

Summation of Glaisher- and Apéry-like Series

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Abstract

To the author's knowledge, new techniques for the summation of certain Glaisher-like series (with terms involving $\zeta(n+1)$) and Apéry-like series (with terms involving $(n!)^2/(2n)!$) are demonstrated.

1 Introduction

In 1913, Glaisher [5] gave the following sums:

$$\begin{aligned}\sum_{n=1}^{\infty} [\zeta(2n+1) - 1] &= \frac{1}{4} \\ \sum_{n=1}^{\infty} [\zeta(6n+4) - 1] &= \frac{1}{12} \\ \sum_{n=1}^{\infty} \frac{1}{2^{2n}} \zeta(2n) &= \frac{1}{2} \\ \sum_{n=1}^{\infty} \frac{1}{2^{2n+1}} \zeta(2n+1) &= \ln 2 - \frac{1}{2} \\ \sum_{n=1}^{\infty} \frac{n}{16^n} \zeta(2n+1) &= 1 - G\end{aligned}$$

where G is Catalan's constant defined by

$$G = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} = 0.915965594177\dots$$

Additionally, Glaisher gave many other such series which the author has termed Glaisher-like, that is, series with terms involving the zeta series

$$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z}. \quad (1.1)$$

In Section 2, an interesting class of Glaisher-like series involving Lucas sequences, which the author has not been able to find in the literature, will be evaluated.

In 1979, Apéry [1] proved the irrationality of $\zeta(3)$ —and in the same manner, the irrationality of $\zeta(2) = \pi^2/6$ —by making use of the identities

$$\zeta(3) = \frac{5}{2} \sum_{n=1}^{\infty} \frac{(n!)^2 (-1)^{n+1}}{(2n)! n^3} \quad (1.2)$$

$$\zeta(2) = 3 \sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)! n^2}. \quad (1.3)$$

(For the interested reader—van der Poorten [12] gave a very entertaining report on Apéry's proof entitled "A proof that Euler missed...".) Following Apéry's proof, many series of a similar form,

$$\sum_n \frac{(n!)^2}{(2n)!} f(n) = \sum_n \binom{2n}{n}^{-1} f(n),$$

which are commonly referred to as Apéry-like series, have been considered by van der Poorten [11] [13], Leschiner [7], Lehmer [6], Zucker [14], J. Borwein and Bradley [Searching Symbolically for Apéry-like formulae for values of the Riemann Zeta function] and J. Borwein, Broadhurst, and Kamnitzer [3]. Berndt and Joshi [2] in a review of Chapter 9 of Ramanujan's 2nd notebook have also recorded many similar formulas. Section 3 will present many general evaluations of Apéry-like series, of which most appear to be new.

In the Section 3, we will make free use of Oldham and Spanier's *The Fractional Calculus* [10]. In particular, we will use the following results:

$$\frac{d^{\frac{1}{2}} f}{dx^{\frac{1}{2}}} = \frac{f(0)}{\sqrt{\pi x}} + \int_0^x \frac{f^{(1)}(y) dy}{\sqrt{\pi(x-y)}} \quad (1.4)$$

$$\frac{d^{-\frac{1}{2}} f}{dx^{-\frac{1}{2}}} = \int_0^x \frac{f(y) dy}{\sqrt{\pi(x-y)}} \quad (1.5)$$

and for $n = 0, 1, 2, \dots$

$$\frac{d^{\frac{1}{2}} x^n}{dx^{\frac{1}{2}}} = \frac{(n!)^2 (4x)^n}{(2n)! \sqrt{\pi x}} \quad (1.6)$$

$$\frac{d^{-\frac{1}{2}} x^n}{dx^{-\frac{1}{2}}} = \frac{(n!)^2 (4x)^{n+\frac{1}{2}}}{(2n+1)! \sqrt{\pi}}. \quad (1.7)$$

We will also make use of n -th order polylogarithm $\text{Li}_n(z)$, defined recursively by

$$\text{Li}_n(z) = \int_0^z \frac{\text{Li}_{n-1}(z)}{z} dz. \quad (1.8)$$

where $\text{Li}_1(z) = -\ln(1-z)$. If $|z| \leq 1$, then

$$\text{Li}_n(z) = \sum_{m=1}^{\infty} \frac{z^m}{m^n}. \quad (1.9)$$

For more properties, see Lewin's *Polylogarithms and associated functions* [8].

