Sightings of the Rare Primitive Pythagorean Triangles

Report

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1) Introduction

In my research I investigated some fundamental properties of Pythagorean triples and I was interested in how rare the Pythagorean triangles are, with integer sides. In this report I try to give a summary of some major findings. There will be some theoretical issues as well as results of several computer studies.

Definitions

Pythagorean triples: Integer triples a, b, c, satisfying the equation $x^2 + y^2 = z^2$

Pythagorean triangle: A right triangle, all sides of which are integers.

I looked at the number of primitive Pythagorean triangles, with largest sides less than 100, 200, 400.

By enumeration I got the following:

Size of sides less than: **N**	100	200	400
# of primitive Pythagorean triangles: **F(N)**	23	49	97

(see details later in #4).

These numbers show how extremely rare these triangles are.

I will continue finding F(N) with increasing values of N.(see details in #4)

In addition I found very interesting the fact that the ratio of N and F(N) in all above three cases it is close to 4.00. Is this true for all size N?

I will look into this problem later, but first some preliminaries are presented.

2) Preliminaries

In this section some theoretical results will be given which will be used later in the study.

I. Assume that M and N are relative prime and M * N is a square. Than M and N both are squares.

Proof By the assumption M * N = $P_1^{2K_1} * P_2^{2K_2} * ... * P_s^{2K_s}$. Since M and N are relative prime they do not have common prime. That is no factor $P_i^{2K_i}$ can be divided up between the two factors. Therefore both M and N are the products of full factors of $P_i^{2K_i}$, so both are squares. ■

II. If any two of the Pythagorean triples have a common divisor than the third number is a multiple of this common divisor

Proof

We consider three cases:

i) $x = n * a$, $z = n * b$, then $z^2 = x^2 + y^2 = n^2(a^2 + b^2)$ so z is a multiple of n.

ii) $x = n * a$, $z = n * b$, then $y^2 = z^2 - x^2 = n^2(b^2 - a^2)$, so y is a multiple of n.

iii) $y = n * a$, $z = n * b$, then $x^2 = z^2 - y^2 = n^2(b^2 - a^2)$, so x is a multiple of n. ■

III. Primitive Pythagorean triples can be obtained by the following formula:

$x = 2 * a * b$, $y = a^2 - b^2, z = a^2 + b^2$ where a and b are relative prime.

Proof

Let x, y, z be such Pythagorean triples, than $x^2 = z^2 - y^2 = (z + y) * (z - y)$.

First we show, that $z + y$ and $z - y$ must not have common divisor other than 2.

In contrary assume that: $z + y = p * A$

$z - y = p * B$ with some odd prime p. Adding and subtracting these equations give relations:

$$z = \frac{p(A + B)}{2}, y = \frac{p(A - B)}{2}$$

From the previous observation x has to be also a multiple of p. This contradicts the assumption that x, y, z have no common divisors.

We have to consider two cases:

i) $z - y$ = odd number, then $z + y = z - y + 2 * y$ is also odd. Since $x^2 = (z+y)*(z-y)$ where $z + y$, and $z - y$ have no common divisor, both are squares by observation I. Hence $z - y = a^2$ and $z + y = b^2$ implying, that $z = \frac{a^2 + b^2}{2}, y = \frac{b^2 - a^2}{2}$, and $x^2 = z^2 - y^2 = a^2 * b^2$, so $x = a * b$. Notice that $b - a$ and $b + a$ are both even numbers since a and b are both odd. We can write $b - a = 2 * u$ and $a + b = 2*v$ implying that $b = u + v$ and $a = v - u$. Therefore $x = a * b = v^2 - u^2$,

$y = \frac{b^2 - a^2}{2} = 2*u*v$, and $z = \frac{a^2 + b^2}{2} = u^2 + v^2$, what was to be proved.

ii) $z - y$ = even number, then $z + y = z - y + 2 * y$ is also even. Then with relative primes a and b,

$z - y = 2 * a$ and $z + y = 2 * b$,

so $z = a + b$ and $y = b - a$. Then $x^2 = z^2 - y^2 = 4*a*b$.

so $x = 2 * c$ and $4*c^2 = 4 * a * b$, showing that $c^2 = a*b$. From the observation I we conclude that $a = u^2$ and $b = v^2$, and so $c = u * v$. Then:

$$x = 2*\sqrt{a*b} = 2*u*v$$
$$y = b - a = v^2 - u^2$$
$$z = a + b = v^2 + u^2$$

what was to be proved. ■

Remark If we allow common divisors of Pythagorean triples, than the formula is:

$x = 2*a*b*n, y = (a^2 - b^2)*n, z = (a^2 + b^2)*n$, where a and b are relative primes and n is arbitrary positive integer.

