Static solutions of an elastic rod in a helical shape without twist

Benjamin Armbruster*
Mentor: Alain Goriely†
Undergraduate Research Assistantship
Department of Mathematics, University of Arizona

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Abstract

Motivated by observations about twining plants we study the static equilibrium solutions of untwisted, elastic rods in a helical shape. This work is an extension and specialization of [2]. We extend it by considering rods with intrinsic curvature and we specialize it by only considering helical solutions. Biology motivates these considerations.

1 Introduction

Only in the last ten years has there been serious work to model and understand the behavior of twining plants. This modeling is done on one of two levels: the kinematic (concerned with describing what happens) and the physical (concerned with explaining why). Though, in practice this is often a matter of degree. A good example of the former type of work is [3]. In [3], Silk shows that the helical shape of climbing plants is produce by differing growth rates across a cross section of the stem. Nevertheless, many phenomena including some described by Darwin ([1]) have not been fully understood.

We motivate our investigation by one phenomenon in particular. In [1], Darwin observed that

> The view just given further explains, as I believe, a fact observed by Mohl (p. 135), namely, that a revolving shoot, though it will twine round an object as thin as a thread, cannot do so round a thick support. I placed some long revolving shoots of a Wistaria close to a post between 5 and 6 inches in diameter, but, though aided by me in many ways, they could not wind round it. This apparently was due to the flexure of the shoot, whilst winding round an object so gently curved as this post, not being sufficient to hold the shoot to its place when the growing surface crept round to the opposite surface of the shoot; so that it was withdrawn at each revolution from its support.

Our aim is to see whether a physical model of a plant exhibits this phenomenon. It is natural to model a climbing plant as an elastic rod constrained to lie on a cylinder (the pole). To be specific, our goal is to determine whether such an elastic rod fails to have helical equilibrium solutions when the curvature of the pole is much less than the intrinsic curvature of the rod and whether this is accompanied by a normal force of the pole which is too small.

Our model considers the (constant) intrinsic curvature of the vine (the natural tendency of the plant to curl even without a support) and twisting. We ignore growth and the fact that a particular part of the stem loses its elasticity with time. These may be critical flaws of our model as plant growth is clearly a dynamic phenomenon. Less problematically, we assume the pole is frictionless (in [1] Darwin made experiments with smooth poles) and anisotropies.

*email: barmbrus@email.arizona.edu
†email: goriely@math.arizona.edu

This problem is at once an extension and specialization of the problem addressed in Heijden's paper, [2], on the equilibrium solutions of elastic rods constrained to a cylinder. We extend Heijden's work by considering rods with intrinsic curvature and specialize it by only considering helical solutions. Mathematically however, we work directly with the local director frame of the rod and ignore the cylindrical coordinates Heijden introduces (as they only complicate matters). The next section of this report will present our model. The third section will derive the twistless solutions. The conclusion of the report begins by describing how to derive the general helical solutions with twist (without actually deriving them as they are not pretty). That section ends with a discussion of the results and plans for further research.

2 Model

The model specifies the geometry, that is the mathematical representation of the rod and its helical shape, and the physics. Starting with the geometry, we parameterize the rod by arclength, s. For simplicity we also assume the rod has an infinitesimal cross-section and hence can be described by its centerline, $\mathbf{r}(s)$.

2.1 Director frame

Since the physics generally works with local properties of the rod, the director frame (a generalized Frenet frame) is convenient to work in. This is a right-handed orthonormal set of vectors, $\{\mathbf{d}_1, \mathbf{d}_2, \mathbf{d}_3\}$, which changes with s. The tangent vector is $\mathbf{d}_3$, that is $\mathbf{r}' = \mathbf{d}_3$, where $'$ denotes differentiation with respect to arclength. Finally, $\mathbf{d}_1$ and $\mathbf{d}_2$ are chosen along the principles axes of inertia in the normal cross section of the rod in order to simplify the constitutive relation discussed later. Along with the director frame come the director frame equations:

$$\mathbf{d}_i' = \mathbf{u} \times \mathbf{d}_i \qquad i = 1, 2, 3. \tag{1}$$

The vector $\mathbf{u}$ is analogous to the curvature and torsion in the Frenet equations. These equations, (1), are used to differentiate vectors described in the director frame. Consider for example $\mathbf{a}(s)$. We may denote vectors throughout the document by the tuple of their components (denoted by subscripts) in the director frame, e.g. (a_1, a_2, a_3), or (u_1, u_2, u_3). It is simple to derive the tuple for $\mathbf{a}'$:

$$\begin{aligned} \mathbf{a}' &= (a_1\mathbf{d}_1)' + (a_2\mathbf{d}_2)' + (a_3\mathbf{d}_3)' && (2)\\ &= a_1'\mathbf{d}_1 + a_1\mathbf{u} \times \mathbf{d}_1 + a_2'\mathbf{d}_2 + a_2\mathbf{u} \times \mathbf{d}_2 + \cdots && (3)\\ &= (a_1', a_2', a_3') + (u_1, u_2, u_3) \times (a_1, a_2, a_3). && (4) \end{aligned}$$

2.2 Helix

A helix is characterized by constant curvature and torsion. Using the Frenet equations one can relate (u_1, u_2, u_3) to curvature, torsion, and the Frenet vectors. So starting with the fact that $\mathbf{d}_3$ is the tangent vector and the Frenet equations, one can derive

$$\textbf{normal} = \frac{(u_2, -u_1, 0)}{\sqrt{u_2^2 + u_1^2}} \tag{5}$$

$$\textbf{binormal} = \frac{(u_1, u_2, 0)}{\sqrt{u_2^2 + u_1^2}} \tag{6}$$

It turns out that the normal vector to a helix is also normal to the enclosed cylinder. Further one can show that a twisted helix (or helical rod in our case) can be described by

$$(u_1, u_2, u_3) = (\kappa \sin\phi, \kappa\cos\phi, \tau + \phi') \tag{7}$$

where κ and τ are the curvature and torsion respectively of the helix and $\phi(s)$ denotes the intrinsic twist. In that case $\phi = 0$ is defined as the case when $\mathbf{d}_2$ lies in the plane tangent to the cylinder. The angle ϕ specifies how far $\mathbf{d}_1$ and $\mathbf{d}_2$ are rotated around $\mathbf{d}_3$. Another result is that a rod with curvature κ is constrained to a cylinder (pole) of radius κ^{-1}.

The rod may have some intrinsic curvature, a particular relaxed shape. We assume for mathematical simplicity that this is an untwisted helix. This is realistic biologically as Darwin, [1], observed that twining plants form a helix even in the absence of a support (such as a pole). Hence the rod's intrinsic curvature can be described by a vector $\mathbf{u}^o$ whose components are are constant (along s) in the director frame.

2.3 Elasticity and balance of forces

We start the discussion of the rod's physical properties by assuming that it is weightless. Even though we assume it is weightless, we cannot neglect the internal forces and moments (torques) in the rod; if we did, then it wouldn't be elastic. Hence let $\mathbf{n}$ be the internal force and $\mathbf{m}$ the internal moment (both generally vary along the rod). Now the elastic property of the rod is described by a constitutive relation analogous to the one-dimensional Hooke's law:

$$\mathbf{B}(\mathbf{u} - \mathbf{u}^o) = \mathbf{m} \tag{8}$$

where $\mathbf{B}$ is the bending stiffness tensor. Due to our choice of $\mathbf{d}_1$ and $\mathbf{d}_2$, $\mathbf{B}$ is diagonal. It is natural to assume the rod is isotropic and hence its bending stiffness is the same in all the directions in the cross section of the rod. So, we let $\mathbf{B} = diag(B, B, C)$ in the director frame (though no complications arise if the rod is anisotropic).