Note Throughout this handout, we use ϕ to denote the golden ratio, $\frac{1+\sqrt{5}}{2}$.

2 Summation of Glaisher-like series

Theorem 2.1 Let $\{U_n\}$ be a Lucas sequence defined by $U_0 = 0$, $U_1 = 1$ and

$$U_n = PU_{n-1} - QU_{n-2} \quad (2.1)$$

where $P, Q \in \mathbb{R}$ and $D = \sqrt{P^2 - 4Q} \neq 0$. Then

$$\begin{aligned} \sum_{n=1}^N U_n \zeta(n+1) &= \frac{1}{D} \left[\psi \left(1 - \frac{P}{2} + \frac{D}{2} \right) - \psi \left(1 - \frac{P}{2} - \frac{D}{2} \right) \right] \\ &\quad - \sum_{y=1}^{\infty} \frac{\frac{U_{N+1}}{y^N} - Q \frac{U_N}{y^{N+1}}}{y^2 - Py + Q}, \end{aligned} \quad (2.2)$$

where $\psi(x) = \frac{d}{dx} \ln \Gamma(x)$ is the digamma function. If $D = 0$, then

$$\sum_{n=1}^N U_n \zeta(n+1) = \psi' \left(1 - \frac{P}{2} \right) - \sum_{y=1}^{\infty} \frac{\frac{U_{N+1}}{y^N} - Q \frac{U_N}{y^{N+1}}}{y^2 - Py + Q}. \quad (2.3)$$

Proof Since $\frac{x}{1 - Px + Qx^2}$ is the generating function for the Lucas sequence defined in (2.1), we readily obtain,

$$\frac{x}{1 - Px + Qx^2} = \sum_{n=1}^N U_n x^n + \frac{U_{N+1} x^{N+1} - Q U_N x^{N+2}}{1 - Px + Qx^2}. \quad (2.4)$$

Putting $x = 1/y$ in (2.4) gives

$$\frac{1}{y^2 - Py + Q} = \sum_{n=1}^N \frac{U_n}{y^{n+1}} + \frac{\frac{U_{N+1}}{y^N} - Q \frac{U_N}{y^{N+1}}}{y^2 - Py + Q}. \quad (2.5)$$

which implies that

$$\sum_{n=1}^N U_n \zeta(n+1) = \sum_{y=1}^{\infty} \frac{1}{y^2 - Py + Q} - \sum_{y=1}^{\infty} \frac{\frac{U_{N+1}}{y^N} - Q \frac{U_N}{y^{N+1}}}{y^2 - Py + Q}. \quad (2.6)$$

Thus, for $D \neq 0$, the theorem immediately follows by noting that

$$\sum_{y=1}^{\infty} \frac{1}{y^2 - Py + Q} = \frac{1}{D} \left[\psi \left(1 - \frac{P}{2} + \frac{D}{2} \right) - \psi \left(1 - \frac{P}{2} - \frac{D}{2} \right) \right], \quad (2.7)$$

which is obtained through partial fraction decomposition of the summand and the use of the identity

$$\sum_{y=1}^{N-1} \frac{1}{y - \alpha} = \psi(N - \alpha) - \psi(1 - \alpha), \quad (2.8)$$

as $N \rightarrow \infty$. For $D = 0$, the theorem follows by taking the limit as $D \rightarrow 0$ in (2.2) and applying the limit definition of the central derivative. ■

From this result, we can obtain the asymptotic representation of $\sum_{n=1}^N U_n \zeta(n+1)$ by evaluating (2.2)(2.3), taking only the nonzero terms in the infinite series on the right hand side when $N \rightarrow \infty$. Seven interesting examples are the following:

1) Taking $P = 1$ and $Q = -1$ we have the Fibonacci numbers F_n and we find that

$$\sum_{n=1}^N F_n \zeta(n+1) \sim \frac{\pi\sqrt{5}}{5} \tan \frac{\pi\sqrt{5}}{2} + F_{N+2}. \quad (2.9)$$

Making use of the identity $\sum_{n=1}^N F_n = F_{N+2} - 1$ we have that

$$\sum_{n=1}^{\infty} F_n [\zeta(n+1) - 1] = 1 + \frac{\pi\sqrt{5}}{5} \tan \frac{\pi\sqrt{5}}{2}. \quad (2.10)$$

