

CONDUCTORS OF ℓ -ADIC REPRESENTATIONS

DOUGLAS ULMER

(Communicated by Matthew A. Papanikolas)

ABSTRACT. We give a new formula for the Artin conductor of an ℓ -adic representation of the Weil group of a local field of residue characteristic $p \neq \ell$.

1. INTRODUCTION

Our aim in this note is to give a new formula for the Artin conductor of an ℓ -adic representation of the Galois group of a non-archimedean local field of residue characteristic $p \neq \ell$; see Theorem 1 in Section 9. The proof that this formula is equivalent to the standard one is a simple unwinding of the definitions, but the new formula has the virtue that it does not require any auxiliary constructions such as semi-simplification or Weil-Deligne representations, and it works equally well for representations with finite or infinite inertial image. We also take the occasion to correct a typographical error in a canonical reference.

We would have imagined that formula (9.1) was well known, and make no claim of priority, but a proof that it is equivalent to other definitions does not seem to appear in the literature.¹ We hope having a complete treatment will be useful to the community.

Here is a brief explanation of the main issue: Let F be a local field with Weil group W_F . The conductor of a representation ρ of W_F depends only on the restriction of ρ to the inertia group I_F , and it is defined in the first instance in [Art31] only for representations which factor through finite quotients of W_F . Since the image of an ℓ -adic representation restricted to inertia need not be finite, further discussion is required. This problem was solved by Serre, who in [Ser70] gave a definition of the conductor of an ℓ -adic representation using the key fact that there is a subgroup of inertia of finite index on which the representation is unipotent. This last fact was conjectured by Serre and Tate and proven by Grothendieck; see [ST68, Appendix]. Later, Deligne gave a definition using what is now known as a Weil-Deligne representation, a technical device with several uses in the Langlands program; see [Del73] and [Tat79].

For the convenience of the reader, we discuss the previous definitions of the conductor (Artin's definition in Sections 2 through 4, Serre's in Sections 5 and 6, and Deligne's in Sections 7 and 8) with full definitions, but minimal proofs. In Section 9 we give the new formula, and in Section 10 we give two applications, one of which motivated this work.

Received by the editors May 14, 2015 and, in revised form, June 29, 2015.

2010 *Mathematics Subject Classification*. Primary 11F80.

¹The formula appears as a definition in [DDT97, §2.1] and in the notes [Wie12], but without proof that it is equivalent to the standard definition.

2. GALOIS GROUPS AND REPRESENTATIONS

Throughout the paper, F will be a non-archimedean local field with residue field of characteristic p and cardinality q . In other words, F is a finite extension of either \mathbb{Q}_p or $\mathbb{F}_p((t))$. Let \overline{F} be a separable closure of F , G_F the Galois group $\text{Gal}(\overline{F}/F)$, and Φ a geometric Frobenius element, i.e., an element of G_F which induces the inverse of the q -power Frobenius automorphism of the residue field of \overline{F} . Let W_F be the Weil group of F , the subgroup of G_F inducing integral powers of the q -power Frobenius on the residue field, and let I_F be the inertia subgroup of G_F , the subgroup acting as the identity on the residue field. We give W_F the topology characterized by the requirement that I_F be an open subgroup.

Let ℓ be a prime number distinct from p and let E be a finite extension of \mathbb{Q}_ℓ , the ℓ -adic numbers. An ℓ -adic representation with values in E is a continuous homomorphism

$$\rho_\ell : W_F \rightarrow \text{GL}_n(E)$$

where $\text{GL}_n(E)$ is given the topology induced by the metric (ℓ -adic) topology on E . (When $E \neq \mathbb{Q}_\ell$, some authors refer to ρ_ℓ as a λ -adic representation.)

We are concerned with (the exponent of) the Artin conductor of ρ_ℓ , which we denote $a(\rho_\ell)$ and call simply the conductor. It will be defined in Section 6 below.

3. RAMIFICATION GROUPS

In this section, we review the lower and upper ramification filtrations on Galois groups. See [Ser79, Ch. IV] for more details.

Let K be a finite Galois extension of F with group $G = \text{Gal}(K/F)$. We write \mathcal{O}_K for the ring of integers of K , π_K for a generator of the maximal ideal of \mathcal{O}_K , and v_K for the valuation of K with $v_K(\pi_K) = 1$.

