Modeling the Invariant Density Curve of a Noisy Chaotic Map

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Abstract

The goal of the research is to analytically determine the invariant density curve for a given chaotic map altered by noise and to compare it to a numerically generated invariant density curve for the same map. The existence of an analytic solution allows one to predict the overall behavioral trend, if not the exact orbit, of a map and is potentially tremendously useful in the modeling of any system for which maps are used. I will consider both additive and parametric noise, focusing primarily on the latter.

Introduction

A goal of science is to accurately model the surrounding world and confirm testable hypotheses by comparing predictions from a theory with the results of nature. To make such predictions the scientist connects the relevant fundamental laws to a real world system through the creation of a mathematical model. Mathematicians have found use in describing such systems in discrete time using difference equations, or iterative functions, where each time step of the function is determined relative to the previous step, and only the previous step (e.g. there is no explicit dependence upon the value of the function two iterations ago) [1, 2].

The general form of an iterative function is as follows:

$$x_{n+1} = f(x_n)$$

This type of function requires knowledge of the initial state of the physical (or mathematical) system in order to be implemented. Since each step of an iterative function relies on the previous step, one finds that the paths of certain difference equations are heavily dependent on the initial value of the variable(s) [1]. This sensitive dependence occurs if the iterative function is chaotic. If chaotic the orbit never quite reaches the initial position and the system is characteristically non-conservative [1]. The measure of the chaotic nature of a difference equation is the Lyapunov exponent, λ, which is a measure of the difference in the orbit and the initial case after a time $t - t_0$. For two trajectories with the initial separation distance δ_0 at time t_0 the separation distance at time t is approximately given by:

$$|\delta(t)| \approx e^{-\lambda t}|\delta_0|$$

[1]. By definition, in a chaotic system $\lambda \geq 0$.

Because difference functions require the computation of many iterations a computer is indispensible to the analysis of such functions. Unfortunately the output of a chaotic iterative function, due to its high level of sensitivity on initial conditions, cannot be very well predicted. For this reason one may opt to view the outputs of the function by plotting a histogram over the defined domain of the function and binning the elements that fall within intervals of a certain size. The size of the bins is given by the following:

$$size\ of\ bins = \frac{size\ of\ domain}{number\ of\ intervals}$$

The histogram plot is important because in the infinite limit of the number of intervals and iterations the peaks of the histogram combine to become a smooth curve with does not depend on the particular x_0 of a given orbit. The normalized version of this curve is called the invariant density of the system which is equivalent to a probability density, thus if the area under the curve of an interval on the x axis is equivalent to the probability that an iteration will happen within the interval. This is, in other words, the fraction of the total time spent in that interval by the orbit (note that over all space this area, or integral, is equal to unity). The invariant density is a useful object of study since one can plot the invariant density for one map using any initial condition and find that the same curve is reproduced regardless of the initial condition(s). Changing the parameters of the map will change the invariant density curve so, using this method, one is allowed to compare chaotic maps of different parameters without having to bother about the map's sensitivity to initial conditions.

A common problem that arises when one models a physical system is that the surroundings often have unpredictable effects upon the system of study. One cannot completely isolate the system of study from this noise and one may not be able to quantify or measure these unpredictable effects, the presence of which make the model, and analysis of the model, inaccurate [1, 4].

However, it is possible to mirror these effects with an additive noise term and attempt to analyze its affect on the system by the comparison of the resultant invariant density curves of the noise and no-noise mapping. The map of the iterative function including the noise term (where ε is a constant magnitude term and r_n is a randomly generated number uniformly between -1 and 1) is the following:

$$x_{n+1} = f(x_n) + \epsilon r_n$$

When noise is added to a system it is important that the magnitude of the noise is chosen appropriate to the magnitude of the output of the function so that the noise affects the curve but does not dominate it. Once this noise has been added one must ask an important question: is the original function still valid? If the invariant density curve is still retained as the magnitude of the noise (ε) goes to zero then the original function is an accurate portrayal of reality [4]. Throughout the paper I will refer to this additive noise as "noise in dynamics".

Maps have been used in several instances with significant success. Perhaps the most famous is logistic map which has been used to model population growth. It is given below where r is a birth or growth rate and is positive (I should mention that this model has found success in modeling certain populations, such as in animal groups, but limited or no success in other, more complicated, populations such as bacterial growth.) [2]

$$x_{n+1} = rx_n(1 - x_n)$$

Another way to use noise to mimic unpredictability is to use another type of noise, parametric noise, or parameters which are time dependant. In the example of the logistic map this would mean that the parameter, r, would change its value randomly every iteration, becoming r_n. A change in parameter is a realistic thing to study since (again in the logistic map example) it might mean an unexpected drought or catastrophe which could alter the growth rate of a population. When compared to an analytic treatment of the noise, if the affect of the parametric noise on the shape of the function goes to zero as the variation of the noise tends towards zero then one can say that the analytical function accurately portrays the behavior of the map. I will refer to the use of parametric noise as "noise in parameters" throughout the course of this paper.

