Random Permutation Matrices

An Investigation of the Number of Eigenvalues Lying in a Shrinking Interval

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September 24, 2000

Abstract

When an $n \times n$ permutation matrix is chosen at random, each of its n eigenvalues will lie somewhere on the unit circle. We investigate the average number of these that fall in an interval that shrinks as the size of the matrix increases, and compare the results against the case where n points are chosen independently.

1 Introduction

A permutation matrix is any $n \times n$ matrix that has exactly one 1 in each row and column, with all other entries being 0. Here is an example of a 6×6 permutation matrix:

$$P = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

All the eigenvalues of a permutation matrix lie on the (complex) unit circle, and one might wonder how these eigenvalues are distributed when permutation matrices are chosen at random (that is, uniformly from the set of all $n \times n$ permutation matrices). Some work has already been done in studying the eigenvalues of permutation matrices. Diaconis and Shahshahani [3] looked at the trace (sum of the eigenvalues), and Wieand [5],[4] investigated the number of eigenvalues that lie in a fixed arc of the unit circle. In both cases, the asymptotic behavior for large n was determined.

Roughly speaking, the number of eigenvalues that lie in a fixed interval on the unit circle will be proportional to the size of the interval and to the dimension n of the matrix. In this paper, the goal will be to allow n to increase while decreasing the size of the interval, so that the number of eigenvalues lying in it should remain fairly constant on average. In particular, we look at the number of eigenvalues $X_{n,a}$ lying in the interval $I_n = \left(e^{2\pi i a},\, e^{2\pi i(a+l/n)}\right]$ when an $n \times n$ permutation matrix is chosen at random, and we find the limit of the mean of $X_{n,a}$ as $n \to \infty$ in the case when a is rational.

The paper will be organized as follows. The next section provides some background about permutations and gives some probabilistic results that will be used later. Section 3 discusses the eigenvalues of permutation matrices and provides a formula for $X_{n,a}$. Section 4 draws some comparisons between the distribution of eigenvalues and the distribution of random points on the unit circle. Section 5 provides a technical result that will be used in section 6 to find the limit of $E[X_{n,a}]$. The conclusion takes another look at the function derived in section 6, and then presents some open questions about $X_{n,a}$.

2 Background About Permutations and Probability

For our purposes, a permutation can be thought of as a one-to-one mapping of the set of integers $\{1, 2, \ldots, n\}$ onto itself. The group of all permutations of n numbers is known as the symmetric group, S_n, and it is a simple matter to verify that there are $n!$ permutations in S_n. In standard notation, a permutation $\sigma \in S_n$ is written as

$$\sigma = \begin{pmatrix} 1 & 2 & \cdots & n \\ \sigma(1) & \sigma(2) & \cdots & \sigma(n) \end{pmatrix},$$

where $\sigma(1)$ is the image of 1 under σ, $\sigma(2)$ is the image of 2, and so on.

2.1 Cycles and Cycle Structure

A permutation can also be written in a way that groups together the images of a given number under repeated applications of σ. For example, the permutation

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 3 & 6 & 4 & 7 & 5 & 9 & 1 & 8 & 2 \end{pmatrix}$$

can be written

$$\sigma = (1\ 3\ 4\ 7)(2\ 6\ 9)(5)(8).$$

The first group of numbers in parentheses indicates that 1 gets mapped to 3, 3 gets mapped to 4, 4 gets mapped to 7, and 7 gets mapped back to 1. Each of the other groupings is interpreted in a similar way. These groups of numbers are called *cycles*, and this notation for permutations is referred to as *cycle notation*. Following are several facts relating to cycles and cycle notation.

- A cycle of k numbers is referred to as a k-cycle; for example, (1 3 4 7) is a 4-cycle.
- A cycle of one number indicates that the number is mapped to itself, and 1-cycles are often referred to as *fixed points*.
- If a permutation σ is applied k times, then the numbers in each k-cycle in σ will return to their starting positions.
- The number of times that a permutation σ must be applied in order to return all numbers to their starting positions is known as the *order* of σ, and will equal the least common multiple of the lengths of all the cycles in σ.
- It does not matter which number is written first in a cycle, as long as the order of the numbers is preserved. For example, $(1\ 3\ 4\ 7) = (4\ 7\ 1\ 3)$, but $(1\ 3\ 4\ 7) \neq (1\ 4\ 3\ 7)$.
- The cycles in a permutation can be written in any order. If desired, one can apply any of a number of systematic approaches to keep the notation consistent.

It is useful to define a vector, $(C_1, C_2, \ldots, C_n)$, called the *cycle structure* of σ, where each entry C_k gives the number of k-cycles in σ. Thus, our sample permutation above has a cycle structure of (2, 0, 1, 1, 0, 0, 0, 0, 0). Two things to notice about cycle structure are

1. The sum of all the values of C_k gives the total number of cycles in σ, and
2. Since there are n numbers in σ, the lengths of all the cycles must add up to n. That is, for $\sigma \in S_n$,
$$\sum_{k=1}^{n} kC_k = n. \tag{1}$$

2.2 Probability and Cycle Structure

At this point, one could ask various questions about cycle structure, such as "How many permutations are there with a given cycle structure?" or, "What is the cycle structure of a 'typical' random permutation σ?" That is, how many fixed points, how many 2-cycles, etc. will σ have, on average?

When the phrase "random permutation" is used in this paper, it means that each permutation in S_n is equally likely to be chosen. Thus, the probability of picking any one permutation is $1/n!$. Using this, the mean, or expected value, of any random variable V defined on S_n will be

$$E[V] = \frac{1}{n!} \sum_{\sigma \in S_n} V(\sigma), \tag{2}$$

and the variance of V will be

$$Var[V] = \frac{1}{n!} \sum_{\sigma \in S_n} (V(\sigma) - E[V])^2. \tag{3}$$

Notice that the values $C_1, C_2, \ldots, C_n$ for a permutation picked from S_n are just random variables, and the expectation of these values might provide some insight into the questions posed above. Using standard group theory arguments, it can be shown that the probability of picking a permutation with a particular cycle structure, say $(\vartheta_1, \vartheta_2, \ldots, \vartheta_n)$, is

$$P(C_1 = \vartheta_1, C_2 = \vartheta_2, \ldots, C_n = \vartheta_n) = \begin{cases} \prod_{k=1}^{n} \frac{1}{k^{\vartheta_k} \vartheta_k!} & \text{if } \sum_{k=1}^{n} k\vartheta_k = n \\ 0 & \text{otherwise.} \end{cases} \tag{4}$$

