THE GLOBAL GAN–GROSS–PRASAD CONJECTURE FOR FOURIER–JACOBI PERIODS ON UNITARY GROUPS

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ABSTRACT. We prove the Gan–Gross–Prasad conjecture for Fourier–Jacobi periods on unitary groups and an Ichino–Ikeda type refinement. Our strategy is based on the comparison of relative trace formulae formulated by Liu. We develop the full coarse spectral and geometric expansions of the relative trace formulae, and compute relevant spectral terms via zeta integrals and truncated periods. We compare all geometric terms and characterize the local geometric comparison in terms of spectral data.

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Part 1. Introduction

1. Statement of the main results

In the 1990s, Gross and Prasad [GP92, GP94] formulated some conjectures on the restriction problems for orthogonal groups. These conjectures were later extended to all classical groups by Gan, Gross and Prasad in their book [GGP12]. Ichino and Ikeda [II10] gave a refinement of the conjecture of Gross and Prasad, and this refinement was further extended to other cases [Har14, Liu16, Xue16, Xue17]. These conjecture are usually referred to as the Gan–Gross–Prasad (GGP) conjectures and have attracted significant amount of research. These conjectures describe the relation between certain period integrals of automorphic forms on classical groups and the central values of some L-functions.

There are two kinds of period integrals in the GGP conjecture: Bessel periods and Fourier–Jacobi periods. Fourier–Jacobi periods usually involve a theta functions while Bessel periods do not. Specifying to the case of unitary groups, Bessel periods are on the unitary groups $\mathrm{U}(n) \times \mathrm{U}(m)$ where $n+m$ is odd, while the Fourier–Jacobi case is when $n+m$ is even. The cases of Bessel periods on unitary groups, together with their refinements, are now completely settled, by the combination of the work of many people, cf. [JR11, Yun11, Zha14a, Zha14b, Xue19, Zyd20, BP21a, BP21b, CZ21, BPLZZ21, BPCZ22, BPC] for an incomplete list. The proof follows the approach of Jacquet and Rallis via the comparison of relative trace formulae.

Inspired by the work of Jacquet and Rallis, Liu [Liu14] proposed a relative trace formula approach towards the Fourier–Jacobi case of the conjecture. Following this approach, we previously in [Xue14, Xue16] proved some cases of the GGP conjecture for $\mathrm{U}(n) \times \mathrm{U}(n)$ under various local conditions. The goal of this paper is to work out the comparison of these relative trace formulae in general, and prove the GGP conjecture for Fourier–Jacobi periods on unitary groups. We prove the following results in this manuscript.

(1) The global GGP conjecture for $\mathrm{U}(n) \times \mathrm{U}(m)$, where $n+m$ is even, as stated in [GGP12, Conjecture 26.1].
(2) An exact analogue of a conjecture of Ichino and Ikeda stated in [II10, Conjecture 1.4] in the context of Fourier–Jacobi periods, which is a refinement of the global GGP conjecture.

1.1. Arthur parameters and weak base change.

Let E/F be a quadratic extension of number fields, and $\mathbb{A}_E$ and $\mathbb{A}$ be their rings of adeles respectively. Denote by c the nontrivial element in the Galois group $\mathrm{Gal}(E/F)$. For any positive integer k, set $G_k := \mathrm{Res}_{E/F}\,\mathrm{GL}_k$, where $\mathrm{Res}_{E/F}$ is the Weil restriction of scalars. Fix a nontrivial additive character $\psi : F\backslash\mathbb{A} \to \mathbb{C}^\times$. Let $\eta : F^\times\backslash\mathbb{A}^\times \to \{\pm 1\}$ be the quadratic character attached to the extension E/F via global class field theory. Let $\mu : E^\times\backslash\mathbb{A}_E^\times \to \mathbb{C}^\times$ be a character such that $\mu|_{\mathbb{A}^\times} = \eta$.

1.1.1. *Regular Hermitian Arthur parameters.*

Let Π be an irreducible automorphic representation of $G_n(\mathbb{A})$. Let $\Pi^\vee$ be the contragredient of Π, and Π^{c} be the automorphic representation of $G_n(\mathbb{A})$

whose space of realizations is given by $\{\varphi(g^{c}) \mid \varphi \in \Pi\}$. We put $\Pi^{*} = \Pi^{\vee,c}$ and say that Π is conjugate self-dual if $\Pi \simeq \Pi^{*}$. Following [BPC, Section 1.1], we shall say that an irreducible automorphic representation Π of $G_n(\mathbb{A})$ is a *discrete Hermitian Arthur parameter* of G_n if there are a partition $n = n_1 + \cdots + n_r$ and irreducible cuspidal automorphic representations π_i of $G_{n_i}(\mathbb{A})$ for $i = 1, \dots, r$, such that

- Π is isomorphic to the full induced representation $\mathrm{Ind}_P^{G_n}(\pi_1 \otimes \dots \otimes \pi_r)$ where P is a parabolic subgroup of G_n with Levi factor $G_{n_1} \times \dots \times G_{n_r}$,
- π_i is conjugate self-dual with central character trivial on $A^{\infty}_{G_{n_i}}$ (the neutral component of the $\mathbb{R}$-points of the maximal $\mathbb{Q}$-split torus of $\mathrm{Res}_{F/\mathbb{Q}} G_{n_i}$) and the Asai L-function $L(s, \pi_i, \mathrm{As}^{(-1)^{n+1}})$ has a pole at $s = 1$ for all $1 \leq i \leq r$,
- the representations π_i are mutually non-isomorphic for $1 \leq i \leq r$.

