Solvable Quintics : A New Approach

by

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INTRODUCTION

This project has been an attempt to advance recent work in the theory of equations. One paper in particular, by Kalman and White, discusses a technique for solving equations up to the fourth degree; see [1]. Making use of circulant matrices, the technique offers simplified, easy-to-remember solutions to the cubic and quartic. Circulant matrices are defined in [1]; here is one example:

$$C = \begin{bmatrix} a & b & c & d \\ d & a & b & c \\ c & d & a & b \\ b & c & d & a \end{bmatrix}$$

It was my hope that, since this method helps unify solutions to low-degree polynomial equations (of degree 1,2,3,4), it might be helpful in approaches to fifth-degree equations.

The problem of finding solutions to quintics is two-fold: first, one must determine for which forms of the quintic solutions can be found; second, one must actually solve for the roots of that form. In seeking to narrow the scope of this project, I have chosen to focus on those quintics of the form:

(1) $$p(x) = x^5 + ax + b = 0$$

that can be solved by radicals, which have been characterized in a paper by Spearman and Williams [2]. The conditions attained in [2] are, unfortunately, complicated and difficult to work with. In the present work I have tried to use the method of circulant matrices to derive a systematic method of determining conditions on (1) that permit a solution by radicals, and subsequently to find that solution. This approach leads to a set of non-linear equations that must be solved simultaneously. The difficulties in solving this system have prevented the establishment of criteria characterizing those quintics that can be solved by radicals; rather, the following gives an approach that can, at least in principle, be used to find those roots. There is, of course, no way to solve the general quintic, but the circulant matrix method does give new insights into this very difficult problem.

THE GENERAL METHOD

In solving the quadratic, cubic, and quartic by circulant matrices, it is necessary to begin with the general form (see [1]), and as such we begin with the general quintic:

(2) $$p(x) = x^5 + \beta x^3 + \gamma x^2 + \delta x + \varepsilon$$

Solving for the roots of a quartic requires finding the characteristic polynomial of the circulant matrix:

$$C = \begin{bmatrix} 0 & b & c & d \\ d & 0 & b & c \\ c & d & 0 & b \\ b & c & d & 0 \end{bmatrix}$$

Continuing with this method for the quartic, it is necessary to find where the characteristic polynomial of the matrix *C* is equal to the general quartic. By doing so one arrives at a system of equations relating b,c,d to the coefficients of the general quartic; for a solution to this system, see[1]. If one wishes to solve for any specific quartic, the values for b,c,d can be easily determined from the coefficients of that quartic. Kalman and White demonstrate that the eigenvalues of the matrix *C* are then equivalent to the roots of the quartic. Eigenvalues are easily found for circulant matrices; they are obtained for an *n x n* matrix by applying the *n*th roots of unity to the polynomial *q,* where *q* is read from the first row of *C*:

$$q(t) = 0 + bt + ct^2 + dt^3$$

The roots of any quartic then, are *q(1), q(-1), q(i),* and *q(-i)*.

Thus, if the circulant matrix approach is to work for the quintic, we must first find the characteristic polynomial of the circulant matrix:

$$C = \begin{bmatrix} 0 & b & c & d & e \\ e & 0 & b & c & d \\ d & e & 0 & b & c \\ c & d & e & 0 & b \\ b & c & d & e & 0 \end{bmatrix}$$

This characteristic polynomial equals det($xI - C$), or:

$$\det \begin{bmatrix} x & -b & -c & -d & -e \\ -e & x & -b & -c & -d \\ -d & -e & x & -b & -c \\ -c & -d & -e & x & -b \\ -b & -c & -d & -e & x \end{bmatrix}$$