This formula can be used to prove a number of facts about the random variables C_k. The results below are due to Goncharov [1]. (Also see Diaconis and Shahshahani [3].)

$$E[C_k] = \begin{cases} \frac{1}{k} & \text{if } k \leq n \\ \\ 0 & \text{otherwise,} \end{cases} \tag{5}$$

$$E[C_j C_k] = \begin{cases} \frac{1}{jk} & \text{if } j + k \leq n \\ \\ 0 & \text{otherwise} \end{cases} \tag{6}$$

if $j \neq k$, and

$$Var[C_k] = \begin{cases} \frac{1}{k} & \text{if } k \leq n/2 \\ \\ \frac{1}{k} - \frac{1}{k^2} & \text{if } n/2 < k \leq n \\ \\ 0 & \text{otherwise.} \end{cases} \tag{7}$$

3 Permutation Matrices and $X_{n,a}$

For each $\sigma \in S_n$, let M_σ be the $n \times n$ matrix constructed by the following rule:

$$(M_\sigma)_{ij} = \begin{cases} 1 & \text{if } j = \sigma(i) \\ 0 & \text{otherwise.} \end{cases} \tag{8}$$

That is, the i^{th} row of M_σ has a 1 in the column $\sigma(i)$ and 0's in all the others. It is easy to verify that M_σ is a permutation matrix (as defined in the introduction), and that this rule in fact defines a one-to-one correspondence between S_n and the $n \times n$ permutation matrices. (With M_σ defined in this way, a matrix that is left-multiplied by M_σ will have its rows permuted according to σ, and a matrix that is right-multiplied by M_σ will have its columns permuted according to the inverse of σ.)

Using some elementary facts about S_n and the properties of determinants, it is not difficult to show that, if σ has a cycle structure of $(C_1, C_2, \ldots, C_n)$, then the characteristic polynomial of M_σ is

$$p(\lambda) = \det(M_\sigma - \lambda I) = (-1)^n \prod_{k=1}^{n} (\lambda^k - 1)^{C_k}, \tag{9}$$

which results because every k-cycle in σ contributes a factor of $(-1)^k(\lambda^k - 1)$ to $p(\lambda)$. The zeros of $\pm(\lambda^k - 1)$ are just the k^{th} roots of unity, which are 1, $e^{2\pi i/k}$, $e^{4\pi i/k}$, $\ldots$, $e^{2(k-1)\pi i/k}$. (These are just points on the unit circle that are spaced at an angle of $2\pi/k$ apart.) Since each k-cycle generates this set of k eigenvalues, if σ has C_k k-cycles, then M_σ has C_k copies of these eigenvalues.

Because of this relationship, the random variable $X_{n,a}$ can be written in terms of the cycle structure $(C_1, C_2, \ldots, C_n)$. In order to do this, the following notation will be needed. These definitions also will be used in later sections for determining the limits on sums.

Definition 1 (Floor, Ceiling, and Fractional Part) *For all real numbers x, the largest integer less than or equal to x is denoted by $\lfloor x \rfloor$, read floor of x. Similarly, the smallest integer greater than or equal to x is denoted by $\lceil x \rceil$, read ceiling of x. In addition, the fractional part of x, written $\{x\}$, is defined to be the difference $x - \lfloor x \rfloor$. (Notice that $0 \le \{x\} < 1$ for all x.)*

Now, consider an arbitrary interval $I = \left(e^{2\pi i a}, e^{2\pi i b}\right]$ on the unit circle, with $0 < b - a \le 1$. Of the k eigenvalues corresponding to a k-cycle, $\lfloor kb \rfloor - \lfloor ka \rfloor$ of them will lie in the interval. Thus, for an arbitrary permutation σ, the number of eigenvalues of M_σ in I is given by $\sum_{k=1}^{n} C_k(\sigma)(\lfloor kb \rfloor - \lfloor ka \rfloor)$. To determine the number of eigenvalues of M_σ in I_n, simply replace b with $a + l/n$ to obtain

$$X_{n,a}(\sigma) = \sum_{k=1}^{n} C_k(\sigma) \left(\left\lfloor k \left(a + \frac{l}{n} \right) \right\rfloor - \lfloor ka \rfloor \right). \tag{10}$$

The mean of $X_{n,a}$ is then

$$
\begin{aligned}
E_{S_n}[X_{n,a}] &= E_{S_n}\left[\sum_{k=1}^{n} C_k \left(\left\lfloor k\left(a+\frac{l}{n}\right)\right\rfloor - \lfloor ka \rfloor\right)\right] \\
&= \sum_{k=1}^{n} E_{S_n}[C_k] \left(\left\lfloor k\left(a+\frac{l}{n}\right)\right\rfloor - \lfloor ka \rfloor\right) \\
&= \sum_{k=1}^{n} \frac{1}{k} \left(\left\lfloor k\left(a+\frac{l}{n}\right)\right\rfloor - \lfloor ka \rfloor\right). \qquad (11)
\end{aligned}
$$

The restriction that $b-a \leq 1$ is to ensure that none of the eigenvalues are counted more than once. In the case of $X_{n,a}$, this is equivalent to requiring that $n \geq l$. Thus, equations (10) and (11) are valid for any value of $l > 0$, provided that n is large enough. Now, if $n \leq l$, then I_n is guaranteed to wrap around the unit circle at least once, and therefore will contain all the eigenvalues of M_σ. That is, $X_{n,a} = n$ when $n \leq l$, and from now on, it will be assumed that $n > l$. Also notice that because the interval lies on a circle, every value of a corresponds to a value in the range $[0, 1)$. Although this point is not essential, it can be assumed that $0 \leq a < 1$ without losing generality.

4 Preliminary Observations

When a permutation is picked with uniform probability from S_n and the eigenvalues of M_σ are plotted, the result is that n points on the unit circle have been chosen "at random", in the sense that the outcome of this experiment is not known beforehand. Obviously, though, not every every point on the circle is equally likely to be picked. In fact, only a finite set of points is possible, and the probability of picking a particular point depends on its location.

