Assignment 5, due March 2

1. Let \( f \) be a continuous map of the unit interval into itself, such that \( f(0) = 0 \) and \( f(0) = 1 \). Show that \( f \) is homotopic to the identity map modulo points 0 and 1, i.e. that there exists a homotopy between \( f \) and identity which is constant at the endpoints.

2. Let \( h : X \to X' \) be a continuous map of topological spaces. For a topological space \( Y \) and a continuous map \( \alpha : X' \to Y \), let \( \Phi(\alpha) = \alpha \circ h \) — a map from \( X \) to \( Y \). Prove that if \( \alpha \) and \( \beta \) are homotopic, then so are \( \Phi(\alpha) \) and \( \Phi(\beta) \).

3. a) Suppose \( X \) is a topological space and \( A \) — its path-connected subspace. Prove that if \( A \) is a deformation retract of \( X \), then \( X \) is path-connected.

b)* Is the conclusion necessarily true if we only assume that \( X \) is a retract of \( A \)?

4. Prove that the zero-dimensional sphere \( \{-1, 1\} \) is not a retract of the one-dimensional disc \([-1, 1]\).

5. Define the Möbius strip \( M \) as a square with two opposite sides identified with the same orientation (identifying them with opposite orientations would give a cylinder). Find a (homeomorphic image of) circle \( C \) in \( M \) which is a deformation retract of \( M \).

6*. Let \( T \) be a torus and \( x \in T \). Prove that \( T \setminus \{x\} \) contains a deformation retract, homeomorphic to a wedge of two circles (a figure “eight”).