This determinant is:

$$\begin{aligned} &x^5 + (-5be - 5cd)x^3 + (-5bc^2 - 5b^2d - 5ce^2 - 5d^2e)x^2 \\ &+ (-5b^3c + 5b^2e^2 - 5bcde - 5bd^3 - 5c^3e + 5c^2d^2 - 5de^3)x \\ &- (b^5 + c^5 + d^5 + e^5) + 5b^3de - 5b^2c^2e - 5b^2cd^2 + 5bc^3d \\ &+ 5bce^3 - 5bd^2e^2 - 5c^2de^2 + 5cd^3e \end{aligned} \tag{3}$$

What we are ultimately looking for here is a circulant matrix having (2) for its characteristic polynomial, so we need to find where (2) equals (3). Set (2) equal to (3), and a system of equations arises:

$$\begin{aligned} &be + cd = -\beta/5 \\ &bc^2 + b^2d + ce^2 + d^2e = -\gamma/5 \\ &b^3c + bcde + bd^3 + c^3e + de^3 - b^2e^2 - c^2d^2 = -\delta/5 \\ &-(b^5 + c^5 + d^5 + e^5) + 5b^3de - 5b^2c^2e - 5b^2cd^2 + 5bc^3d \\ &+ 5bce^3 - 5bd^2e^2 - 5c^2de^2 + 5cd^3e = \varepsilon \end{aligned} \tag{4}$$

One simplification can be made by squaring the first of these equations and substituting into the third

$$b^2e^2 + c^2d^2 + 2bcde = \beta^2/25;$$

another can be made by substituting the product of the first two equations into the fourth

$$(be + cd) \cdot (bc^2 + b^2d + ce^2 + d^2e) = \beta\gamma/25,$$

leading to a slightly simpler system of equations:

$$
\begin{aligned}
&be+cd=-\beta/5 \\
&bc^2+b^2d+ce^2+d^2e=-\gamma/5 \\
&b^3c+bd^3+c^3e+de^3+3bcde=-\delta/5+\beta^2/25 \\
&-(b^5+c^5+d^5+e^5)+10(b^3de+bc^3d+bce^3+cd^3e)-\beta\gamma/5=\varepsilon
\end{aligned} \tag{5}
$$

This system should not prove solvable for all *β,γ,δ,ε,* as not all quintics are solvable by radicals. We know that some are; let us move on to a narrower example.

QUINTICS OF THE FORM P(x)=x⁵+ax+b

For this paper it is more suitable to consider:

(6) $$p(x)=x^5+\delta x+\varepsilon$$

and so we will write it that way. Though we do not yet have values for δ and ε, we know that $\beta=\gamma=0$, leading to yet another system:

$$
\begin{aligned}
&be+cd=0 \\
&bc^2+b^2d+ce^2+d^2e=0 \\
&b^3c+bd^3+c^3e+de^3+3bcde=-\delta/5 \\
&-(b^5+c^5+d^5+e^5)+10(b^3de+bc^3d+bce^3+cd^3e)=\varepsilon
\end{aligned} \tag{7}
$$

It would now be helpful if we could solve for some of the variables in terms of the others. Looking at (7), we see that $b=-cd/e$, and substituting for b in the second equation:

(8) $$\left(\frac{-d}{e}\right)c^3+\left(\frac{d^3}{e^2}\right)c^2+(e^2)c+d^2e=0$$

which is a cubic in c, and in principle solvable by various methods. Using MAPLE to solve for c, we see that (8) has the roots which are functions of *d* and *e*, though they are all lengthy ones. At least one of these is a real function of *d* and *e*, but they are all to complicated to be easily worked with – some simplification must be found to permit further analysis. We can say that *c* is expressable as some real function of *d* and *e*; that is, $c=f(d,e)$. Using this, simplify the last two equations in (7) as:

(9)

$$[f(d,e)]^4(\frac{-d^3}{e^2})+[f(d,e)]^3(e)+[f(d,e)]^2(-3d^2)$$
$$+f(d,e)(\frac{-d^4}{e})+de^3=-\delta/5$$
$$-\left[[f(d,e)]^5(1-\frac{d^5}{e^5})+d^5+e^5\right]$$
$$+10\left[[f(d,e)]^4(\frac{-d^2}{e})+[f(d,e)]^3(\frac{-d^4}{e^2})+[f(d,e)]^2(-de^2)+[f(d,e)](d^3e)\right]=\varepsilon$$

Since we know that for some δ and ε there can be found the roots of (6), then for those δ and ε there exist some d and e that satisfy both equations in (9). The complexity of (9), unfortunately, causes us to fall short of the goal of a simple method for determining solvability. It seems clear, however, that such criteria could be found, as the following examples help show.