The integer r and the automorphic representations $(\pi_i)_{1 \leq i \leq r}$ are unique (up to permutation), and we define the group

$$S_{\Pi} = (\mathbb{Z}/2\mathbb{Z})^{r}. \tag{1.1}$$

We will also need a more general notion of Arthur parameters. We shall say that an irreducible automorphic representation Π of $G_n(\mathbb{A})$ is a *regular Hermitian Arthur parameter* of G_n if

- Π is isomorphic to the full induced representation $\mathrm{Ind}_P^{G_n}(\pi_1 \otimes \dots \otimes \pi_r \otimes \Pi_0 \otimes \pi_r^{*} \otimes \dots \otimes \pi_1^{*})$ where P is a parabolic subgroup of G_n with Levi factor $G_{n_1} \times \dots \times G_{n_r} \times G_{n_0} \times G_{n_r} \times \dots \times G_{n_1}$ and $n_0 + 2(n_1 + \dots + n_r) = n$,
- Π_0 is a discrete Hermitian Arthur parameter of G_{n_0},
- π_i is an irreducible cuspidal automorphic representation of $G_{n_i}(\mathbb{A})$, with central character trivial on $A^{\infty}_{G_{n_i}}$ for $1 \leq i \leq r$,
- the representations $\pi_1, \dots, \pi_r, \pi_1^{*}, \dots, \pi_r^{*}$ are mutually non-isomorphic.

The representation Π_0 is then uniquely determined by Π, and is called the *discrete component* of Π. We set

$$S_{\Pi} = S_{\Pi_0}. \tag{1.2}$$

The parabolic P depends on an ordering of the representations $\pi_1, \dots, \pi_r, \pi_r^{*}, \dots, \pi_1^{*}$, and we fix one. Denote by $\mathfrak{a}_{P,\mathbb{C}}$ the complex vector space of unramified characters of $P(\mathbb{A})$. Consider the real subspaces $\mathfrak{a}_P$ and $i\mathfrak{a}_P$ of real and unitary characters respectively. Let w be the permutation matrix that exchanges the blocks G_{n_i} corresponding to π_i and π_i^{*} for all $1 \leq i \leq r$. Set

$$\mathfrak{a}_{\Pi,\mathbb{C}} := \{\lambda \in \mathfrak{a}_{P,\mathbb{C}} \mid w\lambda = -\lambda\}, \tag{1.3}$$

and $i\mathfrak{a}_{\Pi} := \mathfrak{a}_{\Pi,\mathbb{C}} \cap i\mathfrak{a}_P$. For any $\lambda \in \mathfrak{a}_{\Pi,\mathbb{C}}$, consider the full induced representation

$$\Pi_{\lambda} := \mathrm{Ind}_P^{G_n}((\pi_1 \otimes \dots \otimes \pi_r \otimes \Pi_0 \otimes \pi_r^{*} \otimes \dots \otimes \pi_1^{*}) \otimes \lambda).$$

If $\lambda \in i\mathfrak{a}_{\Pi}$, then Π_{λ} is irreducible.

We have similar notions for products of groups. Let n and m be integers, and set $G = G_n \times G_m$. A *G-regular Hermitian Arthur parameter* of $G_n \times G_m$ is an irreducible automorphic representation of the form $\Pi = \Pi_n \boxtimes \Pi_m$ where Π_n and Π_m are regular Hermitian Arthur parameters of G_n and G_m respectively. If Π_n and Π_m are discrete, we shall say that Π is *discrete*, while if only Π_n is discrete, Π shall be *semi-discrete*.

In the corank zero case, we will be interested in parameters satisfying an additional regularity condition. Assume that $n = m$. Let $\Pi = \Pi_n \boxtimes \Pi'_n$ be a G-regular Hermitian Arthur parameter. Define $i\mathfrak{a}_{\Pi} = i\mathfrak{a}_{\Pi_n} \oplus i\mathfrak{a}_{\Pi'_n}$. Write $\Pi_n = \mathrm{Ind}_{P_n}^{G_n}(\pi_1 \boxtimes \dots \boxtimes \pi_r)$ and $\Pi'_n = \mathrm{Ind}_{P'_n}^{G_n}(\pi'_1 \boxtimes \dots \boxtimes \pi'_{r'})$, for some parabolic subgroups $P_n, P'_n \subset G_n$ and some automorphic representations of smaller G_l. We shall say that Π is *(G, H, η^{-1})-regular* if for all $1 \leq i \leq r$ and $1 \leq j \leq r'$ the representation $\eta^{-1}\pi_i$ is not isomorphic to the contragredient of π'_j.

Remark 1.1. This condition is inspired by [BPC]. In this paper, H will be used to denote the diagonal subgroup $G_n \subset G$, and (H, η^{-1})-regular will refer to the fact that these Arthur parameters have nice properties with respect to the regularized Rankin–Selberg period over H (see Section 10). Note that a semi-discrete Hermitian Arthur parameters is necessarily (G, H, η^{-1})-regular, as otherwise some $L(s, \pi_i, \mathrm{As}^{(-1)^l})$ would have a pole at $s = 1$ for $l = n, n+1$ which is not possible.

1.1.2. *Weak base change.* We now consider weak base change from unitary groups. Let (V, q_V) be a nondegenerate skew c-Hermitian vector space, or simply a skew-hermitian space, of dimension n over E. This means that V is a vector space over E of dimension n, and q_V is a sesquilinear form on V, linear in the first variable, and satisfies $q_V(x, y) = -q_V(y, x)^{\mathsf{c}}$. Let $\mathrm{U}(V)$ be the corresponding unitary group, i.e. the group of isometries of V. Let $Q \subset \mathrm{U}(V)$ be a parabolic subgroup with Levi subgroup M_Q isomorphic to $G_{n_1} \times \dots \times G_{n_r} \times \mathrm{U}(V_0)$ where V_0 is a nondegenerate subspace of V. Let σ be an irreducible cuspidal automorphic representation of $M_Q(\mathbb{A})$, with central character trivial on A_Q^{∞}. We write $\sigma = \tau_1 \boxtimes \dots \boxtimes \tau_r \boxtimes \sigma_0$ according to this decomposition. Then σ_0 is called the *discrete component* of σ.