The ramification filtration on G in the lower numbering is defined by the requirement that

$$\sigma \in G_i \iff v_K(\sigma(x) - x) \geq i + 1 \quad \forall x \in \mathcal{O}_K$$

for i an integer ≥ -1 . Clearly $G_{-1} = G$, G_0 is the inertia subgroup of G , and $G_i = 0$ for all sufficiently large i . By convention, if $r \geq -1$ is a real number, we set $G_r = G_i$ where i is the smallest integer $\geq r$.

Let $\varphi : [-1, \infty) \rightarrow [-1, \infty)$ be the continuous, piecewise linear function with $\varphi(-1) = -1$, slope 1 on $[-1, 0)$, and slope $1/[G_0 : G_i]$ on $(i - 1, i)$. Let $\psi = \varphi^{-1}$, the inverse function. The upper numbering of the ramification filtration on G is given by

$$G^s = G_{\psi(s)} \quad \text{and} \quad G^{\varphi(r)} = G_r.$$

Note that the breaks in the upper numbering (i.e., the values s so that $G^{s+\epsilon} \neq G^s$ for all $\epsilon > 0$) are in general rational numbers, not necessarily integers.

The upper numbering is adapted to quotients in the following sense: if L/F is a Galois extension with $L \subset K$ and $H = \text{Gal}(L/F)$, so that H is a quotient of G , then the upper numbering satisfies

$$H^s = \text{Im}(G^s \rightarrow H).$$

This property allows us to define a ramification filtration on $G_F = \text{Gal}(\overline{F}/F)$ by declaring that

$$G_F^s = \{ \sigma \in G_F \mid \sigma|_K \in \text{Gal}(K/F)^s \forall K \}$$

where K runs through all finite Galois extensions of F . Clearly we have $G_F^{-1} = G_F$ and $G_F^0 = I_F$.

We define

$$G_F^{>0} = \bigcup_{\epsilon > 0} G_F^\epsilon$$

where the union is over all positive real numbers ϵ . We also write P_F for $G_F^{>0}$ and call this the wild inertia group of F . It is known to be a pro- p group, and the quotient I_F/P_F is isomorphic as a profinite group to $\prod_{\ell \neq p} \mathbb{Z}_\ell$.

4. ARTIN CONDUCTOR

In this section we review the definition of the Artin conductor of a representation of $G = \text{Gal}(K/F)$ where K/F is a finite Galois extension. The standard reference for this material is [Ser79, Ch. VI].

Let $\rho : G \rightarrow \text{GL}_n(E)$ be a representation where E is a field of characteristic zero. We write V for the space where ρ acts, namely E^n , and for a subgroup H of G we write V^H for the invariants under H :

$$V^H = \{v \in V \mid \rho(h)(v) = v \ \forall h \in H\}.$$

Recall the ramification subgroups G_i of the previous section. For a subspace W of V , we write $\text{codim } W$ for the codimension of W in V , i.e., $\dim V - \dim W$. Following Artin [Art31], we define the *Artin conductor* of ρ as

$$(4.1) \quad a(\rho) := \sum_{i=0}^{\infty} \frac{\text{codim } V^{G_i}}{[G_0 : G_i]}.$$

Note that this is in fact a finite sum and that it depends only on the restriction of ρ to G_0 , the inertia subgroup of G . It is true but not at all obvious that $a(\rho)$ is an integer; see [Ser79, Ch. VI, §2, Thm. 1’].

Because the definition of $a(\rho)$ depends only on ρ restricted to inertia, we may extend it to representations ρ which are only assumed to have finite image after restriction to inertia.

We give two alternate expressions for $a(\rho)$ which will be useful in what follows. First, we have

$$a(\rho) = \int_{-1}^{\infty} \frac{\text{codim } V^{G_r}}{[G_0 : G_r]} dr$$

because the integrand is constant on intervals $(i - 1, i)$ and the corresponding Riemann sum for the integral is exactly the sum defining $a(\rho)$. Second,

$$a(\rho) = \int_{-1}^{\infty} \text{codim } V^{G^s} ds.$$

This follows from the previous expression and the definition of the function φ relating the upper and lower numberings. Indeed, if $s = \varphi(r)$, then $ds = \varphi' dr$ and $\varphi'(r) = 1/[G_0 : G_i]$ for $r \in (i - 1, i)$.

This last formula for $a(\rho)$ turns out to be useful as it generalizes without change to ℓ -adic representations. It also makes evident the fact that if ρ factors through $H = \text{Gal}(L/F)$ for some subextension $L \subset K$, then the conductor of ρ as a representation of G is the same as its conductor as a representation of H .