Methods

The primary source of useful information on the behavior of an iterative function is the invariant density curve which is not dependent upon initial conditions and thus provides information on the general behavior of the map as opposed to specific and wildly varying individual orbits. The study includes the numerical generation of curves for both the noise in dynamics and noise in parameters. The noise in parameters case will also be analyzed analytically in order to generate a point of comparison between the numerically generated curve and a theoretical curve.

As I mentioned in the introduction the invariant density curve can be seen as the curve of a histogram as the number of iterations of the function tends to infinity and the size of the intervals over the domain of the histogram bars becomes infinitesimal. For clarity in comparison my graphs generate a curve by connecting the center points of the bars of this histogram to make a smooth curve. This, of course, is an approximation since I cannot iterate the function an infinite number of times but since, with enough iterations, the curves look smooth, this is clearly a decent approximation.

In order to generate these curves I have written a program using Matlab which simply calls on a function from another file and iteratively runs it using a random or inputted initial condition (if you recall in my introduction the curve I am trying to generate does not depend on the initial conditions which is its advantage) as well as inputted magnitudes of noise and parameters and generates an array with a counter which increases for each time the output of the function falls within a certain interval of the array. This array is plotted "sideways" so that the size of the counter increases the height of the graph at each interval. A line is then drawn from the center point of each interval to plot the smooth invariant density curve.

Two things should be mentioned with regards to the program. The noise can push the output of a function past the domain over which the function is defined. To counter this problem the program has built in controls to reflect this value back into the domain before the calculation of the next iteration can be done. It should also be mentioned that the number of iterations or size of bins that one uses change the height of the curve which can make it difficult to compare two curves which were generated using different values of these two. To fix this problem I do not just plot the height of the array but rather:

$$\frac{(number\ of\ bins)*(height\ of\ array)}{(number\ of\ iterations)}$$

to normalize it, so that the plot is a true invariant density curve. I found that my home computer could handle about 5,000,000-10,000,000 iterations depending on the complexity of the function in a reasonable amount of time. In the following sections these curves are plotted in this manner.

Noise in Dynamics

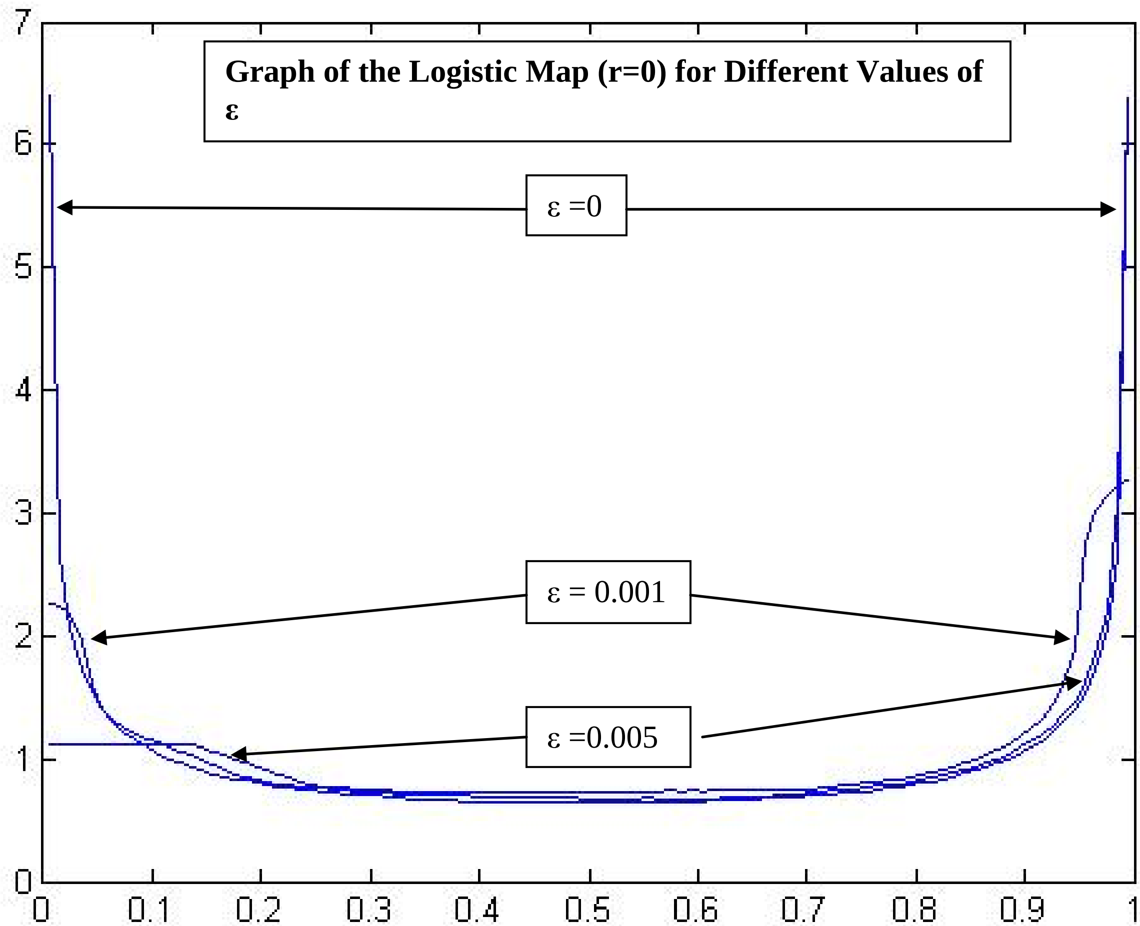
The logistic map [1, 2] is defined over the interval [0,1] as:

$$x_{n+1} = rx_n(1 - x_n)$$

The version of this map that I will use has set r = 4. Noise is added to this map so that it becomes:

$$x_{n+1} = 4x_n(1 - x_n) + \epsilon r_n$$

Here r_n is a uniform random number such that $-1 < r_n < 1$. The curves of this function for ε set to be 0 (no noise), 0.005, and 0.001 is pictured below.

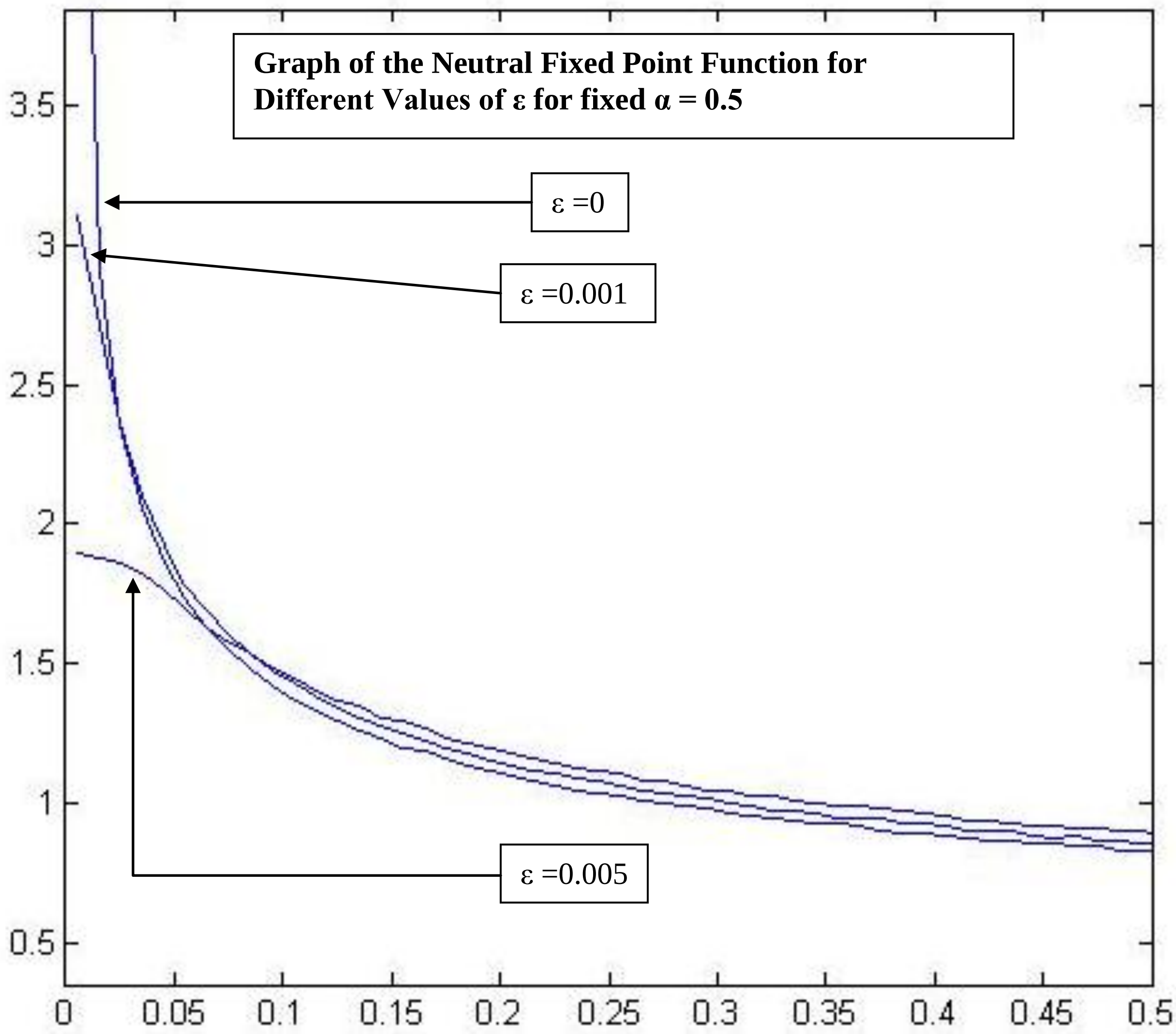


I repeat this method on the circle map with a neutral fixed point [4] which is defined over the interval [0,1] as:

$$x_{n+1} = \begin{cases} x_n + 2^{\alpha} x_n^{\alpha+1}, & 0 \le x_n < \frac{1}{2} \\ 2x_n - 1, & \frac{1}{2} \le x_n \le 1 \end{cases}$$

where $0 < \alpha < 1$

The curves of this function for ε set to be 0 (no noise), 0.005, and 0.001 is pictured below.



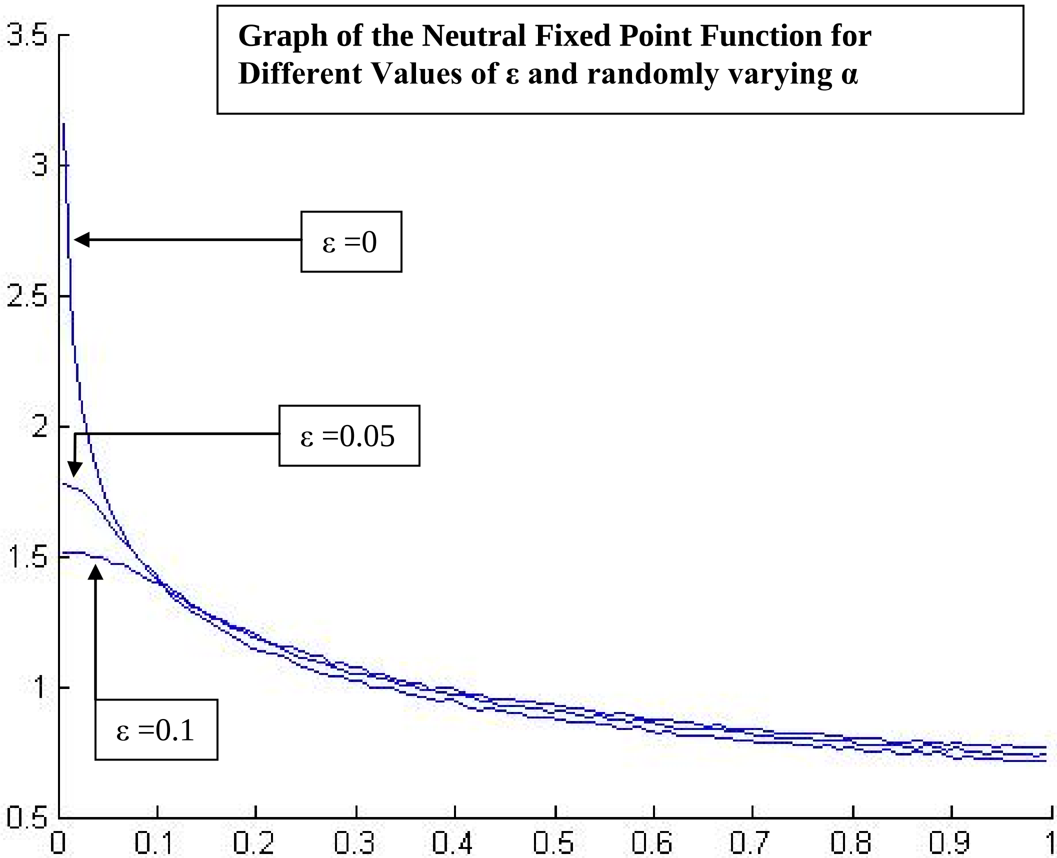
Noise in Parameters

When the parameters themselves, not just the noise, change with each iteration of the function, the general form of the function is:

$$x_{n+1} = f(x_n, \alpha_n)$$

Where α_n is a time dependant parameter.