We shall say that a regular Hermitian Arthur parameter Π of G_n is a *weak base change* of (Q, σ) if there exists a parabolic subgroup P of G_n with Levi factor $M_P = G_{n_1} \times \dots \times G_{n_r} \times G_{n_0} \times G_{n_r} \times \dots \times G_{n_1}$ and a discrete Hermitian Arthur parameter Π_0 of G_{n_0} such that

- Π is isomorphic to the full induced representation $\mathrm{Ind}_P^{G_n}(\tau_1 \boxtimes \dots \boxtimes \tau_r \boxtimes \Pi_0 \boxtimes \tau_r^{*} \boxtimes \dots \boxtimes \tau_1^{*})$,
- for almost all places v of F that split in E, the local component $\Pi_{0,v}$ is the split local base change of $\sigma_{0,v}$.

This implies that Π_0 is the discrete component of Π. If this is satisfied, we shall also say that Π_0 is the weak base change of σ_0. Moreover, we can identify $i\mathfrak{a}_{\Pi}$ with the space of unitary unramified characters of $Q(\mathbb{A})$ and consider the full induced representation $\Sigma_\lambda := \mathrm{Ind}_Q^{\mathrm{U}(V)}(\sigma \otimes \lambda)$.

We extend weak base change to product of groups. Let V and W be nondegenerate skew c-Hermitian vector spaces of dimension n and m respectively. Let $Q = Q_V \times Q_W$ be a parabolic subgroup of $\mathrm{U}(V) \times \mathrm{U}(W)$, and $\sigma = \sigma_V \boxtimes \sigma_W$ be an irreducible cuspidal automorphic representation of $M_Q(\mathbb{A})$. We shall say that Π is a weak base change of (Q, π) if Π_n and Π_m are respectively weak base changes of (Q_V, π_V) and (Q_W, π_W).

1.2. **Fourier–Jacobi periods in corank zero.**

1.2.1. *Fourier–Jacobi periods.* Let (V, q_V) be a nondegenerate n-dimensional skew c-Hermitian space, with skew-Hermitian form q_V. Let $\operatorname{Res} V$ be the F-symplectic space whose underlying vector space is V viewed as a F-vector space, and whose symplectic pairing is given by $\operatorname{Tr}_{E/F} \circ q_V$. Let $\operatorname{Res} V = L + L^\vee$ be a polarization, i.e. L and $L^\vee$ are maximal isotropic subspaces of V such that the pairing $\operatorname{Tr}_{E/F} \circ q_V$ is nondegenerate when restricted to $L \otimes L^\vee$. We have a Weil representation $\omega = \omega_{\psi,\mu}$, realized on the space of Schwartz functions $S(L^\vee(\mathbb{A}))$, cf. Subsection 5.4. It depends on the characters ψ and μ. We denote by $\omega^\vee = \omega_{\psi^{-1},\mu^{-1}}$ the dual representation of ω which we also realize on $S(L^\vee(\mathbb{A}))$. For $\phi \in S(L^\vee(\mathbb{A}))$ we may form the theta function

$$\theta(g, \phi) = \sum_{x \in L^\vee(F)} (\omega(g)\phi)(x), \quad g \in \mathrm{U}(V)(\mathbb{A}).$$

We define similarly $\theta^\vee(g, \phi)$ when $\omega^\vee$ is used.

Put $\mathrm{U}_V := \mathrm{U}(V) \times \mathrm{U}(V)$ and $\mathrm{U}_V' := \mathrm{U}(V)$, viewed as a subgroup of U_V by the diagonal embedding. When the space V in question is clear from the content, we drop the subscript. Let $Q = M_Q N_Q$ be a parabolic subgroup of U_V and π be a cuspidal automorphic representation of $M_Q(\mathbb{A})$ with central character trivial on A_Q^∞. Denote by $\mathcal{A}_{Q,\pi}(\mathrm{U}_V)$ the space of automorphic forms φ on the quotient $A_Q^\infty M_Q(F) N_Q(\mathbb{A}) \backslash \mathrm{U}_V(\mathbb{A})$ such that for every $g \in \mathrm{U}_V(\mathbb{A})$ the function

$$m \in M_Q(\mathbb{A}) \mapsto \delta_Q(m)^{-\frac{1}{2}} \varphi(mg)$$

belongs to π, where δ_Q is the modular character of $Q(\mathbb{A})$. For $\varphi \in \mathcal{A}_{Q,\pi}(\mathrm{U}_V)$ and $\lambda \in i\mathfrak{a}_Q^*$ we can form the Eisenstein series $E(\varphi, \lambda)$. Let $\phi \in S(L^\vee(\mathbb{A}))$. We introduce a regularized period

$$P(\varphi, \phi, \lambda) := \int_{[\mathrm{U}_V']} \Lambda_u^T \left(E(\cdot, \varphi, \lambda) \theta^\vee(\cdot, \phi) \right)(h) \mathrm{d}h. \tag{1.4}$$