Plotting the eigenvalues of a random $n \times n$ permutation matrix can be compared with plotting n independent points chosen uniformly from the set of all points on the unit circle. The purpose of this section is to summarize what happens for independent uniform points, and then to make a few quick observations about the eigenvalue distribution of permutation matrices, providing a brief comparison of the two situations.

4.1 Random Independent Points on the Unit Circle

In order to provide a basis for comparison, we can define a random variable analogous to $X_{n,a}$. Let Y_n be the number of points that land in the interval $I_n = \left(e^{2\pi ia}, e^{2\pi i(a+l/n)}\right]$ when n independent points are picked uniformly.

When the points are picked in this way, there are two intuitive results that follow immediately. First, the distribution of Y_n should not depend on a. (Since the points are equally likely to be chosen from anywhere on the circle, the location of the interval should not matter.) Second, the fraction Y_n/n of points that land in the interval will on average be the same as the ratio of the length of the interval to the circumference of the circle. Thus, by defining I_n to have a length of $2\pi l/n$, the mean of Y_n will have the constant value l, regardless of the value of n.

These results also become apparent by noticing that Y_n is a binomial random variable, as follows. If a single point on the unit circle is chosen at random (uniformly), the probability that it will lie in I_n is $p = l/n$. When n points are chosen independently, the number of points Y_n lying in I_n will, by definition, be binomial with parameters $(n, l/n)$. Binomial random variables are standard in probability, and the mean and variance in this case are known to be

$$E[Y_n] = n\left(\frac{l}{n}\right) = l, \tag{12}$$

and

$$Var[Y_n] = n\left(\frac{l}{n}\right)\left(1 - \frac{l}{n}\right) = l\left(1 - \frac{l}{n}\right). \tag{13}$$

4.2 The Number of Eigenvalues at $e^{i\theta}$

When random points are chosen uniformly on the unit circle, the probability of picking any particular point $e^{i\theta}$ is 0. The eigenvalues of permutation matrices, however, occur only at certain values of θ, so the probability of choosing one of these points is positive, while the probability for any other point is 0.

In order to gain some insight into this problem, define a random variable $Z_{n,\theta}$ to be the number of eigenvalues of $\sigma \in S_n$ equal to $e^{i\theta}$. The variable $Z_{n,\theta}$ is already well understood; see, for example, [1], [2]. Presented here is a brief explanation of what happens to the mean of $Z_{n,\theta}$ as $n \to \infty$.

First consider the case when $\theta = 0$. Every cycle in a permutation σ produces the eigenvalue 1, so the number of eigenvalues at $\theta = 0$ will equal the total number of cycles in σ. Recall that the number of cycles in σ is the sum of all the values of C_k in the cycle structure. Thus,

$$Z_{n,0} = \sum_{k=1}^{n} C_k, \tag{14}$$

and the mean of $Z_{n,0}$ is

$$E\left[Z_{n,0}\right] = \sum_{k=1}^{n} E[C_k] = \sum_{k=1}^{n} \frac{1}{k}. \tag{15}$$

For large n, this sum can be approximated by $\ln n$, resulting in

$$E\left[Z_{n,0}\right] = \ln n + O(1). \tag{16}$$

Similar reasoning can be used to see that in general, if $\theta = 2\pi p/q$ with p and q relatively prime, then

$$E\left[Z_{n,\theta}\right] = \frac{1}{q}\ln n + O(1). \tag{17}$$

If θ is an irrational multiple of 2π, then no eigenvalues can occur there, so $Z_{n,\theta} = 0$ for all n in this case. This behavior is quite different from the uniform case.

4.3 Description of $X_{n,a}$ when $a = 0$ and $l \leq 1$

This is a special case for which very little calculation is involved in determining the behavior of $X_{n,a}$. With $a = 0$, the interval starts at 0 and ends at $2\pi l/n$, and equation (10) simplifies to

$$X_{n,0} = \sum_{k=1}^{n} C_k \left\lfloor \frac{kl}{n} \right\rfloor. \tag{18}$$

Since the largest cycle that can occur in σ is an n-cycle, the first position on the unit circle where an eigenvalue can occur is at $\theta = 2\pi l/n$. If $l < 1$, then $X_{n,0} = 0$ because the interval ends before reaching the first possible eigenvalue. This can also be seen from (18) by noting that $\lfloor kl/n \rfloor = 0$ if $l < 1$.

Now if $l = 1$, then the interval ends exactly where the first eigenvalue can occur, so we have

$$X_{n,0} = \begin{cases} 1 & \text{if } \sigma \text{ has an } n\text{-cycle} \\ 0 & \text{otherwise.} \end{cases}$$

The probability that σ has an n-cycle is just $1/n$, and it is easy to show that in this case,

$$E[X_{n,0}] = \frac{1}{n} \tag{19}$$

and

$$Var[X_{n,0}] = \frac{1}{n}\left(1 - \frac{1}{n}\right) = \frac{n-1}{n^2}. \tag{20}$$

Both the mean and the variance approach 0 as $n \to \infty$.

Remark. A similar analysis shows that an analogous 'gap' occurs around each rational point a. In particular, if $a = p/q$ in lowest terms, then no eigenvalues can fall in the interval I_n if $l < 1/q$. For irrational values of a, this sort of gap does not occur. No matter how small l is, there always will be some values of n that produce eigenvalues in the interval I_n.

These results illuminate some of the differences between the distribution of eigenvalues and that of independent points on the circle. Equations (19) and (20) describe a particularly simple situation, and in this case, the results for random permutations are strikingly different from the results for random independent points. When $l > 1$, the simple argument used here no longer works, and it may not be possible to find explicit formulas for $E[X_{n,a}]$ or $Var[X_{n,a}]$ in terms of n. In that case, when n is small, it is easy to calculate the value of $X_{n,a}$, and of $E[X_{n,a}]$ or other quantities describing the distribution of eigenvalues. As n increases, however, exact results require more and more computation, and it is more useful to try to find general trends that will provide a picture of what is happening. The following sections use a different approach to find the large n limit of $E[X_{n,a}]$, and the goal will be to see whether this limit might resemble the independent points case more closely when the constant l is larger.

5 A Technical Lemma

The following lemma, which is an elementary analysis result, will be needed for calculating the limit of $E[X_{n,a}]$ in the next section.

