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On a Property of Cosets in a Finite Group

WILLIAM YSLAS VÉLEZ

Department of Mathematics, The University of Arizona, Tucson, Arizona 85721

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Let us fix the following notation. Let G be a finite group, H a normal subgroup of G, $g \in G$. Choose $b \in gH$ such that $o(b) \leqslant o(gh)$, for all $h \in H$, where $o(\ )$ denotes the order of that element in G.

In an investigation dealing with radical extensions of fields [1], we showed that if G is abelian, then $o(b) | o(gh)$, for all $h \in H$.

At a recent A.M.S. meeting, we asked I. Kaplansky if he had ever seen this result. Kaplansky had not and in turn asked if this result held for a wider class of groups. Though this result does generalize somewhat, as the following theorems show, the result does not generalize to all groups. The example at the end of this paper is due to H. W. Lenstra, Jr.

THEOREM 1. *If H is nilpotent, then $o(b) | o(gh)$, for all $h \in H$.*

Proof. Let $H = P_1 \cdots P_s$, where each P_i is the unique p_i-Sylow subgroup of H. Set $n = o(gH)$ in G/H. Then $g^n \in H$, so $g^n = g_1 \cdots g_s$, $g_i \in P_i$. Since the g_i all commute with each other and their orders are relatively prime, we have that $o(g) = n \cdot \prod_{i=1}^{s} o(g_i)$.

If $h \in H$, then $h = h_1 \cdots h_s$, $h_i \in P_i$, and $(gh)^n = g^n h^{g^{n-1}} h^{g^{n-2}} \cdots h^g h = \prod_{i=1}^{s} h_i'$, where $h_i' = g_i(\prod_{k=0}^{n-1} h_i^{g^k}) \in P_i$.

Thus $o(gh) = n \prod_{i=1}^{s} o(h_i')$.

Choose $\bar{h}_i \in P_i$ such that $g_i(\prod_{k=0}^{n-1} \bar{h}_i^{g^k})$ has minimal order among all elements of the form h_i'. Since these elements have orders a power of p_i, the minimal order divides the orders of the other elements. Thus if we set $b = g(\bar{h}_1 \cdots \bar{h}_s)$, b has minimal order and its order divides the orders of the other elements in the coset gH.

THEOREM 2. *If $o(gH) = p^s$ in G/H, p a prime, then $o(b) | o(gh)$, for all $h \in H$.*

Proof. Let $g^{p^s} = h \in H$ and $o(h) = p^t m$, $(p, m) = 1$, then $o(g) = p^{s+t} m$. Since $(m, p) = 1$, $\langle gH \rangle = \langle g^m H \rangle$ and $o(g^m) = p^{s+t}$. Choose k so that

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$g^{mk}H = gH$, where of course $(k, p) = 1$. Then $g^{mk} \in gH$ and $o(g^{mk}) = p^{s+t}$. Thus we have shown that the order of every element in gH is divisible by p^s and if $o(gh) = p^r m$, $(p, m) = 1$, then there exists an $h' \in H$ such that $o(gh') = p^r$. Now let $b \in gH$ be such that $o(b) = p^r$ and r is minimal. Clearly $p^r | o(gh)$ for all $h \in H$.

EXAMPLE. Let H be the semidirect product of a cyclic group C_9 of order 9 with the quaternion group Q_8 of order 8, with C_9 acting on Q_8 via an automorphism of order 3. It is easy to see that the center of H is cyclic of order 6. Thus let $Z(H) = \langle a \rangle$, where $a^6 = 1$. Now let $G = \langle H, g \rangle$, where $g^6 = a$ and $gh = hg$, for all $h \in H$. It can then be shown (the author used CAYLEY) that the coset gH contains elements of orders 12 and 18 but no element of order 6.

REFERENCES

1. W. Y. VÉLEZ, A generalization of Schinzel's theorem on radical extensions of fields and an application, *Acta Arith.*, in press.

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