2) Taking $P = 1$ and $Q = 1$ we have the interesting result

$$\sum_{n=0}^{\infty} [\zeta(6n+2) + \zeta(6n+3) - \zeta(6n+5) - \zeta(6n+6)] = \frac{\pi\sqrt{3}}{3} - \frac{2\pi\sqrt{3}}{3e^{\pi\sqrt{3}} + 3} - 1. \quad (2.11)$$

3) Taking $P = 2$ and $Q = -1$ we have the Pell numbers P_n and find that

$$\sum_{n=1}^N P_n \zeta(n+1) \sim -\frac{1}{4} \left[P_N + P_{N+2} \left(1 + \frac{1}{2^{N-1}} \right) - \pi\sqrt{2} \cot \pi\sqrt{2} - 1 \right]. \quad (2.12)$$

4) Taking $P = -2$ and $Q = 1$ we have

$$\sum_{n=1}^N (-1)^{n+1} n \zeta(n+1) \sim \frac{\pi^2}{6} - 1 + \frac{(-1)^{N+1}(2N+1)}{4}. \quad (2.13)$$

5) Taking $P = -1$ and $Q = -1$ we have

$$\sum_{n=1}^{\infty} (-1)^{n+1} F_n [\zeta(n+1) - 1] = \frac{\pi\sqrt{5}}{5} \tan \frac{\pi\sqrt{5}}{2}. \quad (2.14)$$

6) Taking $P = 0$ and $Q = 1$ we have

$$\sum_{n=0}^N [\zeta(4n+2) - \zeta(4n+4)] \sim \frac{\pi}{e^{2\pi} - 1} + \frac{\pi}{2} - \frac{1}{2} [1 + (-1)^N]. \quad (2.15)$$

7) Taking $P = 0$ and $Q = -\frac{1}{2}$ we have

$$\sum_{n=1}^{\infty} \frac{\zeta(2n)}{2^{n-1}} = 1 - \frac{\pi\sqrt{2}}{2} \cot \frac{\pi\sqrt{2}}{2}. \quad (2.16)$$

3 Summation of Apéry-like series

We first consider the simplest Apéry-like series:

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} (4x)^n &= \sqrt{\pi x} \frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}} \sum_{n=0}^{\infty} x^n = \sqrt{\pi x} \frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}} \frac{1}{1-x} \\ &= \frac{\sqrt{1-x} + \sqrt{x} \arcsin \sqrt{x}}{(1-x)^{\frac{3}{2}}}. \end{aligned} \quad (3.1)$$

Next, for $m \in \mathbb{N}$, consider the following infinite series:

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} (n+1)(n+2)\cdots(n+m)(4x)^n \\ &= \sqrt{\pi x} \frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}} \sum_{n=0}^{\infty} (n+1)(n+2)\cdots(n+m)x^n \\ &= \sqrt{\pi x} \frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}} \frac{m!}{(1-x)^{m+1}} = -\frac{m! \sqrt{x} B_x(-\frac{1}{2}, m + \frac{3}{2})}{2(1-x)^{m+\frac{3}{2}}} \\ &= \frac{-m! \sqrt{x}}{2(1-x)^{m+\frac{3}{2}}} \int_0^x y^{-\frac{3}{2}} (1-y)^{m+\frac{1}{2}} dy. \end{aligned} \quad (3.2)$$

Then, if we note that

$$\int^x y^{-\frac{3}{2}}(1-y)^{m+\frac{1}{2}} dy = \frac{\sqrt{1-x}}{\sqrt{x}} \sum_{n=0}^m a_{n,m} x^n - \frac{(2m+1)!}{2^{2m-1}(m!)^2} \arcsin \sqrt{x}, \quad (3.3)$$

where $a_{0,m} = -2$ and

$$\left(\frac{1}{2} - n\right) a_{n,m} + (n-1)a_{n-1,m} = \binom{m+1}{n} (-1)^{n+1}, \quad (3.4)$$

we obtain the result

$$\begin{aligned} & \sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} (n+1)(n+2)\cdots(n+m)(4x)^n \\ &= \frac{\sqrt{x}}{(1-x)^{m+\frac{3}{2}}} \frac{(2m+1)!}{2^{2m}m!} \arcsin \sqrt{x} - \frac{m!}{2(1-x)^{m+1}} \sum_{n=0}^m a_{n,m} x^n. \end{aligned} \quad (3.5)$$

