Definition of the idea of a limit

The limit of $f(x)$ as x approaches c is equal to L if the values of f get closer and closer to L as x gets closer and closer to c. We let δ represent the closeness of c to x, and ϵ the closeness of $f(x)$ to L. Then instead of saying x is "closer and closer to", we use the quantified phrases

$$|f(x) - L| < \epsilon \quad \text{and} \quad |x - c| < \delta.$$

Definition: Given a function f and a real number c in its domain, and a real number L, we say

$$\lim_{x \to c} f(x) = L$$

if for all $\epsilon > 0$ there exists a $\delta > 0$ such that for all x satisfying $0 < |x - c| < \delta$, we have $|f(x) - L| < \epsilon$.

Here's an example of how to use this definition:

Theorem

$$\lim_{x \to 2} x^2 = 4$$

Proof. Suppose $\epsilon > 0$. We want to choose δ such that

$$|x^2 - 4| < \epsilon \quad \text{whenever} \quad |x - 2| < \delta$$

Choose $\delta = \min\{1, \epsilon/5\}$. Assume $|x - 2| < \delta$. Then, since $\delta \leq 1$, then $|x + 2| \leq 5$. Also, we know that

$$|x - 2| < \frac{\epsilon}{5} \quad \text{therefore} \quad |x - 2| < \frac{\epsilon}{|x + 2|}$$

and therefore

$$|x - 2||x + 2| < \epsilon \quad \text{or} \quad |x^2 - 4| < \epsilon.$$

Definition of continuity

A function f is continuous at a point c if

$$\lim_{x \to c} f(x) = f(c).$$

Suppose we have function $f : D \to \mathbb{R}$. We say f is continuous if it is continuous at every point in D.

Suppose we have a continuous function $f : \mathbb{R} \to \mathbb{R}$. This means that for every $c \in \mathbb{R}$, and for every $\epsilon > 0$, there exists a $\delta > 0$ such that for all x, if $|x - c| < \delta$ then $|f(x) - f(c)| < \epsilon$.

Theorem A function $f : \mathbb{R} \to \mathbb{R}$ is continuous if and only if for all open sets $U \subset \mathbb{R}$, $f^{-1}(U)$ is an open set.

Recall that for a function f, the preimage of a set U is

$$f^{-1}(U) = \{x \in \mathbb{R} : f(x) \in U\}.$$

E.g., if $f(x) = x^2$, and $U = (1, 2)$, $f^{-1}(U) = (1, \sqrt{2}) \cup (-\sqrt{2}, -1)$

The theorem says that if a function is not continuous, we should be able to find an open set U such that $f^{-1}(U)$ is not open. For example, consider $f(x) = [x]$. We want to find an open set U such that $f^{-1}(U)$ is not closed. Choose $U = (-1/2, 1/2)$, then $f^{-1}(U) = [0, 1)$.

To prove this theorem, we will use the fact that a set U is open if and only if for every $x \in U$, there exists an $\epsilon > 0$ such that $(x - \epsilon, x + \epsilon) \subset U$.

Proof of Theorem

We want to prove the following are equivalent, given a function $f : \mathbb{R} \to \mathbb{R}$:

1) For every $c \in \mathbb{R}$, and for every $\epsilon > 0$, there exists a $\delta > 0$ such that for all $x \in \mathbb{R}$, if $|x - c| < \delta$ then $|f(x) - f(c)| < \epsilon$.

2) For all open sets $U \subset \mathbb{R}$, $f^{-1}(U)$ is an open set.

Proof that (1) implies (2). Assume (1). Let U be an open subset of $\mathbb{R}$. We want to show that $f^{-1}(U)$ is an open set. Therefore we must show that for all $c \in f^{-1}(U)$, there exists an $\delta > 0$ such that $(c-\delta, c+\delta) \subset f^{-1}(U)$.

Let $c \in f^{-1}(U)$. Then $f(c) \in U$. Since U is open, there exists an $\epsilon > 0$ such that $(f(c)-\epsilon, f(c)+\epsilon) \subset U$. Since f is continuous, there exists a $\delta > 0$ such that for all x, if $|x-c| < \delta$ then $|f(x) - f(c)| < \epsilon$.

We claim that this δ is the δ we are looking for from the first paragraph. That is, we claim that $(c-\delta, c+\delta) \subset f^{-1}(U)$. That is, we claim that for every x such that $|x-c| < \delta$, we have $f(x) \in U$. This is because $|f(x) - f(c)| < \epsilon$ and $(f(c)-\epsilon, f(c)+\epsilon) \subset U$.

Proof that (2) implies (1). Assume (2), i.e, that for all open sets $U \subset \mathbb{R}$, $f^{-1}(U)$ is an open set. Let $c \in \mathbb{R}$ and let $\epsilon > 0$. We want to show (1).

Let $U = (f(c) - \epsilon, f(c) + \epsilon)$. Since $f^{-1}(U)$ is open, and contians x_1. There exists a $\delta > 0$ such that $(c - \delta, c + \delta) \subset f^{-1}(U)$. This is just a reformulation of (1).

Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function.

Theorem 1 (Extreme Value Theorem) On any closed bounded interval $[a, b]$, a continuous function $f : \mathbb{R} \to \mathbb{R}$ has a maximum and a minimum.

Theorem 2(Intermediate Value Theorem) If $f(a) < y < f(b)$, then there exists x such that $a < x < b$ such that $f(x) = y$.

Theorem 3 If $U \subset \mathbb{R}$ is a compact subset, and $f : \mathbb{R} \to \mathbb{R}$ is a continuous function, then $f(U)$ is also compact.

First let's see how Theorem 3 implies Theorem 1. By the Heine-Borel theorem, a set is compact if and only if it is closed and bounded.

Theorem 3 implies Theorem 1 Suppose we know that Theorem 3 is true. Let $[a, b]$ be a closed bounded interval and let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function. By the Heine-Borel theorem, $[a, b]$ is compact. So $f([a, b])$ is compact. Then $f([a, b])$ is closed and bounded. So it has an upper bound, so it has a least upper bound, and since it's closed, the least upper bound is a maximum. Similarly for the minimum.

Theorem 3 If $U \subset \mathbb{R}$ is a compact subset, and $f : \mathbb{R} \to \mathbb{R}$ is a continuous function, then $f(U)$ is also compact.

Proof Let U be a compact subset of $\mathbb{R}$ and let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. We want to show that $f(U)$ is compact. In other words, for every cover of $f(U)$, we want to show that there exists a finite subcover. Let $\mathcal{F}$ be a cover of $f(U)$. Define a cover $\mathcal{G}$ of U, by

$$\mathcal{G} = \{f^{-1}(V) : V \in \mathcal{F}\}.$$

We claim (1) that $\mathcal{G}$ is an open cover of U. Then $\mathcal{G}$ has a finite subcover, say $\{f^{-1}(V_1), f^{-1}(V_2), \ldots, f^{-1}(V_N)\}$. We claim (2) that $\{V_1, V_2, \ldots, V_N\}$ covers $f(U)$, and so is a finite subcover of $\mathcal{F}$.

Proof of claim (1). So we want to show (a) that every element $x \in U$ is contained in one fo the sets in $\mathcal{G}$, and (b) that every set in $\mathcal{G}$ is open. The second condition follows from Theorem Friday. So let's prove (a). So let $x \in U$. Thyen $f(x) \in f(U)$. Then there exists a set $V \in \mathcal{F}$ such that $f(x) \in V$. Then $x \in f^{-1}(V)$.

The proof of claim (2) is left as an exercise.