This method is used to evaluate the circle map with a neutral fixed point with noise. One can consider the hybrid case where a function has both an additive noise term (of magnitude ε) as well as a randomly varying parameter, which is presented here, more as a curiosity than a point of serious study. Here the parameter α_n is chosen uniform randomly between zero and one each iteration and ε is set to be 0 (no noise), 0.05, and 0.1 which yields the following graph:

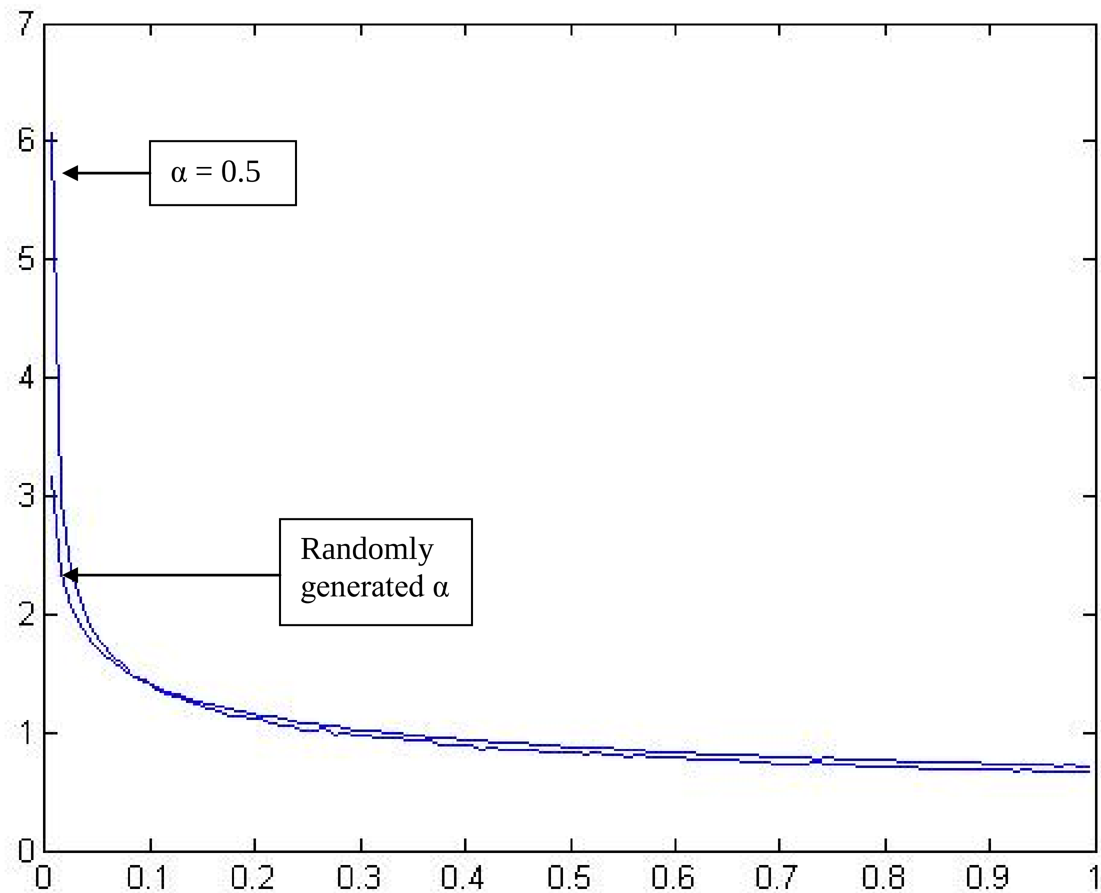


One of the interesting things about this formulation of this map is that, without additive noise, the invariant density curve does not change with different values of x_0 and the initial parameter value even though the parameter itself varies step to step.

The primary area of study is to study the noise in parameters without the additive constant noise. I'll start with a numerical study. The map in this case is the neutral fixed point equation:

$$x_{n+1} = \begin{cases} x_n + 2^{\alpha} x_n^{\alpha+1}, & 0 \le x_n < \frac{1}{2} \\ 2x_n - 1, & \frac{1}{2} \le x_n \le 1 \end{cases}$$

Below is a graph of the invariant density curve for both the case of fixed α = ½ and alpha varying each iteration uniformly random between 0 and 1.



Analytic Approach to the Invariant Density Curve

Here I will work on the derivation of the invariant density curve for the neutral fixed point equation.