Now since 1 and $\{(n+1)(n+2)\cdots(n+m) : m \in \mathbb{N}\}$ form a basis for the vector space of all polynomials, we can find the closed form representation of any series of the type

$$\boxed{\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} (n^{m_1} x_1^n + n^{m_2} x_2^n + \cdots + n^{m_k} x_k^n)} \quad (I)$$

where $m_i \in \mathbb{N} \cup \{0\}$. Examples:

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} = \frac{2\pi\sqrt{3}}{27} + \frac{4}{3} \quad (3.6)$$

$$\sum_{n=0}^{\infty} \frac{(n!)^2 n}{(2n)!} = \frac{2\pi\sqrt{3}}{27} + \frac{2}{3} \quad (3.7)$$

$$\sum_{n=0}^{\infty} \frac{(n!)^2 n^2}{(2n)!} = \frac{10\pi\sqrt{3}}{81} + \frac{4}{3} \quad (3.8)$$

$$\sum_{n=0}^{\infty} \frac{(n!)^2 n^3}{(2n)!} = \frac{74\pi\sqrt{3}}{243} + \frac{10}{3} \quad (3.9)$$

$$\sum_{n=0}^{\infty} \frac{(n!)^2 n^4}{(2n)!} = \frac{238\pi\sqrt{3}}{243} + \frac{32}{3} \quad (3.10)$$

$$\sum_{n=0}^{\infty} \frac{(n!)^2 n^5}{(2n)!} = \frac{938\pi\sqrt{3}}{243} + 42 \quad (3.11)$$

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} (-1)^n = -\frac{4\sqrt{5} \ln \phi}{25} + \frac{4}{5} \quad (3.12)$$

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} (-1)^n n = -\frac{4\sqrt{5} \ln \phi}{125} - \frac{6}{25} \quad (3.13)$$

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} (-1)^n n^2 = \frac{4\sqrt{5} \ln \phi}{125} - \frac{4}{25} \quad (3.14)$$

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} (-1)^n n^3 = \frac{28\sqrt{5} \ln \phi}{625} + \frac{2}{125} \quad (3.15)$$

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} (-1)^n n^4 = \frac{4\sqrt{5} \ln \phi}{3125} + \frac{136}{625} \quad (3.16)$$

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} (-1)^n n^5 = -\frac{1412\sqrt{5} \ln \phi}{15625} + \frac{742}{3125} \quad (3.17)$$

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{(-1)^n}{2^n} = -\frac{4 \ln 2}{27} + \frac{8}{9} \quad (3.18)$$

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{(-1)^n n}{2^n} = -\frac{4 \ln 2}{81} - \frac{4}{27} \quad (3.19)$$

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{(-1)^n n^2}{2^n} = \frac{4 \ln 2}{729} - \frac{32}{243} \quad (3.20)$$

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{(-1)^n n^3}{2^n} = \frac{220 \ln 2}{6561} - \frac{140}{2187} \quad (3.21)$$

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{(-1)^n n^4}{2^n} = \frac{76 \ln 2}{2187} + \frac{40}{729} \quad (3.22)$$

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{(-1)^n n^5}{2^n} = -\frac{196 \ln 2}{59049} + \frac{3836}{19683} \quad (3.22)$$

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} 2^n = \frac{\pi}{2} + 2 \quad (3.23)$$

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} n 2^n = \pi + 3 \quad (3.24)$$

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} n^2 2^n = \frac{7\pi}{2} + 11 \quad (3.25)$$

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} n^3 2^n = \frac{35\pi}{2} + 55 \quad (3.26)$$

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} n^4 2^n = 113\pi + 355 \quad (3.27)$$

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} n^5 2^n = \frac{1787\pi}{2} + 2807 \quad (3.28)$$

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} n^6 2^n = \frac{16717\pi}{2} + 26259 \quad (3.29)$$

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} n^7 2^n = 90280\pi + 283623 \quad (3.30)$$

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} n^8 2^n = \frac{2211181\pi}{2} + 3473315 \quad (3.31)$$

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} n^9 2^n = \frac{30273047\pi}{2} + 47552791 \quad (3.32)$$

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} n^{10} 2^n = 229093376\pi + 719718067 \quad (3.33)$$

The last set of examples—particularly Eq. (3.28), in that $355/113$ is what the ancient Chinese considered the correct approximation of π —leads us to the following

Conjecture. *If $m \in \mathbb{N} \cup \{0\}$, then*

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} n^m 2^n = a_m \pi + b_m, \quad (3.34)$$

where $2a_m, a_{3m+1}, b_m \in \mathbb{N}$, and $c_m = b_m/a_m \rightarrow \pi$ as $m \rightarrow \infty$.