3) Some elementary results

I. One of the Pythagorean triples is a multiple of 3.

Proof Two cases are considered:

i) If a or b is multiple of 3. Then $x = 2 * a * b * n$ is also a multiple of 3

ii) If none of a and b are multiple of 3, then $a = 3 * k \pm 1, b = 3 * l \pm 1$, so

$y = (a^2 - b^2) = (9 * k^2 \pm 6 * k - 9l^2 \mp 6 * l) * n$ is also multiple of 3 ■

II. One is a multiple of 4

Proof

i) If a or b is even, then $x = 2 * a * b * n$ is a multiple of 4.

ii) If both are odd, than $a = 2 * k + 1$, and $b = 2 * l + 1$, implying that

$y = (a^2 - b^2) * n = (4 * k^2 + 4 * k - 4 * l^2 - 4 * l) * n$ is a multiple of 4. ■

III. One is a multiple of 5

Proof

i) If a or b is a multiple of 5, than $x = 2 * a * b * n$ is also divisible by 5.

ii) Otherwise $a = 5 * k \pm 1$, or $a = 5 * k \pm 2$, and $b = 5 * l \pm 1$, or $b = 5 * l \pm 2$

So four cases are possible:

a) If $a = 5*k \pm 1$, and $b = 5*l \pm 1$, then $y = (a^2 - b^2) * n$ is a multiple of 5

b) If $a = 5*k \pm 2$, and $b = 5*l \pm 2$, then $y = (a^2 - b^2) * n$ is a multiple of 5

c) If $a = 5*k \pm 1$, and $b = 5*L \pm 2$, then $z = (a^2 + b^2) * n$ is a multiple of 5

d) If $a = 5*k \pm 2$, and $b = 5*L \pm 1$, then $z = (a^2 + b^2) * n$ is a multiple of 5 ■

IV. The product of these numbers is divisible by 60.

Proof

One is a multiple of 3, the same or other a multiple of 4 and the same or other is a multiple of 5.

So their product is a multiple of 3 * 4 * 5 = 60. ■

V. The area of the Pythagorean triangle is divisible by 6

Proof The area is $\frac{1}{2} * x * y = \frac{1}{2} * 2 * a * b * n * (a^2 - b^2) * n = n^2 * a * b * (a - b) * (a + b)$

Next we show, that this product is a multiple of 2 and 3.

i) If a or b is even, than $a * b$ is even

ii) If a and b are odd, than $a - b$ and $a + b$ are even, so the area is a multiple of 2.

iii) If a or b is a multiple of three, than $a * b$ is a multiple of three.

iv) If neither a nor b is a multiple of three, then $a = 3 * k \pm 1, b = 3 * l \pm 1$,

So $(a - b) * (a + b) = 9 * k^2 \pm 6 * k - 9 * l^2 \mp 6 * l$ is a multiple of 3. So the area is divisible by 3. ■

VI) Any two sides of a primitive Pythagorean triangle are relative prime.

Proof

Observation I of the preliminary section was the same result. ■

4) The number of Pythagorean triangles

I) The number of the primitive Pythagorean triangles having all 3 sides less than 100 is 23.

Complete enumeration shows that, and they are given below:

a	2	3	3	4	4	5	5	5	5	6	6	7	7	7	7	7	7	8	8	8	8	9	9
b	1	1	2	1	3	1	2	3	4	1	5	1	2	3	4	5	6	1	3	5	1	2	4
a^2+b^2	5	10	13	17	25	26	29	34	41	37	61	50	53	58	65	74	85	65	73	89	82	85	97

The largest side is $z = a^2 + b^2$, so it has to be less than a 100.

The same result has been obtained by using the FORTRAN program given in Appendix I.

II) The number of primitive Pythagorean triangles having sides less than 200 is 49.

By enumeration, and also using a slightly modified version of the program given in Appendix I.

III) The number of primitive Pythagorean triangles having sides less than 400 is 97.