Lemma 1 *Suppose $\alpha > \beta \geq 0$. Let (L_n) and (M_n) be sequences of positive integers which satisfy*

$$\frac{L_n}{n} \to L > 0$$

and

$$\frac{M_n}{n} \to M > 0.$$

Then the sum

$$\sum_{k=L_n}^{M_n} \frac{1}{\alpha k + \beta} = \frac{1}{\alpha} \ln(M_n) - \frac{1}{\alpha} \ln(L_n) + A_n,$$

where

$$|A_n| \leq \frac{1}{\alpha L_n}.$$

In particular,

$$\lim_{n\to\infty} \sum_{k=L_n}^{M_n} \frac{1}{\alpha k + \beta} = \frac{1}{\alpha} \ln\left(\frac{M}{L}\right).$$

Proof. First, observe that for any integers $0 < x < y$, the sum

$$\sum_{k=x+1}^{y} \frac{1}{k} = \ln(y/x) + \epsilon_1(x,y), \tag{21}$$

where

$$\frac{1}{2y} - \frac{1}{2x} \leq \epsilon_1(x,y) \leq 0. \tag{22}$$

(This can be seen by comparing the sum with the integral $\int_x^y \frac{1}{t} dt$.)

Next, since $\alpha > \beta \geq 0$, note that

$$\sum_{k=x+1}^{y+1} \frac{1}{\alpha k} < \sum_{k=x}^{y} \frac{1}{\alpha k + \beta} \leq \sum_{k=x}^{y} \frac{1}{\alpha k}. \tag{23}$$

Thus, using (21),

$$\sum_{k=x}^{y} \frac{1}{\alpha k + \beta} = \frac{1}{\alpha} \ln(y/x) + \epsilon_2(x,y), \tag{24}$$

where

$$\frac{1}{\alpha}\left(\frac{1}{y+1} + \epsilon_1(x,y)\right) < \epsilon_2(x,y) \leq \frac{1}{\alpha}\left(\frac{1}{x} + \epsilon_1(x,y)\right). \tag{25}$$

Combining this with (22), the error can be bounded by

$$|\epsilon_2(x,y)| \leq \frac{1}{\alpha x}. \tag{26}$$

Setting $A_n = \epsilon_2(L_n, M_n)$,

$$\sum_{k=L_n}^{M_n} \frac{1}{\alpha k + \beta} = \frac{1}{\alpha} \ln(M_n) - \frac{1}{\alpha} \ln(L_n) + A_n. \tag{27}$$

Finally, note that since $L_n/n \to L > 0$, the sequence $1/(\alpha L_n) \to 0$. Thus $A_n \to 0$, and

$$\lim_{n\to\infty} \sum_{k=L_n}^{M_n} \frac{1}{\alpha k + \beta} = \frac{1}{\alpha} \ln\left(\frac{M}{L}\right).$$

6 Calculation of the Mean

This section is devoted to finding the limit of the mean of $X_{n,a}$. The following theorem is the main result of this paper, and will be proved in sections 6.1 and 6.2.

Theorem 1 *If $a = 0$, then*

$$\lim_{n\to\infty} E[X_{n,a}] = \ln\left(\frac{l^{\lfloor l\rfloor}}{\lfloor l\rfloor!}\right),$$

and if $a = \frac{p}{q}$ with p and q relatively prime, then

$$\lim_{n\to\infty} E[X_{n,a}] = \frac{1}{q}\ln\left(\frac{(ql)^{\lfloor ql\rfloor}}{\lfloor ql\rfloor!}\right).$$

The proof is divided into two parts. The result is proved first for the case when $a = 0$, and then is extended to include any rational a. Although the proof splits naturally in this way, the formula for $a = 0$ actually corresponds to $q = 1$ in the more general case. The case when a is irrational will be looked at in section 6.3.

6.1 The Mean When $a = 0$

In the case where $a = 0$ and $I_n = \left(1,\, e^{2\pi i l/n}\right]$, equations (10) and (11) have a particularly simple form:

$$X_{n,0} = \sum_{k=1}^{n} C_k \left\lfloor \frac{kl}{n} \right\rfloor, \tag{28}$$

and

$$E[X_{n,0}] = \sum_{k=1}^{n} \frac{1}{k} \left\lfloor \frac{kl}{n} \right\rfloor. \tag{29}$$

Observe that $\lfloor kl/n\rfloor$ takes on only integer values, specifically all integers from 0 to $\lfloor l\rfloor$. Thus, it might be useful to group the terms in the sum according to this value. For this purpose, denote the value of $\lfloor kl/n\rfloor$ as j. Now, if $\lfloor kl/n\rfloor = j$, then $j \le \frac{kl}{n} < j+1$, or $j\frac{n}{l} \le k < (j+1)\frac{n}{l}$. The first group of terms, when $j = 0$, does not contribute to the sum. For the last group, when $j = \lfloor l\rfloor$, the upper limit on k is n rather than $(\lfloor l\rfloor + 1)\frac{n}{l}$, so this group will be written separately from the others. Grouping the terms in this way results in

$$E[X_{n,0}] = \sum_{j=1}^{\lfloor l\rfloor -1} \sum_{k=\lceil j\frac{n}{l}\rceil}^{\lceil (j+1)\frac{n}{l}\rceil -1} \frac{j}{k} + \sum_{k=\lceil \lfloor l\rfloor \frac{n}{l}\rceil}^{n} \frac{\lfloor l\rfloor}{k}, \tag{30}$$

where the limits on k are a direct result of the above inequalities and the fact that k must be an integer.

The sums in k are of the form in Lemma 1, with $\alpha = 1$ and $\beta = 0$. In order to find the limit of these sums using the lemma, the following results are needed:

$$\lim_{n\to\infty} \frac{\lceil j\frac{n}{l}\rceil}{n} = \frac{j}{l}; \tag{31}$$

$$\lim_{n\to\infty} \frac{\lceil (j+1)\frac{n}{l}\rceil - 1}{n} = \frac{j+1}{l}; \tag{32}$$

$$\lim_{n\to\infty} \frac{\lceil \lfloor l\rfloor \frac{n}{l}\rceil}{n} = \frac{\lfloor l\rfloor}{l}. \tag{33}$$