For use later we note that the first term in the sum for $a(\rho)$ and the first part of the integrals for it are all equal:

$$\epsilon(\rho) := \int_{-1}^0 \text{codim } V^{G^s} ds = \int_{-1}^0 \frac{\text{codim } V^{G_r}}{[G_0 : G_r]} dr = \text{codim } V^{G_0} = \text{codim } V^{I_F}.$$

It is convenient to break $a(\rho)$ into two parts, $\epsilon(\rho)$ as above, and

$$\delta(\rho) := \int_0^\infty \text{codim } V^{G^s} ds = \int_0^\infty \frac{\text{codim } V^{G_r}}{[G_0 : G_r]} dr = \sum_{i=1}^\infty \frac{\text{codim } V^{G_i}}{[G_0 : G_i]},$$

so that $a(\rho) = \epsilon(\rho) + \delta(\rho)$. One calls $\epsilon(\rho)$ the *tame conductor* of ρ and $\delta(\rho)$ the *wild conductor* or *Swan conductor* of ρ .

The Artin and Swan conductors may also be realized as the inner products of the character of ρ with those of certain representations, known as the *Artin representation* and the *Swan representation* respectively. The existence of representations with this property is essentially equivalent to the fact that the Artin and Swan conductors are integers; see [Ser79, VI, §2].

5. ℓ -ADIC REPRESENTATIONS

Recall that an ℓ -adic representation is a continuous homomorphism

$$\rho_\ell : W_F \rightarrow \text{GL}_n(E)$$

where E is a finite extension of \mathbb{Q}_ℓ and $\text{GL}_n(E)$ is given the ℓ -adic topology. A primary source of such representations is ℓ -adic cohomology. More precisely, if X is a variety over F , then the ℓ -adic étale cohomology groups $H^i(X \times \overline{F}, \mathbb{Q}_\ell)$ (and variants) are equipped with continuous actions of G_F and we may restrict to W_F to obtain ℓ -adic representations as defined above.

Let \mathcal{O}_E be the ring of integers of E and let $\mathfrak{m} \subset \mathcal{O}_E$ be the maximal ideal. It is well known (see, e.g., [Ser06, p. 104]) that every compact subgroup of $\text{GL}_n(E)$ is conjugate to a subgroup of $\text{GL}_n(\mathcal{O}_E)$. Since I_F is a closed subgroup of G_F , it is compact. Thus replacing ρ_ℓ by a conjugate representation if necessary, we may assume that the image of I_F under ρ_ℓ is contained in $\text{GL}_n(\mathcal{O}_E)$.

Now the $\text{GL}_n(\mathcal{O}_E)$ has a finite index subgroup which is a pro- ℓ group, namely, the kernel of reduction $\text{GL}_n(\mathcal{O}_E) \rightarrow \text{GL}_n(\mathcal{O}_E/\mathfrak{m})$. Since $P_F \subset I_F$ is a pro- p group and $\ell \neq p$, it follows that the image of P_F under ρ_ℓ is finite.

On the other hand, it is not in general the case that the image of all of I_F under ρ_ℓ is finite. For example, if ρ_ℓ is the representation of G_F on the Tate module of an elliptic curve over F with split multiplicative reduction, it is known that the restriction of ρ_ℓ to P_F is trivial, and in a suitable basis the restriction of ρ_ℓ to I_F has the form

$$\begin{pmatrix} 1 & \tau \\ 0 & 1 \end{pmatrix}$$

where $\tau : I_F \rightarrow \mathbb{Z}_\ell$ is the projection $I_F \rightarrow I_F/P_F \cong \prod_{r \neq q} \mathbb{Z}_r \rightarrow \mathbb{Z}_\ell$. See [Roh94, §15] for more details.

Since the restriction of ρ_F to inertia need not have finite image, the definition of Artin does not apply directly because the denominators $[G_0 : G_i]$ in (4.1) need not be finite, so further discussion is needed.

The crucial observation that makes it possible to define a conductor for ρ_ℓ is the following, which was conjectured by Serre and Tate and proven by Grothendieck.

Key Fact. If $\rho_\ell : W_F \rightarrow \mathrm{GL}_n(E)$ is a continuous representation, then there is a subgroup $I' \subset I_F$ of finite index such that $\rho_\ell(g)$ is unipotent for every $g \in I'$.

The proof is given in [ST68, Appendix]. It applies more generally to discrete valuation fields F whose residue field has the property that no finite extension contains all roots of unity of ℓ -power order. Note that we may assume that I' is also normal in W_F .