By enumeration, and also using a slightly modified version of the program given in Appendix I gave us the same result.

IV) The number of primitive Pythagorean triangles having sides less than a million is 242,742.

Using a slightly modified version of the program given in Appendix I.

V) By using a slightly modified version of the program given in Appendix I, we have computed the numbers F(N) and G(N) of the primitive and not necessarily primitive Pythagorean triangles with sides less than N for several values of N. The results is given in the following tables:

N	100	200	400	1,000,000
G (N)	32	68	141	391,841

N	F(N)
100	23
200	49
300	72
400	97
500	121
1000	244
10000	2456
20000	4916
30000	7366
40000	9814
50000	12269
60000	14704
70000	17152
80000	19595

N	F(N)
410000	99847
420000	102274
430000	104710
440000	107135
450000	109550
460000	111981
470000	114415
480000	116836
490000	119260
500000	121689
510000	124106
520000	126529
530000	128964
540000	131394

N	F(N)
870000	211304
880000	213721
890000	216145
900000	218559
910000	220979
920000	223403
930000	225826
940000	228245
950000	230660
960000	233080
970000	235488
980000	237907
990000	240316
1000000	242742

90000	22035
100000	24483
110000	26924
120000	29353
130000	31794
140000	34229
150000	36671
160000	39108
170000	41540
180000	43978
190000	46404
200000	48839
210000	51275
220000	53709
230000	56141
240000	58578
250000	61001
260000	63436
270000	65869
280000	68296
290000	70724
300000	73155
310000	75575
320000	78013
330000	80435
340000	82870
350000	85290
360000	87721
370000	90157
380000	92579
390000	95001
400000	97433

550000	133806
560000	136234
570000	138645
580000	141077
590000	143499
600000	145930
610000	148347
620000	150777
630000	153202
640000	155613
650000	158046
660000	160455
670000	162886
680000	165309
690000	167721
700000	170139
710000	172579
720000	174990
730000	177416
740000	179837
750000	182260
760000	184684
770000	187092
780000	189502
790000	191929
800000	194361
810000	196775
820000	199199
830000	201615
840000	204036
850000	206462
860000	208879

2000000	484420
2500000	605097
3000000	725733
3500000	846300
4000000	966868
4500000	1087369
5000000	1207839
5500000	1328308
6000000	1448738
6500000	1569132
7000000	1689537
7500000	1809895
8000000	1930244
8500000	2050588
9000000	2170879
9500000	2291194
10000000	2411514
10500000	2531785
11000000	2652043
11500000	2772311
12000000	2892567
12500000	3012814
13000000	3133034
13500000	3253264
14000000	3373481
14500000	3493674
15000000	3613879
15500000	3734055
16000000	3854246
16500000	3974451
17000000	4094628
17500000	4214799

N	F(N)
18000000	4334923
18500000	4455110
19000000	4575271
19500000	4695433
20000000	4815569
20500000	4935713
21000000	5055799
21500000	5175941
22000000	5296086
22500000	5416195
23000000	5536346
23500000	5656417
24000000	5776530
24500000	5896642
25000000	6016760
25500000	6136854

N	F(N)
29000000	6977413
29500000	7097467
30000000	7217538
30500000	7337601
31000000	7457648
31500000	7577738
32000000	7697774
32500000	7817818
33000000	7937903
33500000	8057928
34000000	8177965
34500000	8298056
35000000	8418040
35500000	8538107
36000000	8658134
36500000	8778158

N	F(N)
40000000	9618298
40500000	9738311
41000000	9858330
41500000	9978337
42000000	10098333
42500000	10218375
43000000	10338362
43500000	10458400
44000000	10578369
44500000	10698359
45000000	10818352
45500000	10938365
46000000	11058372
46500000	11178337
47000000	11298325
47500000	11418287

26000000	6256911
26500000	6377010
27000000	6497085
27500000	6617215
28000000	6737271
28500000	6857323