To start take a smooth function, h(x), averaged over a collection of sample trajectories for some dynamical system. Over many iterations the collection of these trajectories form the invariant density curve which means that this average should not change step to step. Mathematically this is represented by the following equation:

$$\int_0^1 h(x)\, p(x)\, dx = \int_0^1 d\alpha \int_0^1 dx\, h\big(T(x,\alpha)\big)\, p(x) \equiv I$$

For the specific example of the neutral fixed point equation as described by:

$$x_{n+1} = T(x_n, \alpha) \equiv \begin{cases} x_n + 2^{\alpha_n} x^{1+\alpha_n} & 0 \le x_n < \frac{1}{2} \\ 2x_n - 1 & \frac{1}{2} \le x_n < 1 \end{cases}$$

Where α is varied randomly in each step between zero and 1,

The quantity I is given by:

$$I = \int_0^1 d\alpha \int_0^{\frac{1}{2}} dx\, h\big(T(x,\alpha)\big)\, p(x) + \int_0^1 d\alpha \int_{\frac{1}{2}}^1 dx\, h\big(T(x,\alpha)\big)\, p(x)$$

$$I = \int_0^1 d\alpha \int_0^{\frac{1}{2}} dx\, h(x + 2^{\alpha} x^{\alpha+1})\, p(x) + \int_0^1 d\alpha \int_{\frac{1}{2}}^1 dx\, h(2x-1)\, p(x)$$

By effecting the change of variables y = 2x – 1 the system becomes:

$$I = \int_0^1 d\alpha \int_0^{\frac{1}{2}} dx\, h(x + 2^{\alpha} x^{\alpha+1})\, p(x) + \frac{1}{2}\int_0^1 dy\, h(y)\, p\left(\frac{1+y}{2}\right)$$

To proceed further a few approximations must be made. The term $h(x + 2^{\alpha} x^{\alpha+1})$ can be Taylor expanded.

$$h(x + 2^{\alpha} x^{\alpha+1}) = h(x) + 2^{\alpha} x^{\alpha+1} h'(x) + \frac{2^{\alpha} x^{\alpha+1^2}}{2} h''(x) + \cdots$$

For convenience the following two expressions are defined.

$$v(x) \equiv \int_0^1 2^\alpha x^{\alpha+1}\, d\alpha = \frac{x(1-2x)}{-\log(2x)}$$
$$D(x) \equiv \int_0^1 [2^\alpha x^{\alpha+1}]^2\, d\alpha = \frac{x^2(1-4x^2)}{-2\log(2x)}$$

Using the above definitions and the Taylor expansion up to the quadratic terms one gets the following.

$$\int_0^1 h(x)\, p(x)\, dx \approx \int_0^{\frac{1}{2}} \left[h(x) + v(x)h'(x) + \frac{D(x)}{2}h''(x)\right] p(x)dx + \frac{1}{2}\int_0^1 dy\, h(y)\, p\left(\frac{1+y}{2}\right)$$

At this point choose the smooth function h(x), which up until now had no other conditions attached to it, so that it is zero unless x is very small. In other words take h(x ≥ ε) = 0 for some ε « ½. In this case the integral becomes:

$$\int_0^\epsilon h(x)\, p(x)\, dx \approx \int_0^\epsilon \left[h(x) + v(x)h'(x) + \frac{D(x)}{2}h''(x)\right] p(x)dx + \frac{1}{2}\int_0^\epsilon dy\, h(y)\, p\left(\frac{1+y}{2}\right)$$

Making the substitution x = z in the first integral and y = z in the second and then collecting the integrals, the equation becomes:

$$\int_0^\epsilon \left[v(z)h'(z) + \frac{D(z)}{2}h''(z)\right] p(z)dz + \frac{1}{2}\int_0^\epsilon dz\, h(z)\, p\left(\frac{1+z}{2}\right) \approx 0$$

Through integration by parts on the first term the integrals can be grouped together with a common factor of h(z) which yields:

$$\int_0^\epsilon \left[-\frac{d}{dz}[v(z)p(z)] + \frac{1}{2}\frac{d^2}{dz^2}[D(z)p(z)] + \frac{1}{2}p\left(\frac{1+z}{2}\right)\right] h(z)dz \approx 0$$

If the integral is zero over this interval where h(z) is nonzero then the integrand is zero (here I will drop the approximately equal signs) which implies that:

$$-\frac{d}{dz}[v(z)p(z)] + \frac{1}{2}\frac{d^2}{dz^2}[D(z)p(z)] + \frac{1}{2}p\left(\frac{1+z}{2}\right) = 0$$

Since z was close to zero the term $\frac{1}{2}p\left(\frac{1+z}{2}\right)$ is approximately constant so define λ as the source term (the source from the interval ½ to 1 which map back near zero).

$$\lambda \equiv \frac{1}{2}p\left(\frac{1}{2}\right) \approx \frac{1}{2}p\left(\frac{1+z}{2}\right)$$

At this point λ is a constant so whenever there are any other unknown multiplicative constants to λ or additive constants to the equation in the proceeding analysis I will just absorb all such constants into λ to keep things simple.

$$-\frac{d}{dz}[v(z)p(z)] + \frac{1}{2}\frac{d^2}{dz^2}[D(z)p(z)] + \lambda = 0$$

It should be noted that this equation bears a notable resemblance to the Fokker-Planck equation which models the time evolution of a probability density curve [3]. Here the sought after probability density curve is invariant in time so there is no time derivative. The most important difference is the existence of the constant source term, λ, above, which is not considered in the other formulation.