Here $[\mathrm{U}_V'] := \mathrm{U}_V'(F) \backslash \mathrm{U}_V'(\mathbb{A})$ is equipped with the Tamagawa measure (see Subsection 3.4), and the theta function $\theta^\vee$ instead of θ is used here for compatibility with the choice in [GGP12]. The operator Λ_u^T is a variant of the truncation operator introduced by Ichino and Yamana in [IY19]. It is defined in Section 8.5 and depends on a parameter T. The integral is absolutely convergent by Proposition 8.7. If the weak base change of (Q, π) is a (G, H, μ^{-1})-regular Hermitian Arthur parameter Π, it is in fact independent of T by Proposition 11.1. Moreover, if Π is semi-discrete (that is if $\pi = \pi_1 \boxtimes \pi_2$ with π_1 or π_2 being a cuspidal automorphic representation of $\mathrm{U}(V)(\mathbb{A})$), the regularized period simplifies to the absolute convergent integral

$$P(\varphi, \phi, \lambda) = \int_{[\mathrm{U}_V']} E(h, \varphi, \lambda) \theta^\vee(h, \phi) \mathrm{d}h.$$

The linear form $(\varphi, \varphi') \mapsto P(\varphi, \varphi', \lambda)$ belongs to

$$\mathrm{Hom}_{\mathrm{U}'_V(\mathbb{A})}(\pi \otimes \omega^\vee, \mathbb{C}).$$

Thus in order to have a nonzero Fourier–Jacobi period P, this Hom space has to be nonzero in the first place. By the multiplicity one theorems [Sun12, SZ12] its dimension is at most one.

1.2.2. *The Gan–Gross–Prasad conjecture for Fourier–Jacobi periods.* The first main result of this paper is the following, which proves a slight generalization of [GGP12, Conjecture 26.1] for $\mathrm{U}(n) \times \mathrm{U}(n)$. If $\Pi = \Pi_1 \boxtimes \Pi_2$ is a Hermitian Arthur parameter of $G = G_n \times G_n$ and $\lambda = (\lambda_1, \lambda_2) \in i\mathfrak{a}_\Pi^* = i\mathfrak{a}_{\Pi_1}^* \oplus i\mathfrak{a}_{\Pi_2}^*$, we put for $s \in \mathbb{C}$

$$L(s, \Pi_\lambda \otimes \mu^{-1}) = L(s, \Pi_{1,\lambda_1} \times \Pi_{2,\lambda_2} \otimes \mu^{-1}).$$

Theorem 1.2. *Let Π be a (G, H, μ^{-1})-regular Hermitian Arthur parameter of G and let $\lambda \in i\mathfrak{a}_\Pi^*$. Then the following are equivalent.*

(1) *The complete Rankin–Selberg L-function of Π satisfies*

$$L(\tfrac{1}{2}, \Pi_\lambda \otimes \mu^{-1}) \neq 0$$

(2) *There exist a nondegenerate n-dimensional skew $\mathfrak{c}$-Hermitian space V, a parabolic subgroup $Q \subset \mathrm{U}_V$ and an irreducible cuspidal automorphic representation σ of $M_Q(\mathbb{A})$, such that Π is the weak base change of (Q, σ) and the Fourier–Jacobi period*

$$\varphi \otimes \phi \mapsto P(\varphi, \phi, \lambda)$$

does not vanish identically on $\mathcal{A}_{Q,\sigma}(\mathrm{U}_V) \otimes \omega^\vee$.

Remark 1.3. The L-function appearing in the first condition is the completed L-function. The current results towards the Ramanujan conjecture are not enough to eliminate certain poles in the local L-factors.

1.2.3. *Local Fourier–Jacobi periods.* We keep V to be a nondegenerate n-dimensional skew $\mathfrak{c}$-Hermitian space, Q to be a parabolic subgroup of U_V with Levi M_Q and σ to be an irreducible cuspidal automorphic representation of $M_Q(\mathbb{A})$. Take $\lambda \in i\mathfrak{a}_\Pi^*$.

We now consider local objects. Write the restricted tensor product decomposition $\sigma = \bigotimes'_v \sigma_v$. Assume that for all v the local component σ_v is tempered. Set $\Sigma_\lambda := \mathrm{Ind}_Q^{\mathrm{U}(V)}(\sigma \otimes \lambda)$. Let $\Sigma_{\lambda,v} = \mathrm{Ind}_Q^{\mathrm{U}_V} \sigma_v \otimes \lambda$ and $\Pi_{\lambda,v}$ be the local components of Σ_λ and Π_λ. Write $\omega^\vee = \bigotimes'_v \omega^\vee_v$ the decomposition of $\omega^\vee$ into local components.

We endow $L^\vee(\mathbb{A})$ with the Tamagawa measure $\mathrm{d}x$ and we fix a factorization $\mathrm{d}x = \prod \mathrm{d}x_v$. For each place v we give the representation $\omega^\vee_v$ the invariant inner product

$$\langle \phi_v, \phi'_v \rangle_{L^\vee,v} = \int_{L^\vee(F_v)} \phi_v(x_v) \overline{\phi'_v(x_v)} \mathrm{d}x_v, \quad \phi_v, \phi'_v \in \omega^\vee_v.$$

We give the space $\mathcal{A}_{Q,\sigma}(\mathrm{U}_V)$ the Petersson inner product

$$\langle \varphi, \varphi' \rangle_{\mathrm{Pet}} := \int_{A_Q^\infty M_Q(F) N_Q(\mathbb{A}) \backslash \mathrm{U}_V(\mathbb{A})} \varphi(h)\overline{\varphi'(h)}\mathrm{d}h, \quad \varphi, \varphi' \in \mathcal{A}_{Q,\sigma}(\mathrm{U}_V),$$

where $A_Q^\infty M_Q(F) N_Q(\mathbb{A}) \backslash \mathrm{U}_V(\mathbb{A})$ is given the quotient of the Tamagawa measure. We take a factorization $\langle .,. \rangle_{\mathrm{Pet}} = \prod_v \langle .,. \rangle_v$ where $\langle .,. \rangle_v$ is an invariant inner product of $\Sigma_{\sigma,v}$.