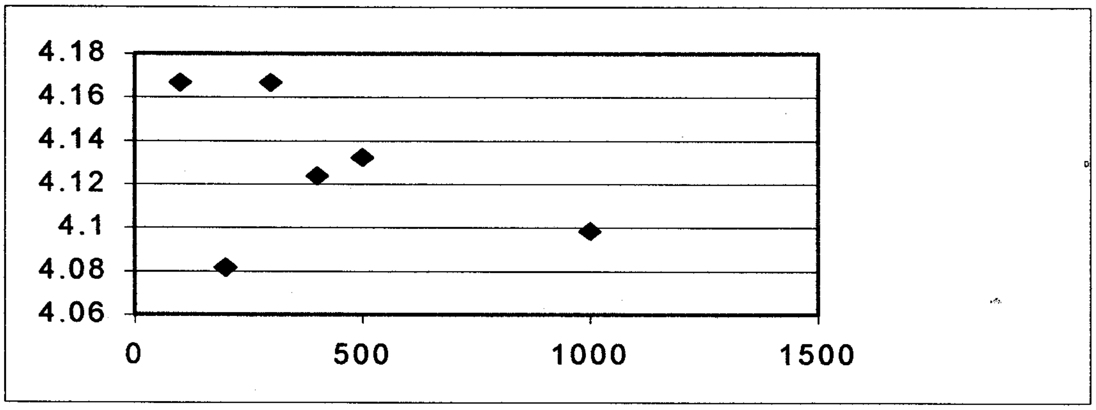
37000000	8898182
37500000	9018186
38000000	9138254
38500000	9258268
39000000	9378250
39500000	9498299

48000000	11538299
48500000	11658305
49000000	11778255
49500000	11898249
50000000	12018221

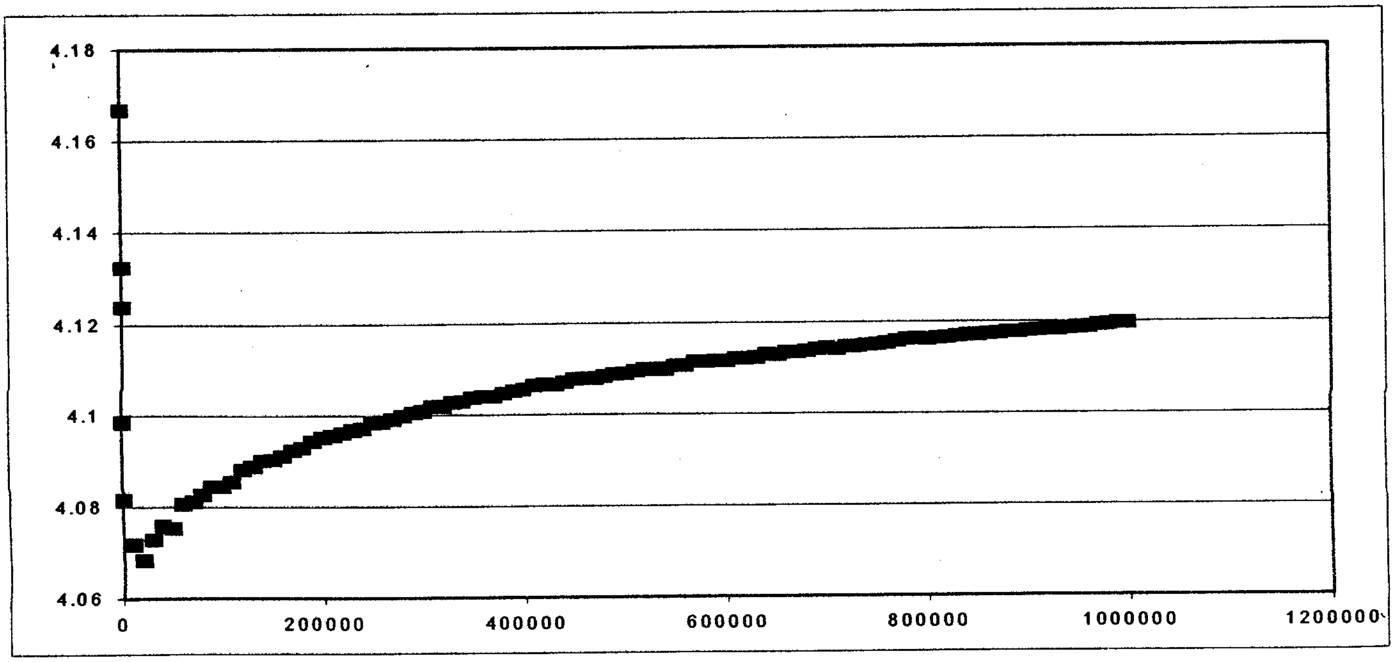
We have also computed and graphed the ratio $\frac{N}{F(N)}$ for the above values.

