A Connected Sum Operation, Maximally Connected Planar Graphs and Colorability

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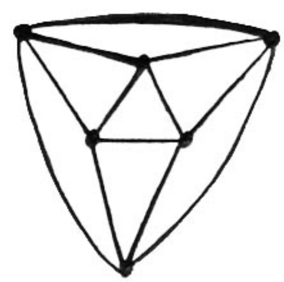
Introduction

The motivation for this project was the pursuit of a simpler strategy of proof for the Four Color Theorem. All current proofs require extensive use of computer algorithms to verify hundreds of cases[1]. The Four Color Theorem states that all planar maps can be colored minimally with at most 4 colors. Planar graphs are graphs that can be drawn on a plane such that no edges cross. The proof of the Four Color Theorem is equivalent to the proof that all planar graphs are 4-vertex colorable[2]. That is to say that for all planar graphs the vertices can be colored in such a way that any vertices sharing an edge (any *adjacent* vertices) are different colors. This led us to the research of the characterization and colorability of planar graphs.

Any planar graph can be found as a subgraph of some simple planar graph, say G, with the same number of vertices but with the maximum number of edges. A simple graph is one that does not contain any edges from a vertex to itself (loops) or any double edges. In such a maximally connected planar graph it follows from the definition that every face is a triangle. That is to say that in any face of a simple graph that is not a triangle there are two vertices not adjacent that could be connected while holding planarity. These simple graphs with the maximum number of edges we call Maximally Connected Planar Graphs (MCPG's).

In our research we take a look specifically at the set of all MCPG's and attempt to prove that they are 4-colorable. Consequently proving that any subgraph of these graphs is 4-colorable since removing edges cannot increase the number of colors required.

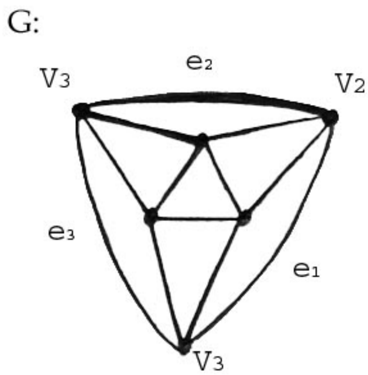
planar graph

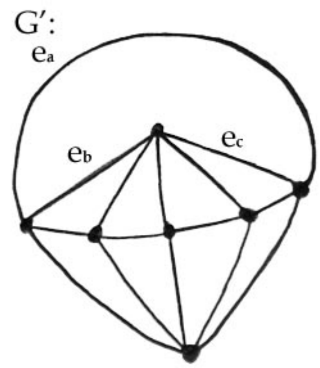


Maximally Connected Planar Graph

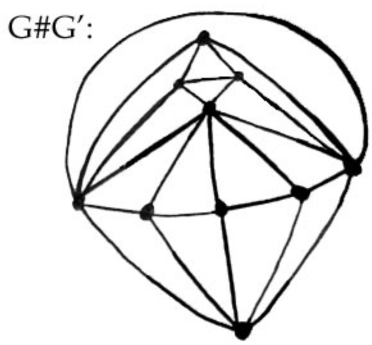
Defining a Connected Sum Operation for MCPG's

A strategy we have employed involves the definition of independent graphs and an operation similar to connected sums of surfaces. The connected sum operation we use is defined in general as the assignment of the vertices of a triangle, T, in one MCPG (G) to the vertices of a triangle, T', in another MCPG (G'). When G is connect summed to G' all vertices of G- T are placed inside the triangle T' and the vertices of T are identified to T'. We denote G connect summed to G' as G#G'. This operation preserves maximal-connectedness.





Let the outer face of G be the traingle T bounded by e_1, e_2, e_3. Then assign v_1, v_2, v_3 to the triangle T' in G' bounded by e_a, e_b, e_c. The following is the connected sum:

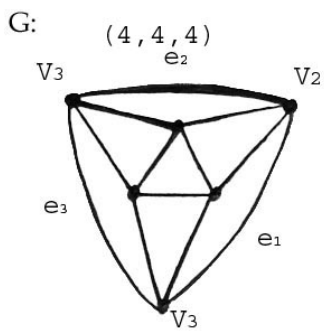


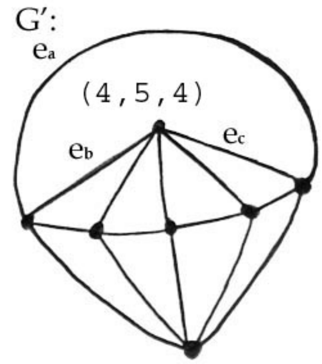
Face value and Existence and Uniqueness for Connected Sum Operation

