-k})(\chi^a) \cong H^0(I, \Omega^1_I \otimes \omega^k)(\chi^{-a'})^*$. Summing up,

Theorem 5.8. *The dimensions of the Hodge cohomology groups* $H^j(\tilde{X},\Omega^i_{\tilde{X}})(\epsilon,\chi^a)$ *are:*

$$\dim H^0(\tilde{X},\Omega^{k+1}_{\tilde{X}})(\epsilon,\chi^a) = \begin{cases} \left(p^{2n-1}-p+2\right)w/2 - p^{n-1}c/2+1 & \text{if } k=a=0 \\ \left(p^{2n-1}-p+2+2kp^{n-1}+2a'\right)w/2 - p^{n-1}c/2 & \text{otherwise,} \end{cases}$$

for $1 \le i \le k$

$$\dim H^{k+1-i}(\tilde{X},\Omega^i_{\tilde{X}})(\epsilon,\chi^a) = \begin{cases} \left(p^{n-1}(p^n-2)+1\right)w & \text{if } a \neq 0 \\ \left(p^{n-1}(p^n-2)-(p-2)\right)w & \text{if } a=0, \end{cases}$$

and

$$\dim H^{k+1}(\tilde{X},\mathcal{O}_{\tilde{X}})(\epsilon,\chi^a) = \begin{cases} \left(p^{2n-1}-p+2\right)w/2 - p^{n-1}c/2+1 & \text{if } k=a=0 \\ \left(p^{2n-1}-p+2+2kp^{n-1}+2a\right)w/2 - p^{n-1}c/2 & \text{otherwise.} \end{cases}$$

Here $a' = p-1-a$ *if* $a \neq 0$ *and* $a'=0$ *if* $a=0$.

Combining Corollary 2.2, Proposition 3.1 and Theorem 5.8 yields Theorem 1.3 for $p > 2$.

Remark: A more elegant approach might be to do the cohomology calculations of [U2] in the crystalline cohomology of $X^\times$ and to prove Nygaard's theorem with projectors for log cohomology. We would then be able to avoid the crutch of the smooth variety $\tilde{X}$ by consistently employing log structures.

6. Another approach in weight 3 In this section we will give another proof of Theorem 1.2 for $k=1$ and any p. Although it is possible to do so, for convenience we will not break up the result according to powers of the Teichmüller character as we are mainly interested in the case $p=2$ (for which Theorem 1.2 and 1.3 are equivalent).

Fix p a prime, n and N positive integers with $(N,p)=1$ and set $k=1$. We assume that $p^nN\geq 3$ (otherwise there are no modular forms to consider). Keeping the notations of §2, we have $\tilde{X}=X=\mathcal{E}$ a smooth elliptic surface over $\mathbf{F}_p$ with elliptic fibration $\pi:X\to I$ where I is the Igusa curve $Ig_1(p^nN)$. Our proof will be an easy consequence of the following (somewhat weakened) result of Milne ([M1], 7.4).

Theorem. *Suppose X is a smooth projective surface over the field of p elements. Let Z be the H^2 zeta function of X:*

$$Z(T)=\det\left(1-\mathrm{Fr}\ T|H^2(X\otimes\overline{\mathbf{F}_p},\mathbf{Q}_\ell)\right)$$

and write $Z(T)=\prod(1-\alpha_iT)$. Fixing a valuation v of $\overline{\mathbf{Q}}$ over p normalized so that $v(p)=1$, we have

$$\sum_{v(\alpha_i)<1}(1-v(\alpha_i))\leq e$$

where $e=\chi(X,\mathcal{O}_X)-1+\dim(Pic\ Var(X))$.

Now let

$$L_i(T)=\det\left(1-\mathrm{Fr}\ T|H^{2-i}(I\otimes\overline{\mathbf{F}_p},R^i\pi_*\mathbf{Q}_\ell)\right).$$

The Leray spectral sequence of π (in ℓ-adic cohomology) degenerates and so we have $Z=L_0L_1L_2$. But L_0 and L_2 have the form $\prod_j(1-\zeta_jpT)$ where the ζ_j are roots of unity, since the corresponding cohomology groups are spanned by the cycle classes of the zero section and of the components of the fibers respectively. Thus the conclusion of Milne's theorem holds with Z replaced by L_1. On the other hand, according to the main theorem of [U2],

$$L_1=\prod_{a=0}^{p-2}H(1,n,N,a)$$

(where the H's are as defined in §1).