The table below summarizes some N and F(N) values, and their ratio is illustrated in the figure.

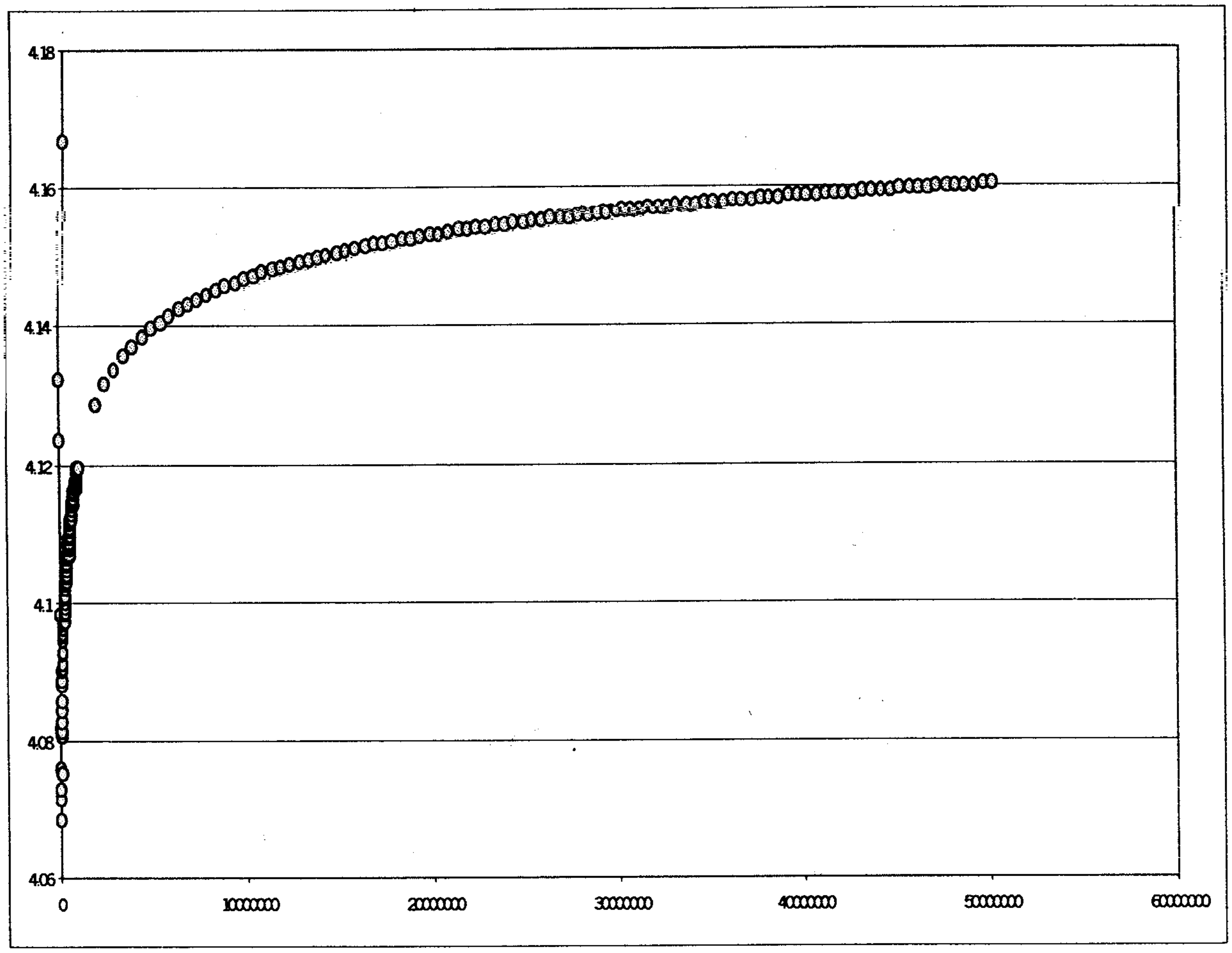
100	23
200	49
300	72
400	97
500	121
1000	244



