Control of the Quadratic Map Using Delay.

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Introduction

The quadratic map,

$$x_{n+1} = a - x_n^2 \qquad a, x_n \in \mathbb{R} \tag{1}$$

is a well studied dynamical system whose behavior has been intensively studied and is in fact a prototypical system in the study of maps and chaotic systems.

This paper examines the effects of modifying this map in an attempt to control its behavior, in some sense, while still maintaining the fundamental structure of the dynamics of the original system (1).

1 Delayed Feedback

The idea of using the output of a system as an input is not a new one (much of the world of modern electronics in fact depends upon it). *Feedback* refers to this effect. In this case, *Delayed* means we wish to explore whether retaining a history of the past of the system will have any effect on its behavior. In general terms, we wish to modify (1) in order to account for its history:

$$x_{n+1} = a - x_n^2 + f(x_n, x_{n-1}, x_{n-2}, ...) \tag{2}$$

f will be seen to be a function of x_n in order to maintain important features of the system (namely its fixed points).

The simplest case for (2) would be

$$x_{n+1} = a - x_n^2 + f(x_n, x_{n-1}) \tag{3}$$

In other words, a history of one iteration will be kept. A simple form for $f(x_n, x_{n-1})$ will be constructed, with the following in mind: it is desired to maintain the fixed points of the original system, since the behavior of a system in general depends in a large part on its fixed points, their stability, and the stable and unstable manifolds of those fixed points.

"Maintain the fixed points of the original system" means that (3) should equal (1) at a fixed point. This means that at a fixed point, it is desirable that $f(x_n, x_{n-1}) = 0$. Since $x_{n-1} = x_n$ at a fixed point, one way to do this is for f to have a factor of $x - x_{n-1}$. We choose f proportional to this factor:

$$f = b(x_n - x_{n-1}) \qquad b \in \mathbb{R} \tag{4}$$

so that

$$x_{n+1} = a - x_n^2 + b(x_n - x_{n-1}) \tag{5}$$

In order to perform analysis on this system (5), it is convenient to introduce a new variable $y_n = x_{n-1}$. Then $y_{n+1} = x_n$. (5) can be rewritten as a two dimensional system

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} a - x_n^2 + b(x_n - y_n) \\ x_n \end{pmatrix} \tag{6}$$

2 Analysis

2.1 Fixed Points

It has already been shown that (6) was constructed so that it retains the fixed (period 1) points of the original system (1). Examining that equation, setting $x_{n+1} = x_n$ yields an equation for the fixed points

$$x = \frac{-1 \pm \sqrt{1+4a}}{2} \tag{7}$$

The stability of these fixed points can be determined in general by calculating the Jacobian of (6) and evaluating it at the fixed points[1].

The Jacobian of (6) is

$$A = \begin{pmatrix} -2x + b & -b \\ 1 & 0 \end{pmatrix} \tag{8}$$

The eigenvalues of (8) are given by

$$\lambda = \frac{(2x - b) \pm \sqrt{(-2x+b)^2 - 4b}}{2} \tag{9}$$

[1] Recall that, for the one dimensional case, one essentially approximates $x_{n+1} = f(x_n)$ by a linear transformation $x_{n+1} = A(x_n)x_n$, and then eigenvalues λ of $A(x_0)$ are sought at the fixed points x_0, and it is assumed that the behavior nearby the fixed points will be dominated by these eigenvalues, so that $x_{n+1} = \lambda x_n$. The values of λ will then determine the stability of the fixed points, i.e. whether x_n increases when near the fixed point ($\lambda > 1$), or whether it decreases ($\lambda < 1$).

In the original system (1) there exists one eigenvalue $\lambda_o = -2x$. Substituting the values for the fixed points (7) it can be seen that one fixed point will always be unstable, while the other will change stability from stable to unstable when $a = \frac{3}{4}$. It will be shown later that a bifurcation occurs at this point as well.

One question to explore is whether the addition of feedback will affect where this change occurs, and if so whether it serves to "control" the system and inhibit the onset of this change in stability and its corresponding bifurcation.

In this analysis we will assume that $b < 1$. By examing (6) it can be seen that this means that the feedback is not "amplified", that is, what is fed into the system from the previous time step will be reduced in magnitude.

The determinant of the Jacobian (8) is b. The product of the eigenvalues will thus also equal b. The change in stability discussed above occurs when $|\lambda| = 1$.

2.1.1 Eigenvalues

Since the product of the eigenvalues of a fixed point equals $b < 1$, the eigenvalues must be real when one of them "crosses" from $|\lambda| < 1$ to $|\lambda| > 1$.

So we wish to examine what happens when $|\lambda| = 1$. There are two cases since λ is real: $\lambda = 1$ and $\lambda = -1$.