These limits are easily attained by noting that, for all real x, $\lceil x\rceil = x + \varepsilon$, where $0 \le \varepsilon < 1$. Applying the lemma then gives

$$\begin{aligned}
\lim_{n\to\infty} E[X_{n,0}] &= \sum_{j=1}^{\lfloor l\rfloor -1} j \ln\left(\frac{j+1}{j}\right) + \lfloor l\rfloor \ln\left(\frac{l}{\lfloor l\rfloor}\right) && (34)\\
&= \ln\left(\prod_{j=1}^{\lfloor l\rfloor -1} \frac{(j+1)^j}{j^j}\right) + \ln\left(\frac{l^{\lfloor l\rfloor}}{\lfloor l\rfloor^{\lfloor l\rfloor}}\right) && (35)\\
&= \ln\left(\frac{\lfloor l\rfloor^{(\lfloor l\rfloor -1)}}{(\lfloor l\rfloor -1)!}\right) + \ln\left(\frac{l^{\lfloor l\rfloor}}{\lfloor l\rfloor^{\lfloor l\rfloor}}\right) && (36)\\
&= \ln\left(\frac{l^{\lfloor l\rfloor}}{\lfloor l\rfloor !}\right). && (37)
\end{aligned}$$

6.2 The Mean When a Is Rational

Now we no longer assume a to be 0 and return to equation (10) for $X_{n,a}$:

$$X_{n,a} = \sum_{k=1}^{n} C_k \left(\left\lfloor ka + \frac{kl}{n}\right\rfloor - \lfloor ka\rfloor\right). \tag{38}$$

A little thought shows that in general,

$$\lfloor x+y\rfloor = \begin{cases} \lfloor x\rfloor + \lfloor y\rfloor & \text{if } \{x\}+\{y\} < 1 \\ \lfloor x\rfloor + \lfloor y\rfloor + 1 & \text{otherwise,} \end{cases} \tag{39}$$

for any real numbers x and y. Applying (39) to the first term in (38) gives

$$\left\lfloor ka + \frac{kl}{n}\right\rfloor = \begin{cases} \lfloor ka\rfloor + \lfloor kl/n\rfloor & \text{if } \{ka\} + \left\{\frac{kl}{n}\right\} < 1 \\ \lfloor ka\rfloor + \lfloor kl/n\rfloor + 1 & \text{otherwise,} \end{cases} \tag{40}$$

and so

$$X_{n,a} = \sum_{k=1}^{n} C_k \left(\left\lfloor ka + \frac{kl}{n} \right\rfloor - \lfloor ka \rfloor \right) \tag{41}$$

$$= \sum_{k=1}^{n} C_k \left\lfloor \frac{kl}{n} \right\rfloor + \sum_{\substack{k:\ \{ka\}+\left\{\frac{kl}{n}\right\}\geq 1,\\ 1\leq k\leq n}} C_k. \tag{42}$$

Thus, the number of eigenvalues $X_{n,a}$ in the interval $\left(e^{2\pi ia},\, e^{2\pi i(a+l/n)}\right]$ equals the number of eigenvalues $X_{n,0}$ in the interval $\left(1,\, e^{2\pi il/n}\right]$, plus $\sum C_k$ for values of k such that $\{ka\} + \left\{\frac{kl}{n}\right\} \geq 1$. Taking the expected value gives

$$E[X_{n,a}] = \sum_{k=1}^{n} \frac{1}{k} \left\lfloor \frac{kl}{n} \right\rfloor + \sum_{\substack{k:\ \{ka\}+\left\{\frac{kl}{n}\right\}\geq 1,\\ 1\leq k\leq n}} \frac{1}{k} \tag{43}$$

$$= \sum_{k=1}^{n} \frac{1}{k} \left\lfloor \frac{kl}{n} \right\rfloor + V_n, \tag{44}$$

where V_n denotes the second sum in (43). Now the problem is to find the limit of V_n, which will require determining the values of k for which $\{ka\} + \left\{\frac{kl}{n}\right\} \geq 1$.

Here, we turn our attention to the case when a is rational. Let $a = p/q$ with p and q relatively prime. Then $\left\{\frac{kp}{q}\right\}$ takes on only a finite number of values, namely all fractions of the form x/q, where x is an integer between 0 and $q-1$ (inclusive). Although the order of the x's depends on p, observe that the sequence $(\{kp/q\}) = (x_k/q)$ repeats with a period of q as k increases. This suggests that it may be helpful to group the terms in V_n according to the value of $\left\{\frac{kp}{q}\right\}$.

Now, for each integer $i = 1, 2, \ldots, q$, let w_i be the number between 0 and $q-1$ such that $\left\{\frac{w_ip}{q}\right\} = 1 - \frac{i}{q}$. Since the sequence $(\{kp/q\})$ repeats, whenever $k \equiv w_i \pmod q$, the value of $\left\{\frac{kp}{q}\right\}$ will be $1 - \frac{i}{q}$. Thus, for such k, the condition $\{ka\} + \left\{\frac{kl}{n}\right\} \geq 1$ becomes $\left\{\frac{kl}{n}\right\} \geq \frac{i}{q}$.

Notice that if $i = q$ (corresponding to $k \equiv 0 \pmod q$), this condition becomes $\left\{\frac{kl}{n}\right\} \geq 1$. Since the fractional part is always less than one, this can never be true, and these terms are not counted in the sum. Now, for each value of i from 1 to $q-1$, it needs to be determined which k satisfy $\left\{\frac{kl}{n}\right\} \geq \frac{i}{q}$. Between each pair of consecutive integers from 0 to $\lceil l \rceil$, there is a (possibly empty) set of terms kl/n that satisfy this condition. In particular, the following inequalities identify the groups of terms that will be counted in the sum:

$$\frac{i}{q} \leq \frac{kl}{n} < 1\,,\ 1 + \frac{i}{q} \leq \frac{kl}{n} < 2\,,\ \ldots\,,\ \lfloor l \rfloor - 1 + \frac{i}{q} \leq \frac{kl}{n} < \lfloor l \rfloor\,,\ \lfloor l \rfloor + \frac{i}{q} \leq \frac{kl}{n} \leq l.$$

Except for the last group, the general limits on k are $(j+\frac{i}{q})\frac{n}{l} \leq k < (j+1)\frac{n}{l}$, where j ranges from 0 to $\lfloor l \rfloor - 1$.

Notice that the last group of terms is only present if $\lfloor l \rfloor + \frac{i}{q} \leq l$, or $\frac{i}{q} \leq \{l\}$. Because of this, the limits on i as well as k are different for the last group. The value of i, instead of ranging over all the integers from 1 to $q-1$, only reaches the largest integer that is less than or equal to $q\{l\}$. That is, $1 \leq i \leq \lfloor q\{l\} \rfloor$, which can be rewritten as $1 \leq i \leq \lfloor ql \rfloor - q\lfloor l \rfloor$.