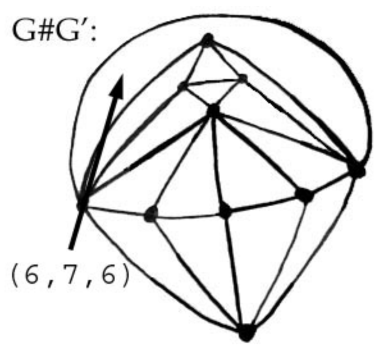
Furthermore to well define the operation there must be a better understanding of existence and uniqueness of face values. Face values can be defined as a 3-tuple (d_1, d_2, d_3) where the degrees of the value corresponds to the three vertex degrees of a given face.

When performing a connected sum operation the face values of T, say (d_1, d_2, d_3), are added to that of T' (d_1', d_2', d_3') such that the each edge on the faces border are counted twice. In this way we can say that the vertices corresponding to T' in G' are now of degree $d_1'+d_1-2$, $d_2'+d_2-2$, and $d_3'+d_3-2$.

Thus when dealing with graphs containing varying face values it becomes necessary to take into account the face value of the two faces being connected since the vertex listing of G#G' is dependent on this relationship. That is to say, there is different degree listings for each distinct pairing of 3-tuples.







Independent Graphs and Their Connected Sum's Effect on Colorability

We define an independent graph as one that is not constructible through any sequence of our connected sum operations. In other words, a MCPG is independent if there does not exist a 3-cycle with vertices on both sides of the 3-cycle. The following theorem is a direct application of this connected sum operation to coloring.

Theorem: The operation of connected sum of any MCPG (G) to another (G’) does not change the coloring of G’ and it follows that if G is n-colorable and G’ is m-colorable then G#G’ is max{m,n}-colorable.

Proof:

Suppose G is n-colorable and that T has vertex coloring a_1, a_2, a_3 and G uses colors $\{a_1, a_2, a_3, ..., a_n\}$.

also suppose that G’ is m-colorable and T' has vertex colorings b_1, b_2, and b_3 and that G' uses colors $\{b_1, b_2, b_3, ..., b_n\}$.

now connect G to G' and set $a_1 = b_1$, $a_2 = b_2$, and $a_3 = b_3$

so now G-T and G' have the same coloring as before.

if n<m set $a_4 = b_4$, $a_5 = b_5$, $a_6 = b_6$, ...$a_n = b_n$ and you can leave b_{n+1},...,m the same coloring making G#G' m-colorable.

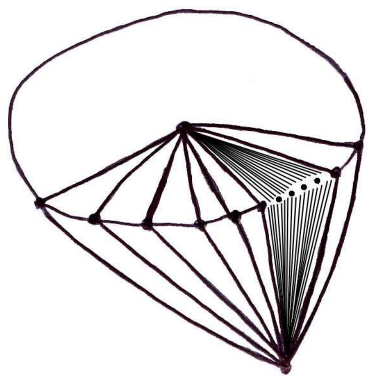
if n>m set $a_4=b_4$,......, $a_m=b_m$ and leave a_{m+1},......, a_n the same color making G#G' n-colorable.

and if n=m the same process gives you n-colorability.

QED

Using the connected sum operation and the above theorem it follows that if an independent graph is 4-colorable then any graph formed by a sequence of connected sums of these graphs will be 4-colorable. Furthermore if you can define a family of graphs that is systematically 4-colorable then that family and all graphs formed by using our operation with these graphs is 4-colorable. Here is such a family defined as the family of *Double Wheels*.

DEFINITION A *Double Wheel* is an n-wheel with a vertex on the outside face adjacent to every vertex on cycle of the wheel.



Any Double Wheel can be colored by alternating colors on the cycle(with 3 colors in an odd cycle and 2 in an even cycle) and then color the two remaining vertices the same color since they are on either side of the closed cycle. As you can tell the face values are the same and therefore create a well defined arena for which to generalize coloring of connected sums. Since these graphs are 4 colorable there follows a simple proof for the 4-colorability of an infinite family using the above theorem for colorability of connected sums..