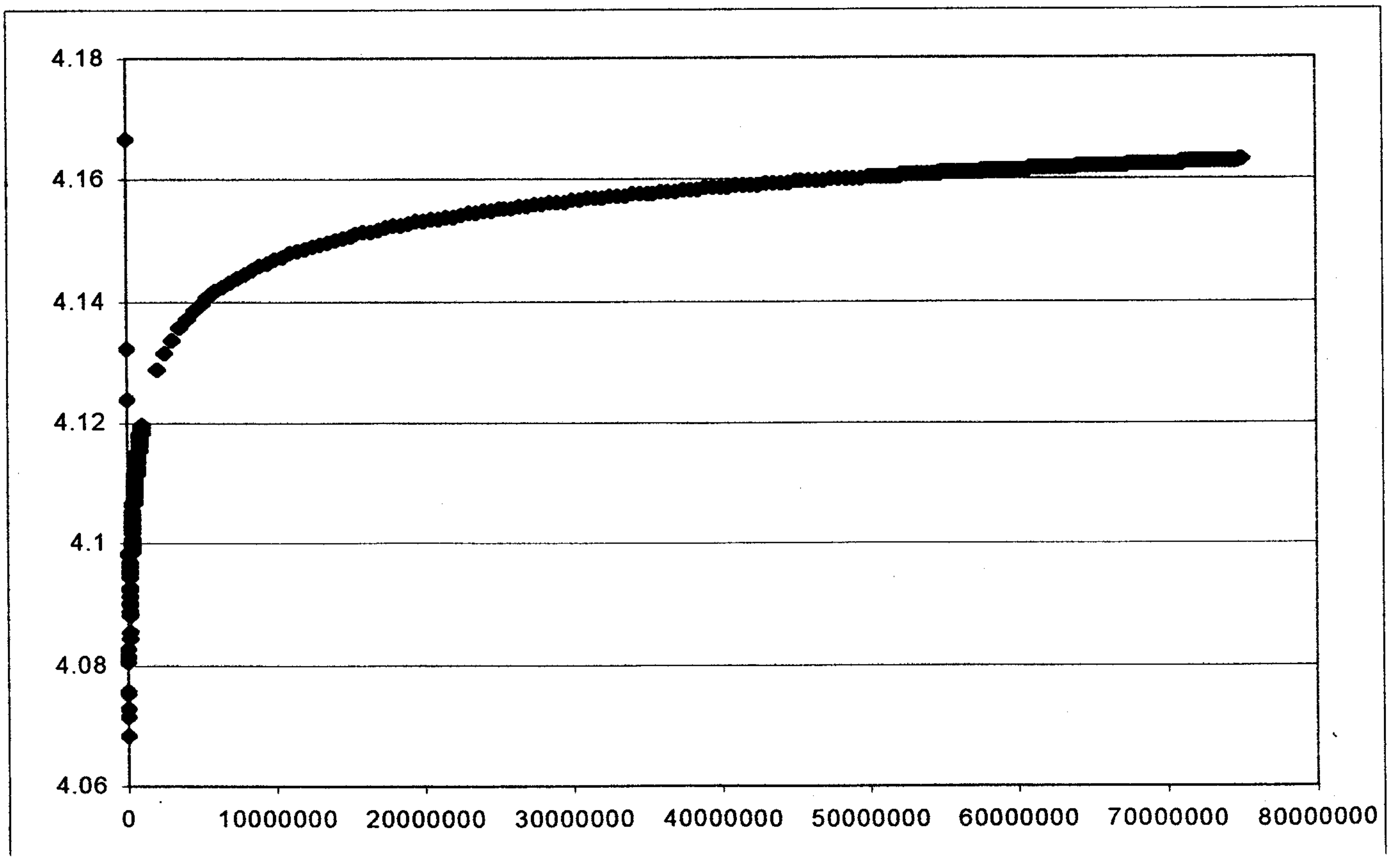
The next figure shows the ratio values for N=0, 1, …, 1,000,000



The following figure shows the values of the ratio when N is up to 50,000,000



The following figure shows the values of the ratio when N is up to 75,000,000



By looking at the last graph we can see, that from N $\approx$ 20,000 this ratio is increasing in N, and has a limit as N $\rightarrow \infty$.

What is this limit, I do not know, but by looking at these graphs it seems that the limit is around 4.165.

V) By using the computer program given in Appendix II we have found the Pythagorean triangle with sides less than 1,000,000 that is the closest to being isosceles. In the method of the program of Appendix I we have generated all such triangles.