I will set λ equal to one for convenience and integrate both sides with respect to z from 0 to y:

$$\Gamma(y) - \Gamma(0) = y$$

I will attempt a series solution to this differential equation where each additive term is smaller than the previous one.

$$p(y) = p_0(y) + p_1(y) + p_2(y) + \cdots$$

To proceed I will first assume that the terms of the equation involving D v are smaller than the terms involving v p. For a first approximation to the solution assume that D = 0 and call the function corresponding this new equation p_0.

$$\frac{y(1-2y)}{-\log(2y)}p_0(y) = y$$

$$p_0(y) = \frac{\log(2\mathrm{y})}{1-2\mathrm{y}}$$

Clearly p_0 is not a solution to the original equation; however it is at least a possible density curve. In order to check that the assumption that D = 0 is valid one must show that the correction terms of the series really are progressively smaller. To do this I first have to plug this formulation into the differential equation:

$$v\,p - \frac{d}{dy}[D\,p] = v\,(p_0 + p_1 + \cdots) - \frac{d}{dy}\left[D\,\left((p_0 + p_1 + \cdots)\right)\right]$$

$$v\,p - \frac{d}{dy}[D\,p] = v\,p_0 + \left[v\,p_1 - \frac{d}{dy}(D\,p_0)\right] + \left[v\,p_2 - \frac{d}{dy}(D\,p_1)\right] + \cdots$$

Thus

$$v\, p_0 = y$$

$$\left[v\, p_1 - \frac{d}{dy}(D\, p_0)\right] = \left[v\, p_2 - \frac{d}{dy}(D\, p_1)\right] = \cdots = 0$$

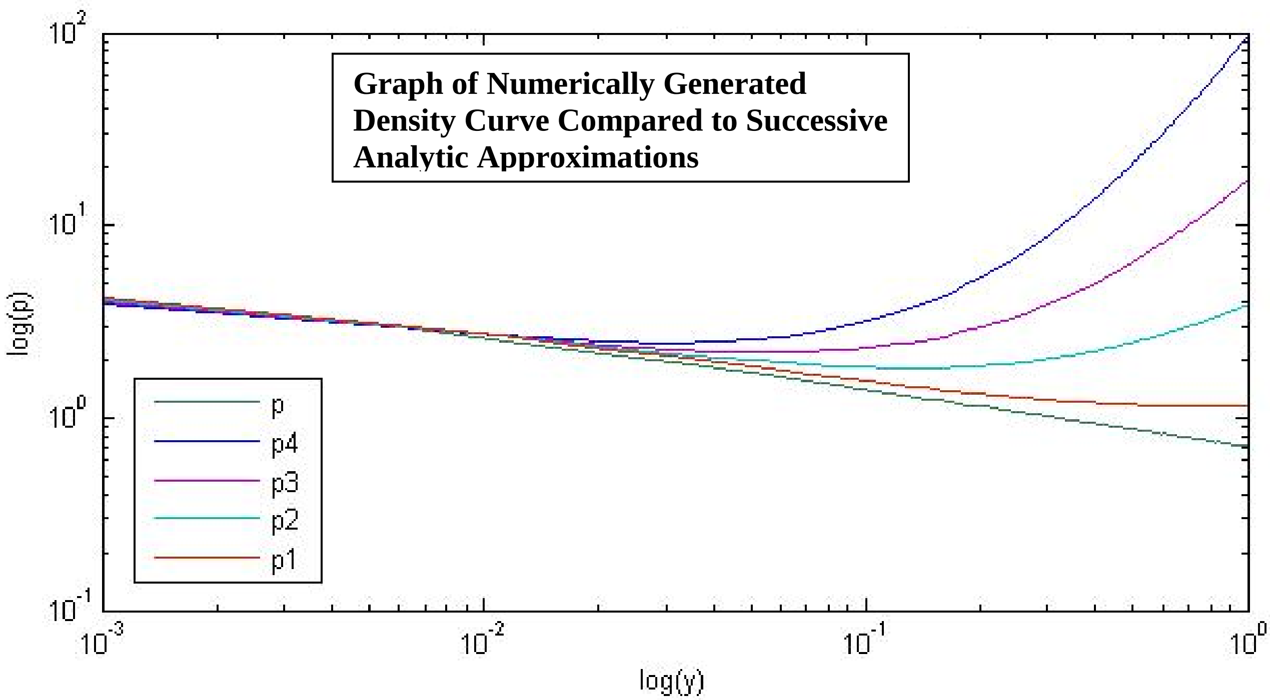
Or for each p_i:

$$p_i = \frac{1}{v}\frac{d}{dy}(D\, p_{i-1})$$

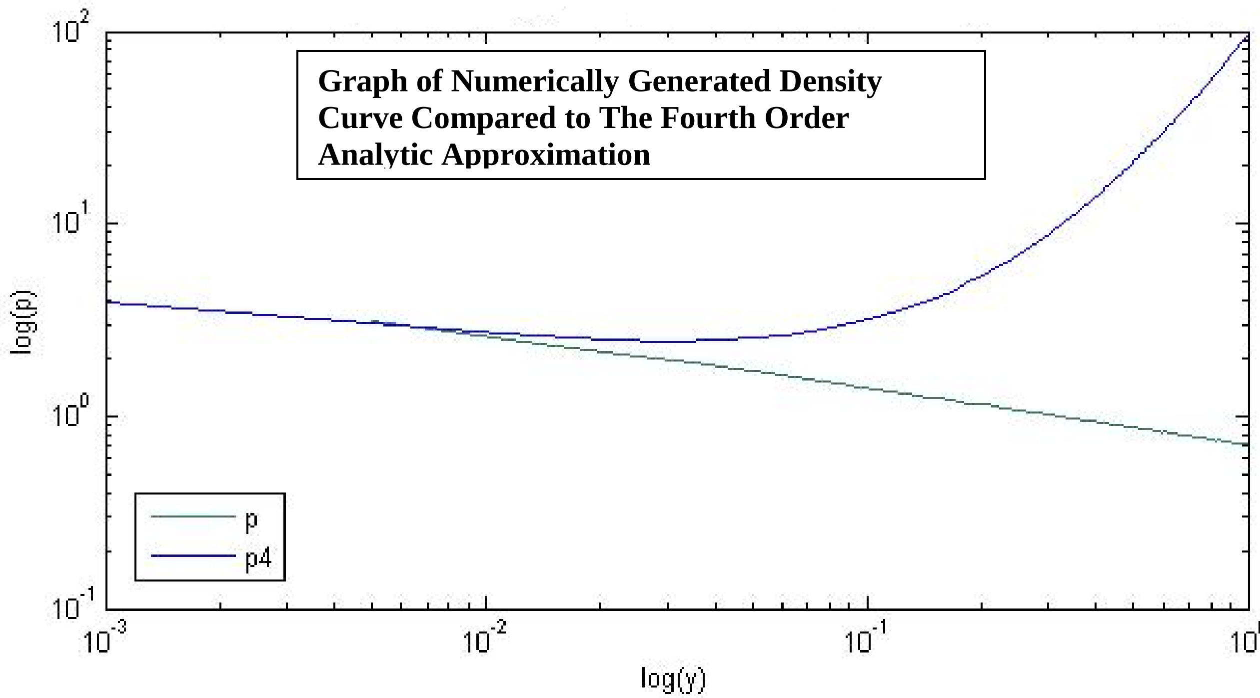
Therefore each p_i can be found recursively from p_{i-1}. I will spare the differentiation details just to mention that each correction term for values of y close to zero decrease in magnitude as i increases. For this reason the assumptions made before are valid and a series approach to solve the differential equation while taking a finite number of terms constitutes a reasonable approximation to the actual solution.

Comparison of Analytic Approach to Numerical Approach

Using the iterative method described in the previous section to find an analytic expression of p up to $p(y) = p_0(y) + p_1(y) + p_2(y) + p_3(y) + p_4(y)$. Using this function (as well as some of the intermediary functions) I determined the constant, λ, by a least squares method to fit the function to the p(y) determined numerically (which creates a vertical shift). The functions, named by their final correction term, are plotted with the numerically generated graph, named p, on a log-log scale below (from the right, top to bottom: p4, p3, p2, p1, p).



Of course the real object of interest is the best approximation, denoted p4, which is compared to the numerically determined invariant density curve in the following graph (from the right, top to bottom: p4, p).



Conclusion

The major assumption of the derivation of the analytic approximation of the invariant density curve is that the function maps the input near zero. With this in mind the analytic function does approximate the numerically generated invariant density curve quite well near the origin, though it clearly does not work well away from it. Ideally this fit can become arbitrarily accurate near zero by adding more terms to the expansion. I ended this expansion at p_4 due to the fact that the number of terms in the derivative of D p_4 is too great to be evaluated by Matlab and the sheer impracticality of differentiating by hand.

This analysis makes it possible to make real predictions on the outcome of the circle map with noise in the parameters in the lower limits of the domain. While it does not generate the lower domain of the circle map it is, at the least, a start in solving the problem of modeling a well behaved curve which, in term, comes from a very volatile mapping. In a more general formulation this method might find utility in the determining the invariant density curves for arbitrary maps with different types of noise and thus find application in modeling.

References

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