6. SEMI-SIMPLIFICATION

Fix an ℓ -adic representation $\rho_\ell : W_F \rightarrow \mathrm{GL}_n(E)$. We write V_ℓ for E^n with its action of W_F via ρ_ℓ .

Let V_{ss} be the semi-simplification of V_ℓ , defined as the direct sum of the Jordan-Hölder factors of V_ℓ as a W_F module, and let $\rho_{ss} : W_F \rightarrow \mathrm{GL}_n(E)$ be the corresponding homomorphism.

It follows from the key fact of the previous section that ρ_{ss} is trivial on a finite index subgroup of I_F . Indeed, by Clifford’s theorem [Cli37, Thm. 1], ρ_{ss} is semi-simple when restricted to any normal subgroup of W_F , and by Kolchin’s theorem [Ser06, V.3*, p. 35], a semi-simple and unipotent representation is trivial. In particular, $\rho_{ss}(I_F)$ is a finite group. Therefore, the wild conductor $\delta(\rho_{ss})$ and the Artin conductor $a(\rho_{ss})$ are well defined.

Following Serre [Ser70], we define

$$a(\rho_\ell) := \epsilon(\rho_\ell) + \delta(\rho_{ss})$$

where as usual $\epsilon(\rho_\ell) = \dim V_\ell - \dim V_\ell^{I_F}$.

Note that if ρ_ℓ restricted to inertia has finite image, then ρ_ℓ and ρ_{ss} have the same restriction to inertia, so this definition agrees with that in Section 4 when they both apply.

Note also that

$$\epsilon(\rho_\ell) - \epsilon(\rho_{ss}) = \dim V_{ss}^{I_F} - \dim V_\ell^{I_F}$$

so we have

$$a(\rho_\ell) = \dim V_{ss}^{I_F} - \dim V_\ell^{I_F} + a(\rho_{ss}).$$

This last formula appears in [Tat79, 4.2.4], but is missing the exponent I_F on V_ℓ .

The wild conductor $\delta(\rho_{ss})$ has two other useful descriptions. We noted above that $\rho_\ell|_{P_F}$ (the restriction to wild inertia) has finite image, so is already semi-simple, and $\rho_{ss}|_{P_F}$ is also semi-simple by the same reasoning (or by Clifford’s theorem). Since $\rho_\ell|_{P_F}$ and $\rho_{ss}|_{P_F}$ have the same character, they are isomorphic and have the same wild conductor. For another description, choose a basis of V_ℓ so that the image of ρ_ℓ lies in $\mathrm{GL}_n(\mathcal{O}_E)$ and let $\bar{\rho}_\ell$ be the reduction modulo \mathfrak{m}_E . Then $\bar{\rho}_\ell$ restricted to P_F is isomorphic to ρ_ℓ restricted to P_F and also gives the same wild conductor. Summarizing:

$$\delta(\rho_{ss}) = \delta(\rho_\ell|_{P_F}) = \delta(\bar{\rho}_\ell).$$

7. WEIL-DELIGNE REPRESENTATIONS

In this section we review (with the minimum of details) the notion of a Weil-Deligne representation. See [Del73, §8] or [Tat79, 4.1] for more details, and [Roh94] for a motivated introduction aimed at arithmetic geometers.

We write $\|\cdot\|$ for the homomorphism $W_F \rightarrow \mathbb{Q}$ which sends Φ to q^{-1} and which is trivial on I_F .

Let V be a vector space over a field of characteristic zero. We define a Weil-Deligne representation of W_F on V as a pair (ρ, N) where $\rho : W_F \rightarrow \text{GL}(V)$ is a homomorphism continuous with respect to the discrete topology on V and $N : V \rightarrow V$ is an endomorphism satisfying

$$\rho(w)N\rho(w)^{-1} = ||w|| N.$$

Continuity of ρ implies that it has finite image restricted to inertia, and the displayed formula implies that N is nilpotent (because its eigenvalues are stable under multiplication by q).

Because ρ has finite image when restricted to I_F , its conductor is defined by the formulas of Section 4. We define the Artin conductor of a Weil-Deligne representation (ρ, N) as

$$a(\rho, N) := a(\rho) + \dim V^{I_F} - \dim V_N^{I_F}.$$

Here V_N is the kernel of N on V , so that

$$V_N^{I_F} = \{v \in V \mid N(v) = 0, \rho(w)(v) = v \ \forall w \in I_F\}.$$

8. ℓ -ADIC REPRESENTATIONS AND WEIL-DELIGNE REPRESENTATIONS

Fix an ℓ -adic representation ρ_ℓ . The key fact stated at the end of Section 5 leads to a description of the behavior of ρ_ℓ restricted to I_F in terms of Weil-Deligne representations.