The Riemann hypothesis and functional equation for varieties over finite fields show that the hypotheses of the following result are satisfied by L_1.

Lemma 6.1. *Suppose $H(T) = \prod(1 - \alpha_i T)$ is a polynomial of degree d whose inverse roots α_i are algebraic integers satisfying $\alpha_i\overline{\alpha}_i = p^2$ and $\{\alpha_i\} = \{\overline{\alpha}_i\}$. If*

$$\sum_{v(\alpha_i)<1} (1 - v(\alpha_i)) \leq e$$

then the Newton polygon of H (with respect to v) lies on or above the Hodge polygon associated to the integers $(e, d-2e, e)$ and these two polygons have the same endpoints.

Proof: That the two polygons have the same endpoints is clear from the hypotheses. To check the statement on the polygons, number the α_i so that $v(\alpha_1) \leq \cdots \leq v(\alpha_d)$. Then since $v(\alpha_i) \geq 0$, we have

$$\sum_{i=1}^{j} v(\alpha_i) \geq \begin{cases} 0 & \text{if } j \leq e \\ j-e & \text{if } j \geq e \end{cases}$$

for all $j \leq \lceil d/2 \rceil$. This is exactly the claim for the first half of the polygon. For the second half, we note that the hypotheses $v(\alpha_i) + v(\overline{\alpha}_i) = 2$ and $\{\alpha_i\} = \{\overline{\alpha}_i\}$ imply that the Newton polygon of H is symmetric with respect to the line $y = (d-e) - x$. Thus the lemma is proved. $\square$

To complete the proof of Theorem 1.2 it remains to compute d and e. The computation of d (i.e., the dimension of a certain space of modular forms) is standard and we merely record its value:

$$d = 2p^{2n-1}(p-1)w - p^{n-1}(p-1)c$$

where g and c are the genus and number of cusps on $X_1(N)$ respectively and $w = g-1+c/2$. To compute e, recall that since X is a non-constant elliptic surface, dim $\mathrm{PicVar}(X) = g_I$, the genus of I. As before, define an invertible sheaf ω on I by $\omega^{-1} = R^1\pi_*\mathcal{O}_X$; then the Leray spectral sequence for π (in coherent cohomology) yields that $\chi(X, \mathcal{O}_X) = w_I$, the degree of ω on I. Now applying the "numerology" of [KM], 12.9.4, we find

$$e = \frac{p^{n-1}(p-1)}{2}\left((p^n+2)w - c\right).$$

Combining Milne's Theorem, 6.1, and the values of d and e, we get a lower bound on the Newton polygon of $\prod_{a=0}^{p-2} H(1,n,N,a)$. The result agrees with Theorem 1.2 when $p > 2$ and proves it for $p = 2$.

Remarks: 1) The theorem of Milne cited above actually asserts the that the sum over eigenvalues is equal to e minus the dimension of a certain group scheme. This dimension measures the failure of the Newton polygon to satisfy a certain "kissing condition" with respect to the Hodge polygon, and it is possible to determine in some cases whether or not this dimension vanishes. We will return to this point in a future paper.

2) The proof of Theorem 1.2 given in this section can also be carried through for the cases $N \le 4$, as long as $p^nN > 2$ (so that there exists a universal curve over some open supset of the Igusa curve). Note that there are no modular forms of weight 3 for $\Gamma_1(2)$, so there is nothing to prove when $p^nN = 2$.

7. The cases $N \le 4$ When $N \le 4$ the moduli problem $\Gamma_1(N)$ is not representable and we do not have a universal curve over all of I, so our previous argument requires modification. In this section we sketch a proof for these cases (when $p > 3$) and state the results.