Indeed, the first ten "convergents" $c_m = b_m/a_m$ are

$$\begin{aligned}
c_0 &= 4 \\
c_1 &= 3 \\
c_2 &= \frac{22}{7} = 3.\overline{142857} \\
c_3 &= \frac{22}{7} = 3.\overline{142857} \\
c_4 &= \frac{355}{133} = 3.14159292035\dots \\
c_5 &= \frac{5614}{1787} = 3.14157806379\dots \\
c_6 &= \frac{52518}{16717} = 3.14159239097\dots \\
c_7 &= \frac{283623}{90280} = 3.14159282233\dots \\
c_8 &= \frac{6946630}{2211181} = 3.14159266021\dots \\
c_9 &= \frac{95105582}{30273047} = 3.14159265171\dots \\
c_{10} &= \frac{719718067}{229093376} = 3.14159265347\dots
\end{aligned}$$

Notice that this conjecture implies the irrationality of π (see Niven [9]) since we have infinitely many pairs of integers $(2a_m, 2b_m)$ such that $0 < |2a_m\pi - 2b_m| < \epsilon$ for any $\epsilon > 0$. (Recently, the author discovered a wonderful paper by Lehmer [6] in which a similar observation is made, along with many interesting evaluations of series involving the central binomial coefficient.)

Now consider the series:

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{(n!)^2 (4x)^n}{(2n)! 2n+1} &= \sqrt{\frac{\pi}{4x}} \sum_{n=0}^{\infty} \frac{(n!)^2 (4x)^{n+\frac{1}{2}}}{(2n+1)! \sqrt{\pi}} = \sqrt{\frac{\pi}{4x}} \frac{d^{-\frac{1}{2}}}{dx^{-\frac{1}{2}}} \sum_{n=0}^{\infty} x^n \\
&= \sqrt{\frac{\pi}{4x}} \frac{d^{-\frac{1}{2}}}{dx^{-\frac{1}{2}}} \frac{1}{1-x} = \frac{\arcsin \sqrt{x}}{\sqrt{x(1-x)}}.
\end{aligned} \tag{3.35}$$

By observing that

$$\sum_{n=0}^{\infty} \frac{(n!)^2 (4x)^n}{(2n)! 2n+1} = 1 + \sum_{n=0}^{\infty} \frac{((n-1)!)^2 n^2 (4x)^n}{(2n-2)!(2n-1)2n(2n+1)}, \tag{3.36}$$

we have, by letting $n-1 = m$,

$$1 + \sum_{m=0}^{\infty} \frac{(m!)^2 (m+1)^2 (4x)^{m+1}}{(2m)!(2m+1)2(m+1)(2m+3)} = \frac{\arcsin \sqrt{x}}{\sqrt{x(1-x)}}. \tag{3.37}$$

From this, we obtain

$$\sum_{n=0}^{\infty} \frac{(n!)^2 (4x)^n}{(2n)! 2n+3} = \left(\frac{2}{x} - 1\right) \frac{\arcsin \sqrt{x}}{\sqrt{x(1-x)}} - \frac{2}{x} \quad (3.38)$$

by partial fraction decomposition. This transformation can be employed successively to obtain the closed form representation of

$$\boxed{\sum_{n=0}^{\infty} \frac{(n!)^2 (4x)^n}{(2n)! 2n+2m-1}} \quad (\text{II})$$

for all $m \in \mathbb{N}$. In particular, we find that

$$\sum_{n=0}^{\infty} \frac{(n!)^2 (4x)^n}{(2n)! 2n+5} = \left(\frac{8}{3x^2} - \frac{4}{3x} - \frac{1}{3}\right) \frac{\arcsin \sqrt{x}}{\sqrt{x(1-x)}} - \frac{8}{3x^2} - \frac{4}{9x} \quad (3.39)$$

$$\sum_{n=0}^{\infty} \frac{(n!)^2 (4x)^n}{(2n)! 2n+7} = \left(\frac{16}{5x^3} - \frac{8}{5x^2} - \frac{2}{5x} - \frac{1}{5}\right) \frac{\arcsin \sqrt{x}}{\sqrt{x(1-x)}} - \frac{16}{5x^3} - \frac{8}{15x^2} - \frac{6}{25x} \quad (3.40)$$