Using these limits to group the terms in the sum, and keeping in mind that for each i we only count values of k such that $k \equiv w_i \pmod q$, we arrive at the following form for V_n when a is rational:

$$V_n = \sum_{i=1}^{q-1} \sum_{j=0}^{\lfloor l \rfloor - 1} \sum_{\substack{k=\left\lceil \left(j+\frac{i}{q}\right)\frac{n}{l} \right\rceil \\ k \equiv w_i \pmod q}}^{\left\lceil (j+1)\frac{n}{l} \right\rceil - 1} \frac{1}{k} + \sum_{i=1}^{\lfloor ql \rfloor - q\lfloor l \rfloor} \sum_{\substack{k=\left\lceil \left(\lfloor l \rfloor+\frac{i}{q}\right)\frac{n}{l} \right\rceil \\ k \equiv w_i \pmod q}}^{n} \frac{1}{k}. \tag{45}$$

Now, every k for which $k \equiv w_i \pmod q$ can be written as $k = qk' + w_i$, for some integer k'. Making this substitution, the sum becomes

$$V_n = \sum_{i=1}^{q-1} \sum_{j=0}^{\lfloor l \rfloor - 1} \sum_{k'=L_{n,ij}}^{M_{n,ij}} \frac{1}{qk'+w_i} + \sum_{i=1}^{\lfloor ql \rfloor - q\lfloor l \rfloor} \sum_{k'=L'_{n,i}}^{M'_{n,i}} \frac{1}{qk'+w_i}, \tag{46}$$

where

$$L_{n,ij} = \left\lceil \frac{1}{q}\left(\left\lceil \left(j+\frac{i}{q}\right)\frac{n}{l} \right\rceil - w_i \right) \right\rceil, \tag{47}$$

$$M_{n,ij} = \left\lfloor \frac{1}{q}\left(\left\lceil (j+1)\frac{n}{l} \right\rceil - 1 - w_i \right) \right\rfloor, \tag{48}$$

$$L'_{n,i} = \left\lceil \frac{1}{q}\left(\left\lceil \left(\lfloor l \rfloor+\frac{i}{q}\right)\frac{n}{l} \right\rceil - w_i \right) \right\rceil, \tag{49}$$

$$M'_{n,i} = \left\lfloor \frac{1}{q}(n - w_i) \right\rfloor. \tag{50}$$

Here, the sums in k' have the form in Lemma 1, this time with $\alpha = q$ and $\beta = w_i$. The relevant limits in this case are

$$\lim_{n\to\infty} \frac{L_{n,ij}}{n} = \frac{1}{ql}\left(j+\frac{i}{q}\right), \tag{51}$$

$$\lim_{n\to\infty} \frac{M_{n,ij}}{n} = \frac{1}{ql}(j+1), \tag{52}$$

$$\lim_{n\to\infty} \frac{L'_{n,i}}{n} = \frac{1}{ql}\left(\lfloor l \rfloor+\frac{i}{q}\right), \tag{53}$$

$$\lim_{n\to\infty} \frac{M'_{n,i}}{n} = \frac{1}{q}. \tag{54}$$

Notice that the values of L_n and M_n in Lemma 1 must be positive. It is possible that the sequences given in (47) through (50) may start off with values of 0, but since they grow without bound, they must eventually remain positive, and the lemma can still be used to find the limit as $n \to \infty$. Doing so gives

$$\begin{aligned}
\lim_{n\to\infty} V_n &= \sum_{i=1}^{q-1}\sum_{j=0}^{\lfloor l\rfloor -1} \frac{1}{q}\ln\left(\frac{j+1}{j+\frac{i}{q}}\right) + \sum_{i=1}^{\lfloor ql\rfloor - q\lfloor l\rfloor} \frac{1}{q}\ln\left(\frac{l}{\lfloor l\rfloor + \frac{i}{q}}\right) && (55)\\
&= \frac{1}{q}\ln\left(\prod_{i=1}^{q-1}\prod_{j=0}^{\lfloor l\rfloor -1}\frac{q(j+1)}{qj+i}\right) + \frac{1}{q}\ln\left(\prod_{i=1}^{\lfloor ql\rfloor - q\lfloor l\rfloor}\frac{ql}{q\lfloor l\rfloor + i}\right) && (56)\\
&= \frac{1}{q}\ln\left(\frac{(q^{\lfloor l\rfloor}\lfloor l\rfloor !)^{(q-1)}}{\prod_{i=1}^{q-1}\prod_{j=0}^{\lfloor l\rfloor -1}(qj+i)}\right) + \frac{1}{q}\ln\left(\frac{(ql)^{(\lfloor ql\rfloor - q\lfloor l\rfloor)}}{\prod_{i=1}^{\lfloor ql\rfloor - q\lfloor l\rfloor}(q\lfloor l\rfloor + i)}\right) && (57)\\
&= \frac{1}{q}\ln\left(\frac{(q^{\lfloor l\rfloor}\lfloor l\rfloor !)^{q}}{(q\lfloor l\rfloor)!}\right) + \frac{1}{q}\ln\left(\frac{(ql)^{\lfloor ql\rfloor}(q\lfloor l\rfloor)!}{(ql)^{(q\lfloor l\rfloor)}\lfloor ql\rfloor !}\right). && (58)
\end{aligned}$$

The last step follows from expanding the two products in the denominators. This shows that $\prod_{i=1}^{q-1}\prod_{j=0}^{\lfloor l\rfloor -1}(qj+i)$ is just the product of all the integers from 1 to $q\lfloor l\rfloor$, excluding multiples of q. Thus

$$\prod_{i=1}^{q-1}\prod_{j=0}^{\lfloor l\rfloor -1}(qj+i) = \frac{\prod_{i'=1}^{q\lfloor l\rfloor} i'}{\prod_{j'=1}^{\lfloor l\rfloor} qj'} = \frac{(q\lfloor l\rfloor)!}{q^{\lfloor l\rfloor}\lfloor l\rfloor !}. \qquad (59)$$