Fix a prime number p, positive integers n and k with $k < p$, a positive integer $N \leq 4$ prime to p, and an auxiliary positive integer $M > 3$ with $(M, N) = 1$ and such that p does not divide the order of $\mathrm{GL}_2(\mathbf{Z}/M\mathbf{Z})$ (which is possible if and only if $p > 3$). Fix also a power χ^a of the basic character $\chi : (\mathbf{Z}/p\mathbf{Z})^\times \to \mathbf{C}$ associated to a fixed valuation v of $\mathbf{Q}(\mu_p)$ over p; we take $0 \leq a \leq p-2$. Let I_M be the complete modular curve attached to the simultaneous moduli problem $Ig(p^n) \times \Gamma_1(N) \times \Gamma(M)$ and let $\tilde{X}_M$ be the desingularization of the k-fold fiber product of the universal curve $\mathcal{E}_M \to I_M$, defined as in Section 2. We have an action of $G = \left((\mathbf{Z}/N\mathbf{Z} \times (\mathbf{Z}/M\mathbf{Z})^2) \rtimes \mu_2\right)^k \rtimes S_k$ on $\tilde{X}_M$ covering the trivial action on I_M, and an action of $\mathrm{GL}_2(\mathbf{Z}/M\mathbf{Z}) \times (\mathbf{Z}/p^n\mathbf{Z})^\times$ on $\tilde{X}_M$ and I_M; (± 1), embedded diagonally in $\mathrm{GL}_2(\mathbf{Z}/M\mathbf{Z}) \times (\mathbf{Z}/p^n\mathbf{Z})^\times$, acts trivially on I_M (but not trivially on $\tilde{X}_M$). The quotient of I_M by the $\mathrm{GL}_2(\mathbf{Z}/M\mathbf{Z})$ is I, the modular curve for the moduli problem $Ig(p^n) \times \Gamma_1(N)$ and the subgroup $(\mathbf{Z}/p\mathbf{Z})^\times \subseteq (\mathbf{Z}/p^n\mathbf{Z})^\times$ acts on I, with (± 1) acting trivially; let Y be the quotient. The curve Y can be interpreted as the modular curve for $\mathcal{P} \times \Gamma_1(N)$ where $\mathcal{P}$ is a certain moduli problem of p-power level which we need not make explicit here.

Let $\epsilon_1 : G \to \pm 1$ be the character which is trivial on the factors $\mathbf{Z}/N\mathbf{Z} \times (\mathbf{Z}/M\mathbf{Z})^2$, the identity on the factors μ_2, and the sign character on the symmetric group S_k. Let ϵ_2 be the trivial character $\mathrm{GL}_2(\mathbf{Z}/M\mathbf{Z}) \to 1$ and let $\epsilon_3 = \chi^a$. Then we have corresponding projectors Π_1, Π_2, and Π_3 in the group ring $\mathbf{Z}_p[G \times \mathrm{GL}_2(\mathbf{Z}/M\mathbf{Z}) \times (\mathbf{Z}/p\mathbf{Z})^\times]$; note that the Π_i commute. For any $G \times \mathrm{GL}_2(\mathbf{Z}/M\mathbf{Z}) \times (\mathbf{Z}/p\mathbf{Z})^\times$-module H we will write $H(\epsilon_i)$ for $\Pi_i H$.

Let $E(k,n,N,\psi)$ be defined as in Section 1 and set

$$H(k,n,N,a) = \prod_{0\le m\le n} \prod_{\substack{\psi:(\mathbf{Z}/p^mN\mathbf{Z})^\times\to\mathbf{C} \\ \psi=\chi^a\eta\theta \\ p^m|cond(\psi)}} E(k,m,N,\psi).$$

Then a trivial modification of the argument of Section 2 (replacing $\Gamma_1(N)$ with $\Gamma_1(N)\times\Gamma(M)$ throughout) gives that

$$H(k,n,N,a) = \det\left(1 - \mathrm{Fr}\, T | H^*_{cris}(\tilde{X}_M/W)(\epsilon_1,\epsilon_2,\epsilon_3)\right).$$

Applying Proposition 3.1, we find that the Newton polygon of $H(k,n,N,a)$, with respect to the valuation v, is bounded below by the Hodge polygon associated to the integers $l_i = \dim\, H^{k+1-i}(\tilde{X}_M,\Omega^i)(\epsilon_1,\epsilon_2,\epsilon_3)$.

Now the arguments of Sections 4 and 5 apply verbatim to show that

$$H^{k+1-i}(\tilde{X}_M,\Omega^i)(\epsilon_1) = \begin{cases} H^1(I_M,\omega^{-k}) & \text{if } i=0 \\ C_k^i & \text{if } 1\le i\le k \\ H^0(I_M,\Omega^1\otimes\omega^k) & \text{if } i=k+1 \end{cases}$$