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{(n!)^2 (4x)^n}{(2n)! 2n+9} &= \left(\frac{128}{35x^4} - \frac{64}{35x^3} - \frac{16}{35x^2} - \frac{8}{35x} - \frac{1}{7}\right) \frac{\arcsin \sqrt{x}}{\sqrt{x(1-x)}} \\ &\quad - \frac{128}{35x^4} - \frac{64}{105x^3} - \frac{48}{175x^2} - \frac{8}{49x}. \end{aligned} \quad (3.41)$$

Examples:

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)! 2n+1} = \frac{2\pi\sqrt{3}}{9} \quad (3.42)$$

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)! 2n+3} = \frac{14\pi\sqrt{3}}{9} - 8 \quad (3.43)$$

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)! 2n+5} = \frac{74\pi\sqrt{3}}{9} - \frac{400}{9} \quad (3.44)$$

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)! 2n+7} = \frac{1774\pi\sqrt{3}}{45} - \frac{16072}{75} \quad (3.45)$$

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)! 2n+9} = \frac{56758\pi\sqrt{3}}{315} - \frac{3602528}{3675} \quad (3.46)$$

$$\sum_{n=0}^{\infty} \frac{(n!)^2 (-1)^n}{(2n)! 2n+1} = \frac{4\sqrt{5} \ln \phi}{5} \quad (3.47)$$

$$\sum_{n=0}^{\infty} \frac{(n!)^2 (-1)^n}{(2n)! 2n+3} = -\frac{36\sqrt{5} \ln \phi}{5} + 8 \quad (3.48)$$

$$\sum_{n=0}^{\infty} \frac{(n!)^2 (-1)^n}{(2n)! 2n+5} = \frac{572\sqrt{5} \ln \phi}{15} - \frac{368}{9} \quad (3.49)$$

$$\sum_{n=0}^{\infty} \frac{(n!)^2 (-1)^n}{(2n)! 2n+7} = -\frac{916\sqrt{5} \ln \phi}{5} + \frac{14792}{75} \quad (3.50)$$

$$\sum_{n=0}^{\infty} \frac{(n!)^2 (-1)^n}{(2n)! 2n+9} = \frac{29308\sqrt{5} \ln \phi}{35} - \frac{3311008}{3675} \quad (3.51)$$

$$\sum_{n=0}^{\infty} \frac{(n!)^2 (-1)^n}{(2n)! 2^n(2n+1)} = \frac{4 \ln 2}{3} \quad (3.52)$$

$$\sum_{n=0}^{\infty} \frac{(n!)^2 (-1)^n}{(2n)! 2^n(2n+3)} = -\frac{68 \ln 2}{3} + 16 \quad (3.53)$$

$$\sum_{n=0}^{\infty} \frac{(n!)^2 (-1)^n}{(2n)! 2^n(2n+5)} = \frac{724 \ln 2}{3} - \frac{1504}{9} \quad (3.54)$$

$$\sum_{n=0}^{\infty} \frac{(n!)^2 (-1)^n}{(2n)! 2^n(2n+7)} = -\frac{34756 \ln 2}{15} + \frac{120464}{75} \quad (3.55)$$

$$\sum_{n=0}^{\infty} \frac{(n!)^2 (-1)^n}{(2n)! 2^n(2n+9)} = \frac{2224364 \ln 2}{105} - \frac{53963072}{3675} \quad (3.56)$$

$$\sum_{n=0}^{\infty} \frac{(n!)^2 2^n}{(2n)! 2n+1} = \frac{\pi}{2} \quad (3.57)$$

$$\sum_{n=0}^{\infty} \frac{(n!)^2 2^n}{(2n)! 2n+3} = \frac{3\pi}{2} - 4 \quad (3.58)$$

$$\sum_{n=0}^{\infty} \frac{(n!)^2 2^n}{(2n)! 2n+5} = \frac{23\pi}{6} - \frac{104}{9} \quad (3.59)$$

$$\sum_{n=0}^{\infty} \frac{(n!)^2 2^n}{(2n)! 2n+7} = \frac{91\pi}{10} - \frac{2116}{75} \quad (3.60)$$

$$\sum_{n=0}^{\infty} \frac{(n!)^2 2^n}{(2n)! 2n+9} = \frac{1451\pi}{70} - \frac{238192}{3675} \quad (3.61)$$