In addition, $\prod_{i=1}^{\lfloor ql\rfloor - q\lfloor l\rfloor}(q\lfloor l\rfloor + i)$ is the product of all the integers from $q\lfloor l\rfloor + 1$ to $\lfloor ql\rfloor$, which is just $(\lfloor ql\rfloor !)/(q\lfloor l\rfloor)!$. When the arguments of the two logarithms are multiplied, some of the terms cancel out, resulting in

$$\begin{aligned}
\lim_{n\to\infty} V_n &= \frac{1}{q}\ln\left[\left(\frac{\lfloor l\rfloor !}{l^{\lfloor l\rfloor}}\right)^{q}\left(\frac{(ql)^{\lfloor ql\rfloor}}{\lfloor ql\rfloor !}\right)\right] && (60)\\
&= \ln\left(\frac{\lfloor l\rfloor !}{l^{\lfloor l\rfloor}}\right) + \frac{1}{q}\ln\left(\frac{(ql)^{\lfloor ql\rfloor}}{\lfloor ql\rfloor !}\right). && (61)
\end{aligned}$$

Adding equations (37) and (61) yields

$$\lim_{n\to\infty} E\left[X_{n,\frac{p}{q}}\right] = \frac{1}{q}\ln\left(\frac{(ql)^{\lfloor ql\rfloor}}{\lfloor ql\rfloor !}\right). \qquad (62)$$

Equation (62) gives the limit of the mean of $X_{n,a}$ for rational a. Notice that this equation involves q, but not p or the numbers w_i. Thus, only the denominator of a matters when the limit is taken. In general, the limit of $E[X_{n,a}]$ is a function of a and l, and will be referred to as $f(a,l)$ in the following sections.

6.3 The Mean When a is Irrational

When a is irrational, $\{a\}$ takes on an infinite number of values, making it difficult to group the terms in V_n and arrive at a limit for the sum. However, notice that any irrational number can be approximated as closely as desired by a sequence of rationals with increasing denominators. That is, for every irrational number a, there exist sequences of integers (p_m) and (q_m), with (q_m) strictly increasing, such that p_m and q_m are relatively prime for all m and

$$\lim_{m\to\infty} \frac{p_m}{q_m} = a.$$

Thus, we could take the formula for $f\left(\frac{p}{q}, l\right)$ given in Theorem 1 and let $q \to \infty$. The hope is that this might provide an idea of what happens for irrational a.

Figure 1 shows the function $f\left(\frac{p}{q}, l\right)$ for several values of q. Notice that as q gets larger, the curves seem to be approaching the line $f = l$. Indeed, this limit can be shown by using Stirling's approximation for $N!$ when N is large. Stirling's formula is

$$N! \approx N^N e^{-N} \sqrt{2\pi N}. \tag{63}$$

Applying (63) to (62), with $N = \lfloor ql \rfloor$,

$$\begin{aligned}
\lim_{n\to\infty} E\left[X_{n,\frac{p}{q}}\right] &= \frac{1}{q} \ln\left(\frac{(ql)^{\lfloor ql \rfloor}}{\lfloor ql \rfloor!}\right) && (64)\\
&\approx \frac{1}{q} \ln\left(\frac{(ql)^{\lfloor ql \rfloor}}{\lfloor ql \rfloor^{\lfloor ql \rfloor} e^{-\lfloor ql \rfloor} \sqrt{2\pi \lfloor ql \rfloor}}\right) && (65)\\
&= \frac{1}{q} \ln\left(\frac{(ql)}{\lfloor ql \rfloor}\right)^{\lfloor ql \rfloor} + \frac{\lfloor ql \rfloor}{q} - \frac{1}{q} \ln\left(\sqrt{2\pi \lfloor ql \rfloor}\right). && (66)
\end{aligned}$$

Now, taking the limit for large q,

$$\begin{aligned}
\lim_{m\to\infty} \lim_{n\to\infty} E\left[X_{n,\frac{p_m}{q_m}}\right] &= \lim_{q\to\infty} \left[\frac{\lfloor ql \rfloor}{q} \ln\left(\frac{(ql)}{\lfloor ql \rfloor}\right) + \frac{\lfloor ql \rfloor}{q} - \frac{1}{q} \ln\left(\sqrt{2\pi \lfloor ql \rfloor}\right)\right] && (67)\\
&= l \cdot 0 + l - 0 && (68)\\
&= l. && (69)
\end{aligned}$$

This result seems to suggest that the limit of the mean of $X_{n,a}$ for irrational a is simply l.

Remark. Note that this is not a proof. The limit of $E[X_{n,a}]$ will be l for every irrational point a if and only if $f(a, l)$ is continuous whenever a is irrational; however, it is not known

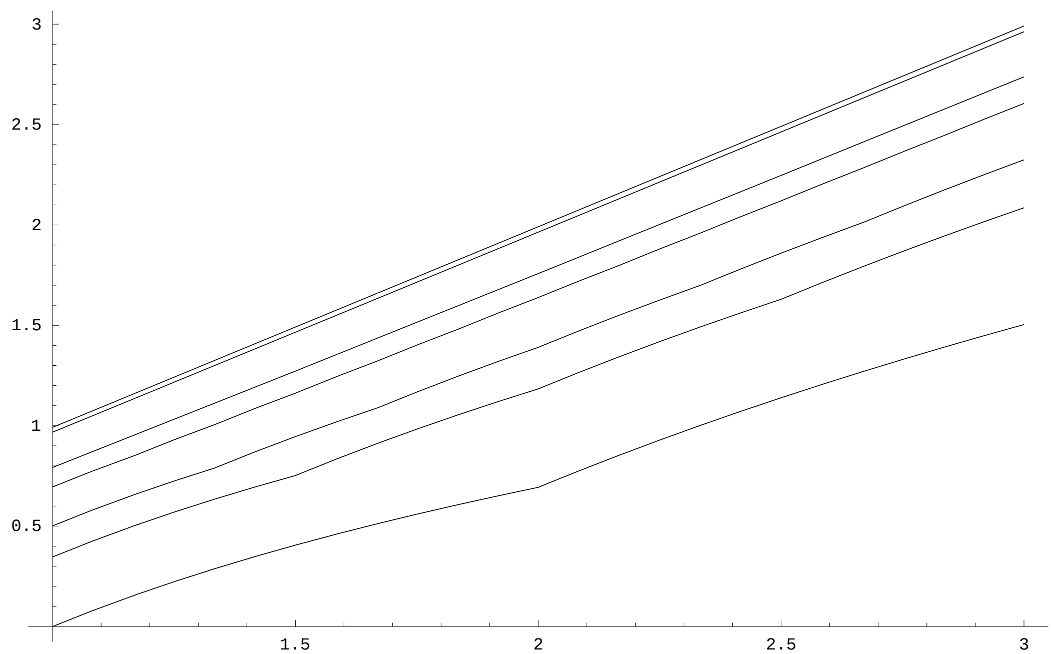

Figure 1: $\lim_{n\to\infty} E[X_{n,a}]$ as a function of l, for various values of q. The curves, from bottom to top are for $q = 1$, $q = 2$, $q = 3$, $q = 6$, $q = 10$, $q = 100$, and $q = 500$.