First, note that because P_F is a pro- p group and $I_F/P_F \cong \prod_{\ell \neq p} \mathbb{Z}_\ell$, there is a non-zero homomorphism $t_\ell : I_F \rightarrow \mathbb{Q}_\ell$ which is unique up to a scalar. It satisfies $t_\ell(w\sigma w^{-1}) = ||w|| t_\ell(\sigma)$ for all $w \in W_F$.

The key fact implies that there is a unique nilpotent linear transformation $N : E^n \rightarrow E^n$ such that for all σ in some finite index subgroup of I_F

$$\rho_\ell(\sigma) = \exp(t_\ell(\sigma)N)$$

as automorphisms of E^n . Here \exp is defined by the usual series $1 + x + x^2/2! + \dots$ and $\exp(t_\ell(\sigma)N)$ is in fact a finite sum because N is nilpotent.

It follows from this (see [Del73, §8]) that there exists a unique Weil-Deligne representation (ρ, N) on $V = E^n$ such that for all $m \in \mathbb{Z}$ and all $\sigma \in I_F$

$$(8.1) \quad \rho_\ell(\Phi^m \sigma) = \rho(\Phi^m \sigma) \exp(t_\ell(\sigma)N).$$

Conversely, given a Weil-Deligne representation (ρ, N) on V , the displayed formula defines an ℓ -adic representation. This correspondence gives a bijection on isomorphism classes. (The correspondence $\rho_\ell \leftrightarrow (\rho, N)$ depends on the choices of t_ℓ and Φ , but after passing to isomorphism classes it is independent of these choices; see [Del73].)

The point of introducing Weil-Deligne representations is that their definition uses only the discrete topology on V , so is convenient for shifting between different ground fields (such as \mathbb{Q}_ℓ for varying ℓ and \mathbb{C}).

Following Deligne [Del73], we define

$$a_D(\rho_\ell) := a(\rho, N) = a(\rho) + \dim V^{I_F} - \dim V_N^{I_F}.$$

We note that $\rho_\ell(I_F)$ is finite if and only if the corresponding $N = 0$, and in this case the definition above reduces to that of Section 4.

Since $\epsilon(\rho) = \dim V - \dim V^{I_F}$, we also have

$$a_D(\rho_\ell) = \dim V - \dim V_N^{I_F} + \delta(\rho).$$

We note also that t_ℓ is trivial on the wild inertia group $P_F = G_F^{>0}$, so ρ_ℓ and ρ are equal on P_F . It follows that $\delta(\rho) = \delta(\rho_\ell|_{P_F}) = \delta(\rho_{ss})$. On the other hand, equation (8.1) (with $m = 0$) implies that $V_N^{I_F} = V_\ell^{I_F}$. Therefore

$$a_D(\rho_\ell) = \dim V - \dim V_\ell^{I_F} + \delta(\rho_\ell) = a(\rho_\ell),$$

in other words, the definitions of Deligne and Serre agree.

9. ANOTHER FORMULA FOR $a(\rho_\ell)$

We can now state the main result of this note. The left hand side of (9.1) appears as a definition in [Wie12, Def. 3.1.27]. This reference seems to include everything needed to prove that Definition 3.1.27 agrees with the definitions of Serre and Deligne, but the proof is not given there.

Theorem 1. *Let ρ_ℓ be an ℓ -adic representation of W_F on V_ℓ with corresponding Weil-Deligne representation (ρ, N) on V and semi-simplification ρ_{ss} on V_{ss} . Let $a(\rho_\ell)$ be the Artin conductor of ρ_ℓ , defined as in Section 6, and let $a_D(\rho_\ell)$ be defined as in Section 8. Then*

$$(9.1) \quad \int_{-1}^\infty \text{codim } V_\ell^{G^s} ds = a(\rho_\ell) = a_D(\rho_\ell).$$

Proof. We saw at the end of Section 8 that $a(\rho_\ell) = a_D(\rho_\ell)$, so we need only check that the integral is equal to $a(\rho_\ell)$. For $-1 < s < 0$, $G^s = G_s = I_F$, so

$$\int_{-1}^0 \text{codim } V_\ell^{G^s} ds = \text{codim } V_\ell^{I_F} = \epsilon(\rho_\ell).$$