where C_k^i is the group of local sections of $\Omega^1_{I_M}\otimes\omega^{2i-2-k}$ modulo those vanishing to order $p^{n-1}(p^n-2)$ at each supersingular point and ω is the invertible sheaf $(R^1\pi_*\mathcal{O}_{\mathcal{E}_M})^{-1}$ on I_M. This isomorphism is compatible with the actions of $\mathrm{GL}_2(\mathbf{Z}/M\mathbf{Z})$ and $(\mathbf{Z}/p\mathbf{Z})^\times$ and so we have to compute the dimensions of the spaces $H^1(I_M,\omega^{-k})(\epsilon_2,\epsilon_3)$, $C_k^i(\epsilon_2,\epsilon_3)$, and $H^0(I_M,\Omega^1\otimes\omega^k)(\epsilon_2,\epsilon_3)$. The computation of these dimensions follows closely the argument in the last part of Section 5, and is essentially straightfoward, albeit tedious. One slightly subtle point is that the invertible sheaf ω does not descend to Y. However, when $N=1$ or 2 (resp. when $N=3$ or 4), there is an invertible sheaf $\tilde{\omega}^2$ (resp. $\tilde{\omega}$) on Y whose pull back to I_M is ω^2 (resp. ω) twisted by a divisor supported on the points parameterizing generalized elliptic curves with level structure which have extra automorphisms. To carry out the computations, one needs to keep track of ramification in the map $I_M \to Y$ and one also needs to know the genus of Y, which can be obtained from the Riemann-Hurwitz formula and Igusa's computation ([Ig]) of the genus of $Ig(p^n)$.

In the statement of the theorem below, $\lfloor x \rfloor$ is the greatest integer $\le x$.

Theorem 7.1. *Fix a prime number $p > 3$ and integers k, n, N, and a with $0 \le k < p$, $n > 0$, $1 \le N \le 4$, $(N, p) = 1$, and $0 \le a \le p-2$; if $N \le 2$, suppose $a \equiv k \pmod 2$. Let $a' = 0$ if $a = 0$ and let $a' = p - 1 - a$ if $a \neq 0$. Also, let $A = a\left(\frac{p^{n-1}-1}{p-1}\right)$ and $A' = a'\left(\frac{p^{n-1}-1}{p-1}\right)$. For $1 \le i \le k$, define integers*

$$\delta_1 = \lfloor \frac{p^{n-1}(p^n-2)}{p-1} \rfloor + \begin{cases} 1 & \text{if } a \neq 0 \\ 0 & \text{if } a = 0 \end{cases}$$

$$\delta_2 = \lfloor \frac{p^{n-1}(p^n-2)}{2(p-1)} \rfloor + \begin{cases} 1 & \text{if } a \neq 0 \text{ and } 2i+2 \equiv k - (-1)^n a \pmod 4 \\ 0 & \text{otherwise} \end{cases}$$

$$\delta_3 = \lfloor \frac{p^{n-1}(p^n-2)}{3(p-1)} \rfloor + \begin{cases} & \text{if } n \text{ odd, } a \neq 0, \text{ and } i \equiv 1 - a - k \pmod 3 \\ 1 & \text{or if } n \text{ even, } a = 0, \text{ and } i \equiv -1 - k \pmod 3 \\ & \text{or if } n \text{ even, } a \neq 0, \text{ and } i \not\equiv a - k \pmod 3 \\ 0 & \text{otherwise.} \end{cases}$$

If $N = 1$, set

$$g = \frac{p^{2n-1} - 12p^{n-1} - p + 26}{24} - \frac{1}{4}\begin{cases} p^{n-1} & \text{if } p \equiv 1 \pmod 4 \\ 1 & \text{if } p \equiv -1 \pmod 4 \end{cases} - \frac{1}{3}\begin{cases} p^{n-1} & \text{if } p \equiv 1 \pmod 3 \\ 1 & \text{if } p \equiv -1 \pmod 3 \end{cases}$$

$$l_0 = g - 1 + \frac{k+a}{2}p^{n-1} - A\left(\frac{p-1}{12}\right)$$

$$- \begin{cases} \lfloor \frac{k+a}{4} \rfloor p^{n-1} & \text{if } p \equiv 1 \pmod 4 \\ \left\lfloor \frac{k+a}{4}p^{n-1} + \frac{ap^{n-1}}{2(p-1)} \right\rfloor - \frac{1}{2}A & \text{if } p \equiv -1 \pmod 4 \end{cases}$$

$$- \begin{cases} \lfloor \frac{k+a}{6} \rfloor p^{n-1} & \text{if } p \equiv 1 \pmod 3 \\ \left\lfloor \frac{k+a}{6}p^{n-1} + \frac{ap^{n-1}}{3(p-1)} \right\rfloor - \frac{1}{3}A & \text{if } p \equiv -1 \pmod 3 \end{cases}$$