2.4 Balance of forces

To complete the physics we have to balance the forces. To make the internal forces balance we relate $\mathbf{n}$ and $\mathbf{m}$,

$$\mathbf{m}' + \mathbf{r}' \times \mathbf{n} = \mathbf{0}. \tag{9}$$

To make the external forces balance,

$$\mathbf{n}' = \mathbf{f} \tag{10}$$

where $\mathbf{f}$ is the external force applied on the rod. Since we neglect forces such as static friction, $\mathbf{f}$ is normal to the cylinder and hence to the helix.

3 Solutions without twist

We will now solve the equations developed in the preceding section in the special case of a rod without twist. Although we developed the model for the general case with twist, solving the equations for the case without twist is both realistic (since that is what Darwin observed for smooth poles in [1]) and instructive of the steps used to solve the general case.

By solving the equations we mean finding the external force on the helix, $\mathbf{f}$, and all the other forces given the properties of the helix, $\mathbf{B}$ and $\mathbf{u}^o$, and the shape we are forcing it into, $\mathbf{u}$. For a rod without twist, ϕ is constant and hence (u_1, u_2, u_3) is too. Since the components of $\mathbf{u}$ and $\mathbf{u}^o$ are constant, the components of m are also constant by the constitutive relation, (8). Hence the moment balance equation, (9), becomes

$$\begin{aligned}(m_1, m_2, m_3) \cdot (\mathbf{d}_1, \mathbf{d}_2, \mathbf{d}_3)' &= \\ -\mathbf{d}_3 \times (\mathbf{d}_1 n_1 + \mathbf{d}_2 n_2 + \mathbf{d}_3 n_3)&\end{aligned} \tag{11}$$

Substituting the director frame equations, (1), we then have component-wise,

$$-u_3 m_2 + u_2 m_3 = n_2 \tag{12}$$
$$-u_3 m_1 + u_1 m_3 = n_1 \tag{13}$$
$$-u_2 m_1 + u_1 m_2 = 0. \tag{14}$$

Now, since the components of $\mathbf{u}$ and $\mathbf{m}$ are constant, n_1 and n_2 are also constant. The last equation, (14), is interesting. After expanding the components of $\mathbf{m}$ using the constitutive relation, (8), we can simplify it to $u_2/u_1 = u_2^o/u_1^o$. Hence the curvature of the rod must be in the same direction as its intrinsic curvature, i.e. $\phi = \phi^o$, in order to remain without twist. (Note, the analogous equation for the anisotropic rod does not have such a simple interpretation.)

Now we will calculate the external applied force. Applying the product rule to the equation balancing the external forces, (10),

$$(n_1, n_2, n_3)' \cdot (\mathbf{d}_1, \mathbf{d}_2, \mathbf{d}_3) + (n_1, n_2, n_3) \cdot (\mathbf{d}_1, \mathbf{d}_2, \mathbf{d}_3)' = \mathbf{f}. \tag{15}$$

Collecting terms after we substitute the director frame equations, (1), (remembering that n_1 and n_2 are constant)

$$-u_3 n_2 + u_2 n_3 = f_1 \tag{16}$$
$$u_3 n_1 - u_1 n_3 = f_2 \tag{17}$$
$$n_3' - u_2 n_1 + u_1 n_2 = f_3 \tag{18}$$

Since we neglect friction, $\mathbf{f}$ is normal to the surface of the pole. This means $\mathbf{f}$ is 0 in the tangent and binormal directions. Hence $f_3 = 0$ and $u_1 f_1 + u_2 f_2 =$

0. Substituting (13) and (14) into the constraint $f_3 = 0$ results in $n_3' = 0$. Hence n_3, the tension, is equal to the tension applied at the ends. Expanding the other constraint tells us nothing new (it is equivalent to (14)).