2.1.2 λ=-1

Assume $\lambda = -1$. Solving (9) for x yields

$$x = -\frac{1}{2} \tag{10}$$

In other words, using (7), the fixed point changes stability when $a = -\frac{1}{4}$. It can be seen from 7 that this is in fact the point at which the fixed points come into existence. Figure 1 is a bifurcation diagram of this system, plotting the stable fixed points of the system versus a. The emergence of fixed points and changes in their stability is discussed in terms of increasing the parameter a.

To examine the stability of the fixed points which emerge here, consider the system (1) without delay. We know that

$$\lambda = -2x = -1 \pm \sqrt{1 + 4a} \tag{11}$$

Once the points are born, it can be seen from (7) that one fixed point will increase in magnitude as a increases, and the other point will decrease. For the point that increases in magnitude, (11) shows that its eigenvalue will be of magnitude less than 1 thus this point will be stable. For the other point, λ will become increasingly large in magnitude, thus this point will remain unstable.

2.1.3 $\lambda = 1$

When $\lambda = 1$, a period doubling bifurcation occurs, and periodic points will emerge as seen in Figure 1.

$$x_{n+1} = \lambda x_n \tag{12}$$

near the fixed point.

For this case, solving (9) for x yields

$$x = b + \frac{1}{2} \tag{13}$$

So it can be seen that the value for x at which this change in stability occurs can be "pushed off" simply by increasing b. This can be seen in comparing Figure 2 to Figure 1. In Figure 2, the addition of the b term has necessitated a larger value of a in order for the onset of the period doubling bifurcation.

2.2 Periodic points

Now we will investigate the existance and behavior of period two points.

In order to investigate the behavior of period 2 points, it is neccessary to create a new map based on the original equation (5).

For a period two point,

$$x_n = x_{n+2} = a - x_{n+1}^2 + b(x_{n+1} - y_{n+1}) \tag{14}$$

$$y_n = y_{n+2} = x_{n+1} \tag{15}$$

where x_{n+1} and y_{n+1} are given by (6).

Plugging these in and dropping subscripts yields the equations to be satisfied at a period 2 point:

$$x = a - (a - x^2 + b(x - y))^2 + b(a - x^2 + b(x - y) - x) \tag{16}$$

$$y = a - x^2 + b(x - y) \tag{17}$$

Plugging (17) into (16) yields an equation which is quartic in x. However, this expression can be simplified since it is known that a period 1 point will also solve this equation. In other words, this equation will be of the form

$$f(x) = 0 \tag{18}$$

and f can be separated into a factor which represents period 1 points and a factor which represents period 2 points. Doing so yields the following expression for the period 2 points:

$$x^2 - (1+2b)x + (1+3b+3b^2-a) = 0 \tag{19}$$

which can be solved for x:

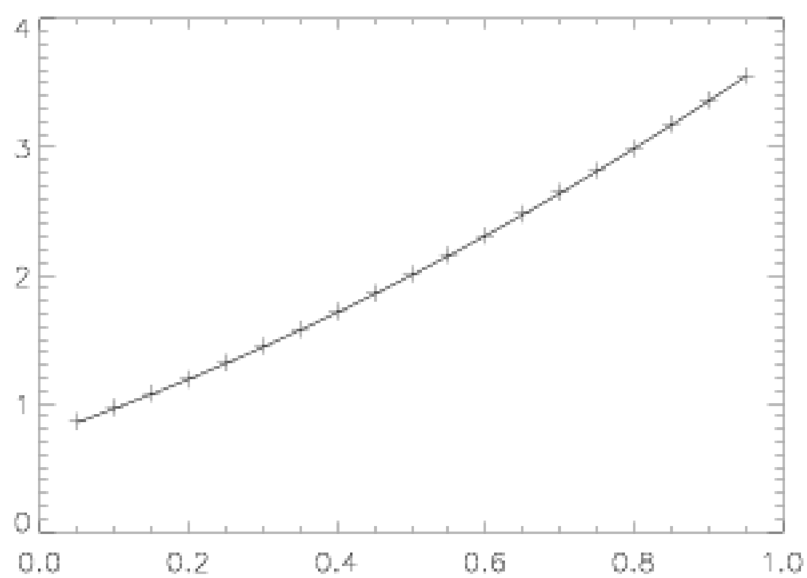
$$x = \frac{(1+2b) \pm \sqrt{(1+2b)^2 - 4(1+3b+2b^2-a)}}{2} \tag{20}$$

Examining the radical term reveals that there won't be a real solution to this expression unless

$$a > b^2 + 2b + \frac{3}{4} \tag{21}$$

In other words, the b delay term requires a larger a forcing term, thus pushing off the onset of the emergence of periodic points when viewed in terms of the the control parameter a. This is directly observable from the following figure which depicts the onset of the bifurcation point in terms of the control parameter a and the b delay term:

Figure 1: The dependence of a upon b



In the figure above, the numerical values calculated are shown as + signs and (21) is shown as a solid line. Note as b approaches 1.00, the predescribed condition from (21) causes non–real solutions.