$$+ \begin{cases} 1 & \text{if } k = a = 0 \\ 0 & \text{otherwise,} \end{cases}$$

$$l_i = \left(\frac{p-1}{12}\right)\delta_1 + \begin{cases} 0 & \text{if } p \equiv 1 \pmod 4 \\ \delta_2 - \frac{1}{2}\delta_1 & \text{if } p \equiv -1 \pmod 4 \end{cases} + \begin{cases} 0 & \text{if } p \equiv 1 \pmod 3 \\ \delta_3 - \frac{1}{3}\delta_1 & \text{if } p \equiv -1 \pmod 3 \end{cases}$$

for $1 \leq i \leq k$, *and*

$$l_{k+1} = g - 1 + \frac{k+a'}{2}p^{n-1} - A'\left(\frac{p-1}{12}\right)$$

$$- \begin{cases} \left\lfloor \frac{k+a'}{4} \right\rfloor p^{n-1} & \text{if } p \equiv 1 \pmod 4 \\ \left\lfloor \frac{k+a'}{4}p^{n-1} + \frac{a'p^{n-1}}{2(p-1)} \right\rfloor - \frac{1}{2}A' & \text{if } p \equiv -1 \pmod 4 \end{cases}$$

$$- \begin{cases} \left\lfloor \frac{k+a'}{6} \right\rfloor p^{n-1} & \text{if } p \equiv 1 \pmod 3 \\ \left\lfloor \frac{k+a'}{6}p^{n-1} + \frac{a'p^{n-1}}{3(p-1)} \right\rfloor - \frac{1}{3}A' & \text{if } p \equiv -1 \pmod 3 \end{cases}$$

$$+ \begin{cases} 1 & \text{if } k = a = 0 \\ 0 & \text{otherwise,} \end{cases}$$

If $N = 2$, *set*

$$g = \frac{p^{2n-1} - 8p^{n-1} - p + 10}{8} - \frac{1}{4}\begin{cases} p^{n-1} & \text{if } p \equiv 1 \pmod 4 \\ 1 & \text{if } p \equiv -1 \pmod 4 \end{cases}$$

$$l_0 = g - 1 + \frac{k+a}{2}p^{n-1} - A\left(\frac{p-1}{4}\right)$$

$$- \begin{cases} \lfloor \frac{k+a}{4} \rfloor p^{n-1} & \text{if } p \equiv 1 \pmod 4 \\ \left\lfloor \frac{k+a}{4}p^{n-1} + \frac{ap^{n-1}}{2(p-1)} \right\rfloor - \frac{3}{2}A & \text{if } p \equiv -1 \pmod 4 \end{cases}$$

$$+ \begin{cases} 1 & \text{if } k = a = 0 \\ 0 & \text{otherwise,} \end{cases}$$

$$l_i = \left(\frac{p-1}{4}\right)\delta_1 + \begin{cases} 0 & \text{if } p \equiv 1 \pmod 4 \\ \delta_2 - \frac{1}{2}\delta_1 & \text{if } p \equiv -1 \pmod 4 \end{cases}$$

for $1 \le i \le k$, *and*

$$\begin{aligned} l_{k+1} &= g - 1 + \frac{k+a'}{2}p^{n-1} - A'\left(\frac{p-1}{4}\right) \\ &\quad - \begin{cases} \left\lfloor \frac{k+a'}{4} \right\rfloor p^{n-1} & \text{if } p \equiv 1 \pmod 4 \\ \left\lfloor \frac{k+a'}{4}p^{n-1} + \frac{a'p^{n-1}}{2(p-1)} \right\rfloor - \frac{3}{2}A' & \text{if } p \equiv -1 \pmod 4 \end{cases} \\ &\quad + \begin{cases} 1 & \text{if } k = a = 0 \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

If $N = 3$, *set*

$$\begin{aligned} g &= \frac{p^{2n-1} - 6p^{n-1} - p + 8}{6} \\ &\quad - \frac{1}{3}\begin{cases} p^{n-1} & \text{if } p \equiv 1 \pmod 3 \\ 1 & \text{if } p \equiv -1 \pmod 3 \end{cases} \end{aligned}$$

$$\begin{aligned} l_0 &= g - 1 + (k+a)p^{n-1} - A\left(\frac{p-1}{3}\right) \\ &\quad - \begin{cases} \left\lfloor \frac{2(k+a)}{3} \right\rfloor p^{n-1} & \text{if } p \equiv 1 \pmod 3 \\ \left\lfloor \frac{2(k+a)}{3}p^{n-1} + \frac{ap^{n-1}}{3(p-1)} \right\rfloor - \frac{4}{3}A & \text{if } p \equiv -1 \pmod 3 \end{cases} \\ &\quad + \begin{cases} 1 & \text{if } k = a = 0 \\ 0 & \text{otherwise,} \end{cases} \end{aligned}$$