EXAMPLE 1 : P(x) = (x-1)^5

We now return to the general quintic and select one that has all real roots. For simplicity, let us consider $p(x) = (x-1)^5$, whose roots are all equal to unity. The circulant matrix approach works for equations of degree four and lower because the eigenvalues of C are equal to the roots of p. This must hold true for fifth-degree equations if the method is to work. The eigenvalues of C are obtained by applying the fifth roots of unity to the poynomial q, read from the first row of C:

(10) $$q(t)=0+bt+ct^2+dt^3+et^4$$

As all the roots are 1, so must all the eigenvalues be; thus:

(11)
$$q(1)=b+c+d+e=1$$
$$q(w)=bw+cw^2+dw^3+ew^4=1$$
$$q(w^2)=bw^2+cw^4+dw^6+ew^8=bw^2+cw^4+dw+ew^3=1$$
$$q(w^3)=bw^3+cw^6+dw^9+ew^{12}=bw^3+cw+dw^4+ew^2=1$$
$$q(w^4)=bw^4+cw^8+dw^{12}+ew^{16}=bw^4+cw^3+dw^2+ew=1$$

But the sum of the last four equations in (11) can be rewritten as:

(12) $$(b+c+d+e)(w+w^2+w^3+w^4)=4$$

And since $1+w+w^2+w^3+w^4=0$, (12) becomes:

(13) $$b+c+d+e=-4$$

which contradicts the first equation in (11) since it cannot be true that $b+c+d+e = -4 = 1$. We see from this example that *b, c, d,* and *e* may, in some cases, never result in the conditions necessary to produce the roots of an equation, even when that equation does have known roots.

EXAMPLE 2 : P(x) = (x-1)^4(x+4)

As a counterexample, we demonstrate a case that does work. All that changes from Example 1 is that in (11) we replace 1 with –4. Now the system is in agreement that $b+c+d+e = -4$, and we can proceed. There are many *b, c, d, e* for which their sum equals –4; let us choose the simple case where
$b = c = d = e = -1$. Inserting those values into (4) to solve for $\beta, \gamma, \delta, \varepsilon$, we get $\beta = -10$, $\gamma = 20$, $\delta = -15$,
$\varepsilon = 4$, which are indeed the coefficients of $p(x) = (x-1)^4(x+4)$ when expanded. So we know that if there were some way to arrive at $b = c = d = e = -1$ from the coefficients of *p(x)*, then we could use the circulant matrix method to find the roots of *p(x)*. The missing element here is a solution, or rather, parameters for solvability, to the systems that relate *b, c, d, e* to $\beta, \gamma, \delta, \varepsilon$.

CONCLUSIONS

The realm of fifth-degree polynomials is not a simple one, and we should not expect any easy answers. Though this project did not lead to any definitive conclusions about the applicability of circulant matrices to quintics, I believe it demonstrates the potential for such applications. What has been accomplished is to show that the same steps can be followed, initially, for using circulant matrices to solve quintics as are followed for solving lower-degree polynomials. A few simplifications have been found to the system of equations (4) for one special form of the quintic; perhaps better, more workable systems would result from applying other forms of the quintic. Example 2 shows that by working from the middle of the equation outward, so to speak, a connection can be made between a quintic with known roots and its roots using the circulant matrix method, and though this connection is not evident as a generality of any sort, further investigation could lead to the discovery of classes of quintics for which this method works.