On the other hand, for $s > 0$, $G^s \subset P_F$ and ρ_ℓ restricted to G^s is isomorphic to ρ_{ss} restricted to G^s . Therefore

$$\int_0^\infty \text{codim } V_\ell^{G^s} ds = \int_0^\infty \text{codim } V_{ss}^{G^s} ds = \delta(\rho_{ss}).$$

It follows that

$$\int_{-1}^\infty \text{codim } V_\ell^{G^s} ds = \epsilon(\rho_\ell) + \delta(\rho_{ss}) = a(\rho_\ell),$$

as desired. □

10. AN APPLICATION TO TWISTING

We give an easy application of the theorem which is the motivation for this work.

Let $\rho_\ell : W_F \rightarrow \text{GL}_n(E)$ be an ℓ -adic representation and let $\chi : W_F \rightarrow E^\times$ be a character. We say “ χ is more deeply ramified than ρ_ℓ ” if there exists a non-negative real number s such that $\rho_\ell(G_F^s) = \{id\}$ and $\chi(G_F^s) \neq \{id\}$. In other words, χ is non-trivial further into the ramification filtration than ρ_ℓ is. Let m be the supremum of the set of s such that χ is non-trivial on G_F^s . It follows from Section 4 that $a(\chi) = m + 1$.

Proposition 1. *If χ is more deeply ramified than ρ_ℓ , then*

$$a(\rho_\ell \otimes \chi) = \text{deg}(\rho_\ell)a(\chi).$$

Proof. Let V_ℓ be the space where W_F acts via ρ_ℓ and let $V_{\ell,\chi}$ be the same space where W_F acts via $\rho_\ell \otimes \chi$. By the theorem we have

$$a(\rho_\ell \otimes \chi) = \int_{-1}^{\infty} \operatorname{codim} V_{\ell,\chi}^{G_F^s} ds.$$

If $s \leq m$, then $V_{\ell,\chi}^{G_F^s} \subset V_{\ell,\chi}^{G_F^m}$ and the latter is zero because $\rho_\ell(G_F^m) = \{id\}$ and $\chi(G_F^m) \neq \{id\}$. Thus in this range the integrand is $\dim V_\ell = \deg(\rho_\ell)$. On the other hand, if $s > m$, then $\rho_\ell \otimes \chi(G_F^s) = \{id\}$ and the integrand is zero. Thus

$$\int_{-1}^{\infty} \operatorname{codim} V_{\ell,\chi}^{G_F^s} ds = \deg(\rho_\ell)(m+1) = \deg(\rho_\ell)a(\chi),$$

as desired. \square

A particularly useful case of the proposition occurs when ρ_ℓ is tamely ramified and χ is wildly ramified, e.g., when χ is an Artin-Schreier character.

A variant of the proposition where χ and ρ_ℓ are both assumed to be irreducible, but χ may be of dimension > 1 , is stated as Lemma 9.2(3) of [DD13]

We end with another application in the same spirit, namely a very simple solution to Exercise 2 in [Ser79, p. 103].

Proposition 2. *Suppose that ρ_ℓ is an irreducible ℓ -adic representation of W_F on V_ℓ , and let m be the supremum of the set of numbers s such that $\rho_\ell(G^s) \neq 0$. Then*

$$a(\rho_\ell) = (\dim V_\ell)(m+1).$$

Proof. Since G^s is a normal subgroup of G_F , the subspace of invariants $V_\ell^{G^s}$ is preserved by G_F . Since V_ℓ is irreducible, we have that $\operatorname{codim} V_\ell^{G^s}$ is 0 if $s > m$ and $\dim V_\ell$ if $s < m$. Therefore

$$a(\rho_\ell) = \int_{-1}^{\infty} \operatorname{codim} V_\ell^{G_F^s} ds = (\dim V_\ell)(m+1),$$

as desired. \square

ACKNOWLEDGMENTS

The author thanks Romyar Sharifi and Emmanuel Kowalski for pointing out [Wie12] and [DDT97] respectively, in both cases after they re-discovered the formula in Theorem 1. The author also thanks David Rohrlich for his comments and encouragement, Jean-Pierre Serre for clarifying the history of the subject, and an anonymous referee for several valuable suggestions.

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SCHOOL OF MATHEMATICS, GEORGIA INSTITUTE OF TECHNOLOGY, ATLANTA, GEORGIA 30332
E-mail address: ulmer@math.gatech.edu