Hence expanding the applied normal force and simplifying,

$$\begin{aligned}\mathbf{f} \cdot \mathbf{normal} &= \frac{u_2 f_1 - u_1 f_2}{\sqrt{u_1^2 + u_2^2}} \\ &= \kappa n_3 + \tau \left[\tau B_1(\kappa - \kappa^o) - \kappa C(\tau - \tau^o)\right].\end{aligned} \tag{19}$$

Note that any change in the tension is proportional to a change in the pole's normal force. In addition to the preceding equation for the external force, the main results of this section are that the tension inside the rod is equal to the applied tension and that to remain twistless, the curvature of the helix must be in the same direction as the rod's intrinsic curvature.

4 Conclusion

With similar steps the general case of a twisted rod can be solved. The first two components of the equation balancing the internal forces, (9), are used to find n_1 and n_2. Then substituting into the external force balance equation, (10), we can find the components of $\mathbf{f}$. We are then left with two unknowns $n_3(s)$ and $\phi(s)$. We have not used the third component of the internal force balance equation, (9), and the fact that $\mathbf{f}$ is 0 in the tangent and binormal directions. Complicating matters is the fact that these are not simple equations but differential equations. The idea is to isolate ϕ'' in the third component of (9). Then one expands the equation $\mathbf{binormal} \cdot \mathbf{f} = 0$ with the equations found earlier for (f_1, f_2, f_3) and ϕ''. This then turns out to be a first order differential equation in $\phi(s)$. While not pretty this can be solved analytically. Now expanding the last constraint, $f_3 = 0$, with the equations for f_3 and ϕ we obtain a first order differential equation in $n_3(s)$. Finally, the solution to this ODE can be substituted in $\mathbf{f} \cdot \mathbf{normal}$ to find the applied external force.

The calculated applied external force, (19), of the twistless case does not confirm Darwin's observations as solutions exist for poles of any size. In particular, if there is no applied tension, then there will be no applied external force for any κ and τ.

However, this need not contradict Darwin's observations. While we showed that equilibrium solutions exist for poles with large radii, these solutions may be unstable (and hence not observed in practice). However, our static model cannot address questions of stability. To determine the stability of equilibrium solutions we must either introduce a dynamic model or an energy functional. Using an energy functional seems simpler since we won't need to change the model (e.g. by turning it into a PDE). A naive energy functional would be

$$\int \|\mathbf{u} - \mathbf{u}^o\|^2 ds. \tag{20}$$

With such a functional and the calculus of variations one could classify equilibrium solutions as stable or unstable depending on whether they correspond to local energy minima or energy maxima or saddles.

Since the applied tension was exogenous to our model, an open question is where exactly this tension comes from. Further research could also use the same model to explore other questions about twining plants. For example, one could look at the solutions to the general twisted rod for any asymptotic behavior which can be examined experimentally. It would also be interesting to see if the general solution exhibits any preference for solutions with little twist and whether this is related to the presence of static friction. As inspiration for further research, we quote Darwin's observations[1] on twist:

> Mohl has remarked (p. 111) that when a stem twines round a smooth cylindrical stick, it does not become twisted. {6} Accordingly I allowed kidney-beans to run up stretched string, and up smooth rods of iron and glass, one-third of an inch in diameter, and they became twisted only in that degree which follows as a mechanical necessity from the spiral winding. The stems, on the other hand, which had ascended ordinary rough sticks were all more or less and generally much twisted. The influence of the

roughness of the support in causing axial twisting was well seen in the stems which had twined up the glass rods; ... there must be some connexion between the capacity for twining and axial twisting. The stem probably gains rigidity by being twisted (on the same principle that a much twisted rope is stiffer than a slackly twisted one), and is thus indirectly benefited so as to be enabled to pass over inequalities in its spiral ascent, and to carry its own weight when allowed to revolve freely. {8}

I have alluded to the twisting which necessarily follows on mechanical principles from the spiral ascent of a stem, namely, one twist for each spire completed. This was well shown by painting straight lines on living stems, and then allowing them to twine

References

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