Appendix I.

```
*       L      ->   loop cunter to total up COUNT
*       M      ->   number to set data goes into which COUNT
*       A      ->   COUNT array dummy value
*       I      ->   loop counter for output printing
*       U      ->   value of larger side
*       V      ->   value of smaller side
*       Z      ->   value of U^2+V^2
*       N      ->   value to limit common devisor calculation
*       J      ->   loop counter for common devisor test variable
*       N1     ->   common devisor test variable
*       N2     ->   common devisor test variable
*       ZMAX   ->   maximum value for Z
*       UMAX   ->   maximum value for U
*       DIV    ->   division modifier
*       COUNT  ->   array that holds final counter values
*       KEY    ->   value that is used to see if U and V have common devisor

      PROGRAM PITA
      IMPLICIT NONE
      INTEGER L,M,A,I,U,V,Z,N,J,N1,N2,ZMAX,UMAX,DIV
      PARAMETER (UMAX=11,ZMAX=100,A=101,DIV=100)
      INTEGER COUNT(A)
      LOGICAL KEY

      OPEN (10,file='pita.out',status='unknown')
      U=1
      COUNT(0)=1      ! since U=1,V=1 is not calculated by program
      DO WHILE (U .LT. UMAX)  !  loop to find all U,V combinations
        V=1
        DO WHILE (V. LT. U)
          KEY=.TRUE.      !  to find if common devisor exists
          N=FLOAT(V)**0.5+1
          DO J=2,N
            N1=MOD(U,J)
            N2=MOD(V,J)
            IF (N1.EQ.0.AND.N2.EQ.0) THEN
              KEY=.FALSE.
              GOTO 100   !  if one common devisor found, data not good
            ENDIF
          ENDDO
100       IF (KEY.OR.V.EQ.1) THEN  ! only calculate if no common devisor
          IF (MOD(U,V).NE.0.OR.V.EQ.1) THEN
            Z=U**2+V**2
            IF (Z .LE. ZMAY) THEN  ! dont care about data outside of ZMAX
             Z= Z-1       !  to place counter in correct array spot
             M = Z / DIV
```

```
100       IF (KEY.OR.V.EQ.1) THEN  ! only calculate if no common devisor
          IF (MOD(U,V).NE.0.OR.V.EQ.1) THEN
            Z=U**2+V**2
            IF (Z .LE. ZMAX) THEN  ! dont care about data outside of ZMAX
              Z= Z-1       !  to place counter in correct array spot
              M = Z / DIV
              COUNT (M) = COUNT (M) + 1
            ENDIF
          ENDIF
          ENDIF
          V= V+1
        ENDDO
        U = U+1
      ENDDO
      DO L=1,A    ! totals up values
        COUNT(L)=COUNT(L)+COUNT(L-1)
      ENDDO
      DO L=0,A-1      ! prints values
        WRITE (10,20) L*DIV,'<U^2+V^2<',(L+1)*DIV,'=',COUNT(L)
20    FORMAT (I10,A,I10,A,I)
      ENDDO
      CLOSE (10)
      END  ! program PITA
```

```
      PROGRAM PIT
      IMPLICIT NONE
      DOUBLE PRECISION A,C,AA,BA,CA,ERROR,ERRORA,D

      ERRORA=1.0
      OPEN (10,FILE='pit.out',status='unknown')
      DO A=1.0,1000000.0
          C=SQRT(A**2+(A-1)**2)
          D=SQRT(A**2+(A-2)**2)
          IF (MOD(C,1.0).EQ.0.0) THEN
            ERROR=ABS((A-(A-1))/A)
            IF (ERROR.LT.ERRORA) THEN
              ERRORA=ERROR
              AA=A
              BA=A-1
              CA=C
            ENDIF
          ENDIF
          IF (MOD(D,1.0).EQ.0.0) THEN
            ERROR=ABS((A-(A-2))/A)
            IF (ERROR.LT.ERRORA) THEN
              ERRORA=ERROR
              AA=A
              BA=A-2
              CA=D
            ENDIF
          ENDIF
      ENDDO
      WRITE(10,20) AA,BA,CA
20    FORMAT (T1,F12.2,T20,F12.2,T40,F12.2)
      CLOSE (10)
      END
```