for certain that this is the case. One way to prove that (69) gives the correct limit would be first to show that $E[X_{n,a}]$ (with a given n) is continuous at an irrational a, so that

$$E[X_{n,a}] = \lim_{m\to\infty} E\left[X_{n,\frac{p_m}{q_m}}\right].$$

Since $f(a,l) = \lim_{n\to\infty} E[X_{n,a}]$, it would then need to be shown that

$$\lim_{n\to\infty}\lim_{m\to\infty} E\left[X_{n,\frac{p_m}{q_m}}\right] = \lim_{m\to\infty}\lim_{n\to\infty} E\left[X_{n,\frac{p_m}{q_m}}\right].$$

It is not difficult to determine where $E[X_{n,a}]$ is continuous when n is fixed, but so far, there is no justification for switching the order of the limits.

There is, however, some evidence to support the result in (69). Computer simulations indicated that $E[X_{n,a}]$ tended toward l when a was irrational, although this may merely reflect the limit in (69) since computers cannot process true irrational numbers. Another indication that this limit might be correct comes from a comparison with some of the results in [4] and [5] involving the number of eigenvalues in a fixed interval. Here, too, the limit depended only on the denominators of the endpoints when they were rational, and in this case, letting the denominators approach infinity did produce the correct result for irrational endpoints under certain circumstances.

7 Conclusion

At this point we can ask how the result in Theorem 1 compares with the distribution of random independent points. Recall that the mean in that case was just l. Figure 2 shows another graph of f vs. l for different values of q, this time plotted over a wider range of l values. Notice that each f appears more and more like a straight line as l increases.

Using Stirling's formula as before, we can obtain an approximation of $f\left(\frac{p}{q}, l\right)$ for large l values. Expanding the first and third terms in (66) as Taylor series reveals that f can be approximated by $l - (\ln l)/2q + O(1)$. Although it is not obvious that f should have this form, the negative sign for the correction term does make sense. Even though the eigenvalues become more evenly distributed as n increases, there are a large number of them at the endpoint $e^{2\pi i p/q}$ that are always excluded from the interval I_n. The correction term may reflect the effect of this exclusion.

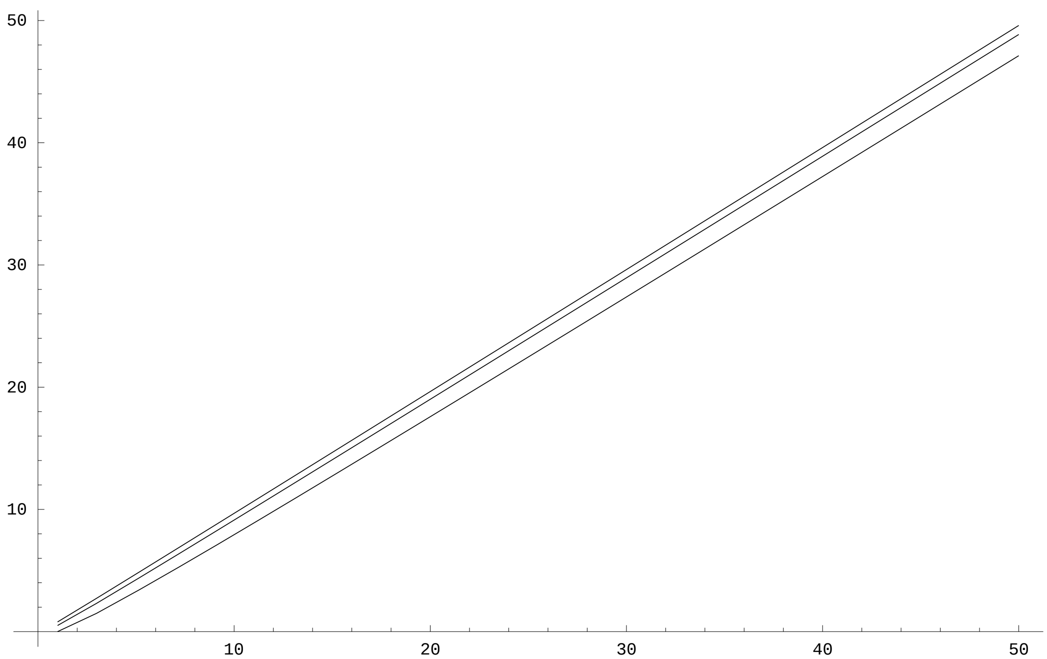

Figure 2: $\lim_{n\to\infty} E[X_{n,a}]$ as a function of l. The curves, from bottom to top are for $q = 1$, $q = 3$, and $q = 10$.

Remark. The results derived in this paper are for half-open intervals, rather than for more standard open or closed intervals. This was done mainly to simplify notation as much as possible. In fact, the limit for an open interval of the form $\left(e^{2\pi ia}, e^{2\pi i(a+l/n)}\right)$ is the same as the limit for the half-open interval studied here. When a is irrational, the limit will be the same for closed intervals as well. However, it is easy to see that for rational a, including the lower endpoint will have a huge effect—the mean no longer would stay bounded because the number of eigenvalues at the endpoint a grows as $(\ln n)/q$.

There are still several questions about $X_{n,a}$ that remain unanswered. One of these is the problem of finding the limit of $E[X_{n,a}]$ for irrational a. Another open question is the variance of $X_{n,a}$. Using the results of section 2.2, a formula can be derived for $Var[X_{n,a}]$ in a way similar to what was done for the mean, although the sum is more complicated. Some work with this sum has shown that the variance is bounded for $a = 0$, and we suspect that it will be bounded for other values of a as well. It would be interesting to try to derive a formula for the limit of the variance, and more generally, to be able to describe the distribution of $X_{n,a}$.

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