$$l_i = \left(\frac{p-1}{3}\right)\delta_1 + \begin{cases} 0 & \text{if } p \equiv 1 \pmod 3 \\ \delta_3 - \frac{1}{3}\delta_1 & \text{if } p \equiv -1 \pmod 3 \end{cases}$$

for $1 \le i \le k$, *and*

$$\begin{aligned} l_{k+1} &= g - 1 + (k+a')p^{n-1} - A'\left(\frac{p-1}{3}\right) \\ &\quad - \begin{cases} \left\lfloor \frac{2(k+a')}{3} \right\rfloor p^{n-1} & \text{if } p \equiv 1 \pmod 3 \\ \left\lfloor \frac{2(k+a')}{3}p^{n-1} + \frac{a'p^{n-1}}{3(p-1)} \right\rfloor - \frac{4}{3}A' & \text{if } p \equiv -1 \pmod 3 \end{cases} \\ &\quad + \begin{cases} 1 & \text{if } k = a = 0 \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

If $N = 4$, set

$$g = \frac{p^{2n-1} - 6p^{n-1} - p + 6}{4}$$

$$l_0 = g - 1 + \left\lfloor \frac{k+a+1}{2} \right\rfloor p^{n-1} - A\left(\frac{p-1}{2}\right) + \begin{cases} 1 & \text{if } k = a = 0 \\ 0 & \text{otherwise,} \end{cases}$$

$$l_i = \left(\frac{p-1}{2}\right)\delta_1$$

for $1 \le i \le k$, and

$$l_{k+1} = g - 1 + \left\lfloor \frac{k+a'+1}{2} \right\rfloor p^{n-1} - A'\left(\frac{p-1}{2}\right) + \begin{cases} 1 & \text{if } k = a = 0 \\ 0 & \text{otherwise.} \end{cases}$$

Then the Newton polygon, with respect to the valuation v fixed above, of the polynomial

$$H(k,n,1,a) = \prod_{0 \le m \le n} \prod_{\substack{\psi : (\mathbf{Z}/p^m N\mathbf{Z})^\times \to \mathbf{C} \\ \psi = \chi^a \eta\theta \\ p^m | cond(\psi)}} E(k,m,1,\psi)$$

(here the second product runs over all characters ψ modulo $p^m N$ whose conductor is divisible by p^m and whose restriction to $(\mathbf{Z}/p\mathbf{Z})^\times$ is χ^a) lies on or above the Hodge polygon associated to the integers $l_0, \ldots l_{k+1}$. Moreover, these two polygons have the same endpoints.

Remarks: 1) If $N \le 2$ and $a \not\equiv k \pmod 2$ (i.e., $\chi^a(-1) \ne (-1)^k$) then there are no modular forms with character ψ and the polynomial $H(k,n,1,a) = 1$.

2) The integer g is the genus of Y. The expression $g - 1$ in Theorem 1.4 is the analogue

of the expression $p^n\phi(p^n)w/2 - \phi(p^n)c/2$ in Theorem 1.3.

3) The integer δ_1 is the number of integers x with $p^{n-1}a' \leq x < p^{n-1}a' + p^{n-1}(p^n - 2)$ and $x \equiv -1 \pmod{p-1}$. The integer δ_2 (resp. δ_3) is the number of integers x in the same range which are congruent to $-1 + \frac{2i-2-k+a'}{2}\phi(p^n)$ modulo $2(p-1)$ when $p \equiv -1 \pmod 4$ (resp. are congruent to $-1 + (2i - 2 - k + a')\phi(p^n)$ modulo $3(p-1)$ when $p \equiv -1 \pmod 3$).

The only cases not covered by Theorems 1.3 and 7.1 are those where $N \leq 4$ and $p \leq 3$. The cases N arbitrary, $k = 1$, p arbitrary can be treated as in Section 6. The cases $k = 2$, $p = 3$, $N = 1, 2$ or 4 seem to require a detailed geometric analysis (as in Section 4) of singularities arising from the bad reduction of the universal curve at the unique supersingular point (when $N = 1$ or 2) or at the irregular cusps (if $N = 4$). We will not pursue this further.

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