Derivation of MCPG Degree Listings from the Euler Formula and an Algorithm Attempt

Another goal in pursuit of a better characterization of MCPG's was to find an algorithm for finding degree listings for simple MCPG's. For this we used a derivation originating with Euler's Formula which states; if G is a plane graph then V-E+F=2[where v is vertices, e is edges, and f is faces]. So from this we made the following derivation:

(Eq. 1) Euler's Formula[1] V-E+F=2

First of all, we found the number of edges in an MCDG:

(Eq. 2) $E = 3V-6$

The number of vertices in a graph, V, is given by

(Eq. 3) $V = X_3 + X_4 + X_5 + \ldots + X_k$

where X_k equals the number of vertices of degree k. Since each edge shares a pair of vertices the sum of the vertex degrees is twice the number of edges in G.

(Eq. 4) $2E = 3X_3 + 4X_4 + 5X_5 + \ldots + kX_k$

(Eg. 2)x2 gives $2E = 6V-12$, It follows that

(Eq. 5) $6V-12 = 3X_3 + 4X_4 + 5X_5 + \ldots + kX_k$

By plugging in (Eq. 3) it follows,

(Eq. 6) $6(X_3 + X_4 + X_5 + \ldots + X_k) – 12 = 3X_3 + 4X_4 + 5X_5 + \ldots + kX_k$

Which leads to the equation

(Eq. 7) $3X_3 + 2X_4 + X_5 + 0X_6 – X_7 – 2X_8 – 3X_9 - \ldots -(k-6)X_k = 12$

From these equations it is possible to find all the possible degree listings, but because Euler's Formula is sufficient but not necessary conditions, it contains “bad” graphs. You can give quantities which satisfy Euler's formula but make no sense as planar graphs. Similarly with our degree listing equation it is possible to have quantities which give no simple planar graph. The most intuitive example is one where the largest degree is greater than or equal to the number of vertices in the graph. For example a graph on six vertices with five degree three vertices and one degree nine vertex. Clearly no such simple planar graph exists yet these numbers are consistent with the Euler Formula Though no working algorithm has been found yet for finding all the MCPG's the following is along the lines of what one might look like.

Algorithm: Suppose you have a degree listing with n vertex degrees. Form an nxn matrix with the columns representing the vertices such that column 1 is the highest degree vertex, column 2 is the next highest degree vertex, etc. Suppose the vertex corresponding to column j is of degree m. For the entries a_j where i=j enter in a “0”. Then for a_{1j}, a_{2j},..., a_{mj} enter in a “1” skipping the diagonal entries(where there is already a “0”). Finally for $a_{(m+1)j}$, ..., a_{nj} fill in a “0”.

$$\begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$ Here is a 5x5 representing the degree listing (4,3,3,3,3)

Now, since the adjacency matrix of any graph has to be symmetric, our goal becomes to find an algorithm to make the matrix symmetric. Various attempts led us to the definition of which permutations do not effect the column degree(that is the number of 1's in the column). Clearly the only permutation would be switching a 1 from column j with a 0 from column j. This does not change the degree of the column, but still allows for a permutation of a matrix into a symmetric matrix. As shown by the figure below.

$$\begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$ first column ok $$\begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$ second column needs

has a higher degree than the second row so we switch the zero in a_{55} with the 1 in the $a_{25.}$ Then again the third row has a higher degree than the third column. This take two moves. first switch a_{43} with a_{53} then switch a_{34} down to a_{44} which give us

$$\begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 \end{bmatrix}$$ finally the switch of a_{44} to a_{54} $$\begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 \end{bmatrix}$$

Furthermore

A development of an algorithm will hopefully give us a filter for equation 7. I also would like to continue searching for a super-family or set of families that span the entire set of MCPG's through our connected sum operation. In addition I would like to explore other properties of MCPG's that may lead to a systematic method of coloring all planar graphs. These involve the degree distribution of the graph and the existence of "double triangles," or triangle pairs that share a common edge and are associated with each pair of adjacent vertices. I am also in the process of defining other operations which take into account the relationship between the two non adjacent edges in these double triangles. All these things will lead to a better understanding of the planarity and the elusive forces which seem to keep planar graphs 4-colorable.

REFERENCES

[1] K. Appel and W. Haken, Every planar map is four colorable, Contemporary Math. 98 (1989).

[2] Wilson, Robert J. Introduction to Graph Theory Fourth Edition. Harlow, England: Pearson Education Limited, 1996.