We take a factorization of the Tamagawa measure $\mathrm{d}h = \prod \mathrm{d}h_v$, and for $\varphi_v \in \Sigma_{\sigma,v}$ and $\phi_v \in \omega_{\overline{\psi}}$ we define the local period

$$P_v(\varphi_v, \phi_v, \psi) := \int_{\mathrm{U}_V^0(F_v)} \langle \Sigma_{\sigma,v}(h_v)\varphi_v, \varphi_v \rangle_v \langle \omega_{\overline{\psi}}(h_v)\phi_v, \phi_v \rangle_{L^-,v} \mathrm{d}h_v.$$

As $\Sigma_{\sigma,v}$ is tempered, this integral is absolutely convergent by Lemma 20.1.

1.2.4. *Factorization of Fourier–Jacobi periods.* We now state a refinement of Theorem 1.2 which is a version of a conjecture of Ichino and Ikeda ([II10]) in our setting.

We keep Q and σ as in the previous section, and furthermore assume that the weak base change Π of (Q,σ) to G is a (G, H, μ^{-1})-regular Hermitian Arthur parameter. Set

$$L(s, \Sigma_\sigma) = \left(s - \frac{1}{2}\right)^{-\dim \mathfrak{a}_\Pi} \prod_{i=1}^{n} L\left(i + s - \frac{1}{2}, \eta^i\right) \frac{L(s, \Pi \otimes \mu^{-1})}{L(s+\frac{1}{2}, \Pi, \mathrm{As}'_G)},$$

where $L(s, \eta^i)$ is the completed Hecke L-function associated to η^i and $L(s, \Pi, \mathrm{As}'_G)$ is the Asai L-function associated to $\mathrm{As}^{(-1)^n} \boxtimes \mathrm{As}^{(-1)^n}$.

Remark 1.4. If π_1 and π_2 are irreducible automorphic representations of $G_{n_1}(\mathbb{A})$ and $G_{n_2}(\mathbb{A})$ respectively with $n_1 + n_2 = n$, and if P is a parabolic subgroup of G_n with Levi subgroup $G_{n_1} \times G_{n_2}$, then we have the induction formula

$$L(s, \mathrm{Ind}_P^{G_n}(\pi_1 \boxtimes \pi_2), \mathrm{As}^{(-1)^n}) = L(s, \pi_1 \times \pi_2^c) L(s, \pi_1, \mathrm{As}^{(-1)^n}) L(s, \pi_2, \mathrm{As}^{(-1)^n}). \tag{1.5}$$

By [BPCZ22, Section 4.1.2], if π is an automorphic cuspidal representation which is non-isomorphic to $\pi^\vee$, then $L(s, \pi, \mathrm{As}^{(-1)^n})$ is regular at $s = 1$. Moreover, if Π_0 is a discrete Hermitian Arthur parameter of G_{n_0} with $n - n_0$ even, induction on (1.5) and Remark 1.1 show that $L(s, \Pi_0, \mathrm{As}^{(-1)^n})$ is also regular at $s = 1$. Therefore, our hypotheses on our Hermitian Arthur parameter Π imply the meromorphic function $L(s, \Pi, \mathrm{As}'_G)$ has a pole of order $\dim \mathfrak{a}_\Pi$ at $s = 1$. In particular, $L(s, \Sigma_\sigma)$ is holomorphic at $s = \frac{1}{2}$.

We denote by $L(s, \Sigma_{\sigma,v})$ the corresponding local L-factor, so that for $\Re(s)$ large enough we have

$$L(s, \Sigma_\sigma) = \left(s - \frac{1}{2}\right)^{-\dim \mathfrak{a}_\Pi} \prod_v L(s, \Sigma_{\sigma,v}),$$

where the product runs over the places v of F. As $\Sigma_{\sigma,v}$ is tempered, $L(s, \Sigma_{\sigma,v})$ has neither a pole nor a zero at $s = \frac{1}{2}$. We now define for $\varphi_v \in \Sigma_{\sigma,v}$ and $\phi_v \in \omega_{\overline{\psi}}$ the local normalized period

$$P_v^\natural(\varphi_v, \phi_v, \psi) := L\left(\frac{1}{2}, \Sigma_{\sigma,v}\right)^{-1} P_v(\varphi_v, \phi_v, \psi).$$

Theorem 1.5. *Let Π and $(Q;\ )$ be as above. Assume that the weak base change Π of $(Q;\ )$ to G is a $(G; H;\ ^{1})$-regular Hermitian Arthur parameter. For $\ \in i\mathfrak{a}_{\Pi}$ and every non-zero and factorizable $' = \otimes'_v \, '_v \in \mathcal{A}_{Q;\ }(\mathrm{U}_V)$ and $\ = \otimes'_v \ _v \in \ $, we have*

$$(1.6) \qquad |P(';\ ;\ )|^2 = |S_\Pi|^{-1} L(\tfrac{1}{2};\Sigma\ ) \prod_v P^{\natural}_v('_v;\ _v;\ ).$$

Remark 1.6. It follows from [Xue16, Appendix D] and our choice of measures and of local inner products that almost all factors in the RHS are equal to 1, noting that for any finite place v we have $L(s;\Pi_{\ ;v};\mathrm{As}'_G) = L(s;\Sigma_{1;\ ;v};\mathrm{Ad})L(s;\Sigma_{2;\ ;v};\mathrm{Ad})$.

