## Using equivalence relations to define rational numbers

Consider the set

$$S = \{(x, y) \in \mathbb{Z} \times \mathbb{Z} : y \neq 0\}.$$

We define a rational number to be an equivalence classes of elements of $S$, under the equivalence relation

$$(a, b) \simeq (c, d) \iff ad = bc.$$

An equivalence class is a complete set of equivalent elements. I.e., it's a set of elements of $S$, all of which are equivalent to each other, and which contains all of the pairs that are equivalent to those pairs. (Stricly speaking we need to use some properties of equivalence relations to check that this makes sense . . . more about that later.)

For example, with the equivalence relation above,

$$\{(1,2),(2,4),(3,6),(-1,-2),\dots\}$$

would be one equivalence class. Another one would be

$$\{(3,4),(6,8),(-3,-4),(-75,-100),\dots\}.$$

We define a rational number to be one of these equivalence classes. A particular element of the equivalence class is called a representative. We can give names to these equivalence classes. For example, we could call the first one

$$\frac{1}{2}.$$

We could call the second one 3/4.

We can think of $\mathbb{Z}$ as being included in $\mathbb{Q}$ via the function

$$i:\mathbb{Z}\to\mathbb{Q} \quad i(a) = \text{the equivalence class containingn } (a,1)$$

Now, there are lots of things we need to check for this definition to make sense. First we need to check that

$$(a, b) \simeq (c, d) \iff ad = bc.$$

really is an equivalence relation.

1. Reflexive: Is $(a, b) \simeq (a, b)$ for all $(a, b) \in S$? Yes, because $ab = ba$ for all $a, b, \in \mathbb{Z}$.

2. Symmetry. Is it true for all $(a, b), (c, d) \in S$ that if $(a, b) \simeq (c, d)$ then $(c, d) \simeq (a, b)$? Yes, because for all $a, b, c, d \in \mathbb{Z}$, if $ad = bc$ then $cb = da$.

3. Transitivity. Is it true for all $(a, b), (c, d), (e, f) \in S$ that if $(a, b) \simeq (c, d)$ and $(c, d) \simeq (e, f)$, then $(a, b) \simeq (e, f)$? Here we want to prove, for all $a, b, c, d, e, f \in \mathbb{Z}$, with $b \neq 0$, $d \neq 0$, and $f \neq 0$, that

$$\text{if } ad = bc \text{ and } cf = de \text{ then } af = be.$$

Well, if $ad = bc$ and $cf = de$ then $adcf = bcde$. We are given $d \neq 0$. So if $c \neq 0$, then we can divide by $cd$ and get $af = be$.

On the other hand, $c = 0$ then $ad = 0$ and $de = 0$. Since $d \neq 0$, this means that $a = e = 0$. Therefore $af = be$ because both sides are 0.

Now, let's return to some unfinished business. We informally defined an equivalence class to be a set of elements of $S$ such that (a) all the elements are equivalent to each other, and (b) it which contains all the elements of $S$ which are equivalent to some element of the equivalence class. If you think about this definition, you will realize that we haven't really proved that these two conditions are compatible, or that equivalence classes exist at all.

A leaner definition is: If $R$ is an equivalence relation on a set $S$, then we define the equivalence class of an element $x \in S$ to be the set of all elements of $S$ equivalent to $x$.

Then we need to prove:

**Theorem**

1. Every element of $S$ is in some equivalence class.

2. If $x, y \in S$, the equivalence classes of $x$ and $y$ are either equal or disjoint (i.e. have empty intersection).

Proof of 1. By reflexive property, every element is in its own equivalence class.