Now consider

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{(n!)^2 (4x)^n}{(2n)! n} &= \sqrt{\pi x} \frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}} \sum_{n=0}^{\infty} \frac{x^n}{n} = -\sqrt{\pi x} \frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}} \ln(1-x) \\ &= \frac{2\sqrt{x} \arcsin \sqrt{x}}{\sqrt{1-x}}. \end{aligned} \quad (3.62)$$

From Eq. (3.63) we have that

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{(n!)^2 (4x)^n}{(2n)! n^2} &= \int_0^x \sum_{n=1}^{\infty} \frac{(n!)^2 (4y)^{n-1}}{(2n)! n} dy \\ &= \int_0^x \frac{2\sqrt{y} \arcsin \sqrt{y}}{y\sqrt{1-y}} dy = 2 \arcsin^2 \sqrt{x}. \end{aligned} \quad (3.63)$$

Using Eqs. (3.63) and (3.64) we find the result

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{(n!)^2 (4x)^n}{(2n)! n+1} &= \sum_{n=0}^{\infty} \frac{((n+1)!)^2 (2n+2)(2n+1)(4x)^n}{(2n+2)! (n+1)^2(n+1)} \\ &= \sum_{m=1}^{\infty} \frac{(m!)^2 2m(2m-1)(4x)^{m-1}}{(2m)! m^3} \\ &= \frac{1}{x} \sum_{m=1}^{\infty} \frac{(m!)^2 (4x)^m}{(2m)! m} - \frac{1}{2x} \sum_{m=1}^{\infty} \frac{(m!)^2 (4x)^m}{(2m)! m^2} \\ &= \frac{2 \arcsin \sqrt{x}}{\sqrt{x}\sqrt{1-x}} - \frac{\arcsin^2 \sqrt{x}}{x}. \end{aligned} \quad (3.64)$$

Equally,

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{(n!)^2 (4x)^n}{(2n)! n+1} &= \sum_{m=0}^{\infty} \frac{(m!)^2 (m+1)(4x)^{m+1}}{(2m)! 2(2m+1)(m+2)} + 1 \\ &= \sum_{m=0}^{\infty} \frac{(m!)^2}{(2m)!} \left[\frac{(4x)^{m+1}}{6(2m+1)} + \frac{(4x)^{m+1}}{6(m+2)} \right] + 1, \end{aligned} \quad (3.65)$$

which, when combined with Eq. (3.39) and Eq. (3.65), yields

$$\sum_{n=0}^{\infty} \frac{(n!)^2 (4x)^n}{(2n)! n+2} = \left(\frac{3}{x} - 1 \right) \frac{\arcsin \sqrt{x}}{\sqrt{x(1-x)}} - \frac{3 \arcsin^2 \sqrt{x}}{2x^2} - \frac{3}{2x}. \quad (3.66)$$

Similarly, we can find the closed form representation of

$$\boxed{\sum_{n=0}^{\infty} \frac{(n!)^2 (4x)^n}{(2n)! n+m}} \quad (III)$$

for all $m \in \mathbb{N}$. In particular, we find that

$$\sum_{n=0}^{\infty} \frac{(n!)^2 (4x)^n}{(2n)! n+3} = \left(\frac{15}{4x^2} - \frac{5}{4x} - \frac{1}{2} \right) \frac{\arcsin \sqrt{x}}{\sqrt{x(1-x)}} - \frac{15 \arcsin^2 \sqrt{x}}{8x^3} - \frac{5}{8x} - \frac{15}{8x^2} \quad (3.67)$$

$$\sum_{n=0}^{\infty} \frac{(n!)^2 (4x)^n}{(2n)! n+4} = \left(\frac{35}{8x^3} - \frac{35}{24x^2} - \frac{7}{12x} - \frac{1}{3} \right) \frac{\arcsin \sqrt{x}}{\sqrt{x(1-x)}} - \frac{35 \arcsin^2 \sqrt{x}}{16x^4} - \frac{7}{18x} - \frac{35}{48x^2} - \frac{35}{16x^3} \quad (3.68)$$

$$\sum_{n=0}^{\infty} \frac{(n!)^2 (4x)^n}{(2n)! n+5} = \left(\frac{315}{64x^4} - \frac{105}{64x^3} - \frac{21}{32x^2} - \frac{3}{8x} - \frac{1}{4} \right) \frac{\arcsin \sqrt{x}}{\sqrt{x(1-x)}} - \frac{315 \arcsin^2 \sqrt{x}}{128x^5} - \frac{9}{32x} - \frac{7}{16x^2} - \frac{105}{128x^3} - \frac{315}{128x^4}. \quad (3.69)$$