Remark 1.7. In the core of the paper, we will use a slightly different normalization of the Tamagawa measure (see Subsection 3.4 and (19.2)). This is accounted for by the factor $\prod_{i=1}^{n} L(i;\ ^{i})$ in the definition of L.

1.3. **Fourier–Jacobi periods in positive corank.** Throughout this section we fix n and m two integers such that $n = m + 2r$ with r a positive integer.

1.3.1. *Jacobi groups.* Let $(V; q_V)$ be a n-dimensional nondegenerate skew c-Hermitian space over E/F and $W \subset V$ be a non degenerate subspace of dimension m. Denote by $\mathrm{U}(V)$ and $\mathrm{U}(W)$ the groups of isometries of V and W respectively. Define

$$U_W := \mathrm{U}(V) \times \mathrm{U}(W).$$

Assume that the orthogonal $W^{\perp}$ of W in V is split. This means that there exists a basis $\{x_i; x_{-i} \mid i = 1;\dots;r\}$ of $W^{\perp}$ such that for all $1 \leq i; j \leq r$ we have

$$q_V(x_i; x_j) = 0;\ q_V(x_{-i}; x_{-j}) = 0;\ q_V(x_i; x_{-j}) = \ _{i;j}.$$

Set $X = \mathrm{span}_E(x_1;\dots;x_r)$ and $X^{\vee} = \mathrm{span}_E(x_{-r};\dots;x_{-1})$, so that $V = X \oplus W \oplus X^{\vee}$. This determines an embedding $\mathrm{U}(W) \subset \mathrm{U}(V)$.

Let $S(W) = \mathrm{Res}\, W \times F$ be the Heisenberg group of $\mathrm{Res}\, W$ (see Subsection 5.3). As $r \geq 1$, there exists an embedding $h : S(W) \to \mathrm{U}(V)$ described in (20.2). Define the Jacobi group

$$J(W) := S(W) \rtimes \mathrm{U}(W);$$

where $U(W)$ acts on $\mathrm{Res}\, W$. Then we have an embedding $J(W) \subset \mathrm{U}(V)$. Let U_{r-1} be the unipotent radical of the parabolic subgroup of $\mathrm{U}(V)$ stabilizing the flag $\mathrm{span}_E(x_1) \subset \mathrm{span}_E(x_1; x_2) \subset \dots \subset \mathrm{span}_E(x_1;\dots;x_{r-1})$. Define the Fourier–Jacobi group

$$H := U_{r-1} \rtimes J(W) \subset U_W;$$

where the embedding is the product of inclusion $H \subset \mathrm{U}(V)$ and the projection $H \to \mathrm{U}(W)$.

1.3.2. *Fourier–Jacobi models.* Let ω_W be the Weil representation of $\mathrm{U}(W)(\mathbb{A})$ associated to the characters and , and denote by $\omega_{\overline{W}}$ its dual. Take a polarization $\operatorname{Res} W = Y \quad Y^{\vee}$ and a realization of $\omega_{\overline{W}}$ on the Schwartz space $S(Y^{\vee}(\mathbb{A}))$. Denote by the Heisenberg realization of the Heisenberg group $J(W)(\mathbb{A})$ associated to the character , and $^{\vee}$ its dual (see Subsection 5.4). It is also realized on $S(Y^{\vee}(\mathbb{A}))$.

Define a algebraic morphism $: U_{r-1} \to \mathbb{G}_a$ by

$$(u) = \mathrm{Tr}_{E/F}\left(\sum_{i=1}^{r-1} q_V(u(x_{i+1}), x_i)\right), \quad u \in U_{r-1}.$$

The action of $J(W)$ by conjugation on U_{r-1} is trivial on , so that we may extend to $H = U_{r-1} \rtimes J(W)$. Consider the character

$$_U(h) := ((h)), \quad h \in H.$$

Define the representation $^{\vee}$ of $H(\mathbb{A})$ by the rule

$$^{\vee}(uhg_W) = \overline{}_U(u) \,^{\vee}(h)\,\omega_{\overline{W}}(g_W), \quad u \in U_{r-1}(\mathbb{A}),\ g_W \in \mathrm{U}(W)(\mathbb{A}),\ h \in S(W)(\mathbb{A}).$$

It is realized on $S(Y^{\vee}(\mathbb{A}))$. Consider the theta series

$$_{\overline{H}}(h, \ ) = \sum_{y \in Y^{\vee}(F)} (^{\vee}(h) \ )(y), \quad h \in H(\mathbb{A}), \quad \in \ ^{\vee}.$$

Remark 1.8. By generalizing these constructions to $r = 0$ (i.e. by taking $H = \mathrm{U}(W)$) we find ourselves in the corank zero situation described in Subsection 1.2.1.

Remark 1.9. In [GGP12, Section 12], Fourier–Jacobi models are defined by a pair $(H, \ )$, where H is a subgroup of U_W and a representation of H. In our setting, H is H and $^{\vee}$ is the dual of .

1.3.3. *Fourier–Jacobi periods.* Set $[H] := H(F)\backslash H(\mathbb{A})$. Let be a cuspidal automorphic representation of $U_W(\mathbb{A})$. For $\varphi \in$ and $\in \ ^{\vee}$, we define the Fourier–Jacobi period to be the absolutely convergent integral

$$P_{[H]}(\varphi, \ ) = \int_{[H]} \varphi(h)\, _{\overline{H}}(h, \ )\,\mathrm{d}h.$$

1.3.4. *The Gan–Gross–Prasad conjecture for Fourier–Jacobi periods in arbitrary corank.* We now state our second main theorem concerning the GGP conjecture for Fourier–Jacobi periods in arbitrary corank.

