Assignment 9, due May 1

1. Prove the **Rank Theorem**: Let \( f : \mathbb{R}^m \to \mathbb{R}^n \) be a smooth function, \( m > n \). Suppose that at a point \( x_0 \in \mathbb{R}^m \) the rank of the tangent map \( df_{x_0} \) equals \( n \). Prove that there exist maps \( \phi \) and \( \psi \), such that \( \phi \) maps a neighborhood of \( x_0 \) diffeomorphically onto a neighborhood of 0 in \( \mathbb{R}^m \), \( \psi \) maps a neighborhood of \( f(x_0) \) diffeomorphically onto a neighborhood of 0 in \( \mathbb{R}^n \) and

\[
\psi \circ f \circ \phi^{-1}(x^1, \ldots, x^m) = (x^1, \ldots, x^n)
\]

for \( x = (x^1, \ldots, x^m) \) in the range of \( \phi \).

2. Let \( f : M^m \to N^n \) be a smooth map between manifolds of dimensions \( m \geq n \) and \( y \)—a regular value of \( f \). Use the rank theorem to prove that \( f^{-1}(y) \) is a smooth submanifold of dimension \( m - n \).

3. Let \( M \) be a manifold (without boundary) and \( f : M \to \mathbb{R} \)—a smooth function, having 0 as its regular value. Prove that the set of points \( x \in M \), where \( f(x) \geq 0 \) is a manifold with boundary and that the boundary is equal \( f^{-1}(0) \).

4. Let \( X \) be an \( m \)-dimensional manifold with boundary \( \partial X \) and \( f : X \to N \)—a smooth function into an \( n \)-dimensional manifold (without boundary), \( m > n \). Let \( y \) be a regular value both for \( f \) and for its restriction to \( \partial X \). Prove that \( f^{-1}(y) \) is an \( m - n \)-dimensional manifold with boundary and that its boundary is equal to the intersection of \( f^{-1}(y) \) with \( \partial X \).

5*. Prove that a compact, one-dimensional manifold with boundary is diffeomorphic to a finite union of closed intervals and/or circles.

6*. Recall that a manifold is orientable if the coordinate charts can be chosen so that all transition maps have positive Jacobi determinants. This definition extends to manifolds with boundary. Prove that if \( X \) is an orientable manifold with boundary, then \( \partial X \) is also orientable.