Examples:

$$\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)! n} = \frac{\pi\sqrt{3}}{9} \quad (3.70)$$

$$\sum_{n=1}^{\infty} \frac{(n!)^2 (-1)^{n+1}}{(2n)! n} = \frac{2\sqrt{5} \ln \phi}{5} \quad (3.71)$$

$$\sum_{n=1}^{\infty} \frac{(n!)^2 (-1)^{n+1}}{(2n)! n 2^n} = \frac{\ln 2}{3} \quad (3.72)$$

$$\sum_{n=1}^{\infty} \frac{(n!)^2 2^n}{(2n)! n} = \frac{\pi}{2} \quad (3.73)$$

$$\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)! n^2} = \frac{\pi^2}{18} \quad (3.74)$$

$$\sum_{n=1}^{\infty} \frac{(n!)^2 (-1)^{n+1}}{(2n)! n^2} = 2 \ln^2 \phi \quad (3.75)$$

$$\sum_{n=1}^{\infty} \frac{(n!)^2 (-1)^{n+1}}{(2n)! n^2 2^n} = \frac{\ln^2 2}{2} \quad (3.76)$$

$$\sum_{n=1}^{\infty} \frac{(n!)^2 2^n}{(2n)! n^2} = \frac{\pi^2}{8} \quad (3.77)$$

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{1}{n+1} = \frac{4\pi\sqrt{3}}{9} - \frac{\pi^2}{9} \quad (3.78)$$

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{1}{n+2} = \frac{22\pi\sqrt{3}}{9} - \frac{2\pi^2}{3} - 6 \quad (3.79)$$

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{1}{n+3} = \frac{109\pi\sqrt{3}}{9} - \frac{10\pi^2}{3} - \frac{65}{2} \quad (3.80)$$

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{1}{n+4} = \frac{508\pi\sqrt{3}}{9} - \frac{140\pi^2}{9} - \frac{1379}{9} \quad (3.81)$$

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{1}{n+5} = \frac{4571\pi\sqrt{3}}{18} - 70\pi^2 - \frac{5525}{8} \quad (3.82)$$

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{(-1)^n}{n+1} = \frac{8\sqrt{5} \ln \phi}{5} - 4 \ln^2 \phi \quad (3.83)$$

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{(-1)^n}{n+2} = -\frac{52\sqrt{5} \ln \phi}{5} + 24 \ln^2 \phi + 6 \quad (3.84)$$

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{(-1)^n}{n+3} = \frac{258\sqrt{5} \ln \phi}{5} - 120 \ln^2 \phi - \frac{55}{2} \quad (3.85)$$

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{(-1)^n}{n+4} = -\frac{3616\sqrt{5} \ln \phi}{15} + 560 \ln^2 \phi + \frac{1169}{9} \quad (3.86)$$

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{(-1)^n}{n+5} = \frac{5423\sqrt{5} \ln \phi}{5} - 2520 \ln^2 \phi - \frac{4667}{8} \quad (3.87)$$

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{(-1)^n}{2^n(n+1)} = \frac{8 \ln 2}{3} - 2 \ln^2 2 \quad (3.88)$$

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{(-1)^n}{2^n(n+2)} = -\frac{100 \ln 2}{3} + 24 \ln^2 2 + 12 \quad (3.89)$$

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{(-1)^n}{2^n(n+3)} = \frac{998 \ln 2}{3} - 240 \ln^2 2 - 115 \quad (3.90)$$

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{(-1)^n}{2^n(n+4)} = -\frac{9316 \ln 2}{3} + 2240 \ln^2 2 + \frac{9688}{9} \quad (3.91)$$

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{(-1)^n}{2^n(n+5)} = \frac{83843 \ln 2}{3} - 20160 \ln^2 2 - \frac{38743}{4} \quad (3.92)$$