Theorem 1.10. *Let Π be a discrete Hermitian Arthur parameter of $G_n \quad G_m$. The following assertions are equivalent.*

(1) *The complete Rankin–Selberg L-function of Π satisfies*

$$L(\frac{1}{2}, \Pi \quad ^{1}) \neq 0$$

(2) *There exist nondegenerate skew* c*-Hermitian spaces* $W \subset V$ *of respective dimensions* m *and* n *and* σ *a cuspidal automorphic representation of* $U_W(\mathbb{A})$ *such that* $W^\perp$ *is split, the weak base change of* σ *to* $G_n \times G_m$ *is* Π *and the Fourier–Jacobi period*

$$\varphi \otimes \phi \mapsto P_{[H]}(\varphi, \phi)$$

does not vanish identically on $\sigma \otimes \omega^\vee$.

1.3.5. *Local Fourier-Jacobi periods in arbitrary corank.* Let $W \subset V$ be nondegenerate skew c-Hermitian spaces $W \subset V$ of respective dimensions m and n such that $W^\perp$ is split. Let σ be an irreducible cuspidal automorphic representation of $U_W(\mathbb{A})$.

Write factorizations $\sigma = \bigotimes'_v \sigma_v$ and $\omega^\vee = \bigotimes'_v \omega^\vee_v$, and assume that for every v the local component σ_v is tempered. As in Subsection 1.2.3, we equip $Y^\vee(\mathbb{A})$ with the Tamagawa measure $\mathrm{d}y = \prod \mathrm{d}y_v$ and define an invariant inner product on $\omega^\vee_v$ by

$$\langle \phi_v, \phi'_v \rangle_{Y^\vee,v} := \int_{Y^\vee(F_v)} \phi(y_v)\overline{\phi'(y_v)}\mathrm{d}y_v, \quad \phi, \phi' \in \omega^\vee_v.$$

We equip $[U_W]$ and $[H]$ with their respective quotient of Tamagawa measures $\mathrm{d}g$ and $\mathrm{d}h$, and take factorizations $\mathrm{d}g = \prod \mathrm{d}g_v$ and $\mathrm{d}h = \prod \mathrm{d}h_v$. We give σ the Petersson inner-product $\langle .,. \rangle_{\mathrm{Pet}}$ defined by

$$\langle \varphi, \varphi' \rangle_{\mathrm{Pet}} := \int_{[U_W]} \varphi(h)\overline{\varphi'(h)}\mathrm{d}h, \quad \varphi, \varphi' \in \sigma,$$

and write a factorization $\langle .,. \rangle_{\mathrm{Pet}} = \prod_v \langle .,. \rangle_v$.

For $f_v \in C_c^\infty(H(F_v))$ (the space of compactly supported smooth function on $H(F_v)$) and $\phi_v \in \omega^\vee_v$ define the local period

$$P_{H,v}(f_v, \phi_v) := \int_{H(F_v)} f_v(h_v)\langle \omega^\vee_v(h_v)\phi_v, \phi_v \rangle_{Y^\vee,v}\mathrm{d}h_v.$$

It follows from Lemma 20.1 that for fixed ϕ_v, the linear form $f_v \mapsto P_{H,v}(f_v, \phi_v)$ extends by continuity to the local Harish-Chandra Schwartz space $\mathcal{C}(H(F_v))$ (see Subsection 20.2.1). As σ_v is assumed to be tempered, its matrix coefficients belong to $\mathcal{C}(H(F_v))$ so that we can define the local Fourier–Jacobi period to be

$$P_{H,v}(\varphi_v, \phi_v) := P_{H,v}(\langle \sigma_v(\cdot)\varphi_v, \varphi_v \rangle_v, \phi_v), \quad \varphi_v \in \sigma_v, \quad \phi_v \in \omega^\vee_v.$$

1.3.6. *Factorization of Fourier–Jacobi periods.* Our last main result is a refinement of Theorem 1.10.

Assume that Π the weak base change of σ to $G_n \times G_m$ is a discrete Hermitian parameter. Consider the product of completed global L-function

$$L(s, \sigma) = \prod_{i=1}^{m} L(i + s - \frac{1}{2}, \eta^i)\frac{L(s, \Pi \otimes \mu^{-1})}{L(s + \frac{1}{2}, \Pi, \mathrm{As}'_{n,m})},$$

with $\mathrm{As}^{0}_{n;m} = \mathrm{As}^{(\ 1)^n} \quad \mathrm{As}^{(\ 1)^m}$, and its local counterpart $L(s; \ _v)$. It follows from Remark 1.4 that $L(s; \ )$ is regular at $s = \frac{1}{2}$. We define the normalized local Fourier–Jacobi period to be

$$P^{\natural}_{H;v}(\,'_v; \ _v) := L(\frac{1}{2}; \ _v) \ ^1 P_{H;v}(\,'_v; \ _v):$$

It follows from Proposition 21.2 that if $' = \ ^{0}_{v}\,'_v$ and $\ = \ ^{0}_{v}\ _v$ are factorizable vectors, then $P^{\natural}_{H;v}(\,'_v; \ _v) = 1$ for almost all v.

Theorem 1.11. *Let and Π be as above. For every factorizable vectors* $' = \ ^{0}_{v}\,'_v \ 2$ *and* $\ = \ ^{0}_{v}\ _v\ 2$ *– we have*

$$jP_{[H]}(\,'; \ )j^2 = jS_\Pi j \ ^1 L(\frac{1}{2}; \ ) \prod_v P^{\natural}_{H;v}(\,'_v; \ _v): \tag{1.7}$$

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2. The relative trace formulae

We take the approach of the relative trace formulae proposed in [Liu14] to prove our main results. This approach involves the following major steps.

(1) Define the modified automorphic kernels, and develop the coarse geometric and spectral expansions of the relative trace formulae. This is the content of Part 2.
(2) Explicitly compute the relevant spectral contributions in terms of the relative characters. This is the content of Part 3.
(3) Compare the geometric sides of the relative trace formulae. This includes the (possibly singular) transfer of test functions and the fundamental lemma. It also involves a comparison of local relative characters and a spectral characterization of transfer. This is the content of Part 4.
(4) Derive the main theorems from the comparison of relative trace formulae. In the corank zero case, Theorem 1.2 and Theorem 1.5 follow rather directly from the comparison, via the spectral isolation technique from [BPLZZ21]. The higher corank situation of Theorem 1.10 and Theorem 1.11 is reduced to the corank zero case by (both local and global) unfoldings. This is the content of Part 5.

We explain in this section the formalism of the relative trace formulae, and compare it with the previous work on the relative trace formulae of Jacquet and Rallis.

2.1. **The relative trace formulae of Jacquet and Rallis.** We first briefly explain the relative trace formulae of Jacquet and Rallis in this subsection.

Let W_n be a nondegenerate n-dimensional c-Hermitian space, and $W_{n+1} = W_n \oplus E$ an $(n+1)$-dimensional c-Hermitian space. Here we write E for the one-dimensional c-Hermitian space with the skew Hermitian form $(x, y) \mapsto xy^{\mathsf{c}}$. This determines an embedding $\mathrm{U}(W_n) \hookrightarrow \mathrm{U}(W_{n+1})$. Let π_i be an irreducible cuspidal automorphic representation of $\mathrm{U}(W_i)$, $i = n, n+1$. The Gan–Gross–Prasad conjecture in this setup studies the nonvanishing properties of the periods integral

$$\int_{[\mathrm{U}(W_n)]} \varphi_n(h)\varphi_{n+1}(h)\mathrm{d}h, \quad \varphi_n \in \pi_n, \quad \varphi_{n+1} \in \pi_{n+1}. \tag{2.1}$$

Jacquet and Rallis proposed in [JR11] the following relative trace formula to attack this conjecture. Let $f_i \in \mathcal{S}(\mathrm{U}(W_i)(\mathbb{A}))$, $i = n, n+1$, be test functions and set $f = f_n \otimes f_{n+1} \in \mathcal{S}(\mathrm{U}(W_n)(\mathbb{A}) \times \mathrm{U}(W_{n+1})(\mathbb{A}))$. Consider the distribution

$$J(f) = \iint_{[\mathrm{U}(W_n)]^2} K_f(h_1, h_2)\mathrm{d}h_1\mathrm{d}h_2, \tag{2.2}$$

where $K_f(g_1, g_2)$ is the usual automorphic kernel function given by

$$K_f(g_1, g_2) = \sum_{\gamma \in \mathrm{U}(W_n)(F) \times \mathrm{U}(W_{n+1})(F)} f(g_1^{-1}\gamma g_2), \quad g_1, g_2 \in [\mathrm{U}(W_n)] \times [\mathrm{U}(W_{n+1})].$$

In the ideal situation, the distribution $J(f)$ decomposes as a sum of period integrals of the form (2.1).

The distribution (2.2) decomposes geometrically into a sum of orbital integrals. Put

$$\varphi(g) = \int_{\mathrm{U}(W_n)(\mathbb{A})} f(h^{-1}, h^{-1}g)\mathrm{d}h.$$

Then formally we have

$$(2.2) = \sum_{\gamma \in \mathrm{U}(W_{n+1})(F) /\!/ \mathrm{U}(W_n)(F)} \int_{\mathrm{U}(W_n)(\mathbb{A})} \varphi(h^{-1}\gamma h)\mathrm{d}h,$$

where $\mathrm{U}(W_{n+1})(F)/\!/\mathrm{U}(W_n)(F)$ stands for the orbits in $\mathrm{U}(W_{n+1})(F)$ under the conjugate action of $\mathrm{U}(W_n)(F)$. This is not absolutely convergent in general and some regularization process is needed.

The orbital integrals appearing in this geometric expansion decompose into product of local orbital integrals. In order to better understand these local orbital integrals, Jacquet and Rallis introduced an infinitesimal variant. Let v be a place of F. Let $\mathfrak{u}(W_{n+1})$ be the Lie algebra of $\mathrm{U}(W_{n+1})$, with receives a conjugation by $\mathrm{U}(W_n)$. Let $\xi \in \mathfrak{u}(W_{n+1})(F_v)$ and $\varphi \in \mathcal{S}(\mathfrak{u}(W_{n+1})(F_v))$ the local Schwartz space. The infinitesimal orbital integral equals

$$\int_{\mathrm{U}(W_n)(F_v)} \varphi(h^{-1}\xi h)\mathrm{d}h. \tag{2.3}$$

It turns out that infinitesimal variants will also show up in our relative trace formula approach in this paper.

2.2. **The relative trace formulae of Liu.** We now introduce the the relative trace formulae we use in this paper. Let the notation be as in Subsection 1.2. Put $\mathrm{U}_V = \mathrm{U}(V) \times \mathrm{U}(V)$ and U_V^0 the diagonal subgroup of U_V. Inspired by the work of Jacquet and Rallis, Liu proposed the following relative formulae in [Liu14] to attack the Gan–Gross–Prasad conjecture for Fourier–Jacobi periods. If $f \in S(\mathrm{U}_V(\mathbb{A}))$ and $\phi_1, \phi_2 \in S(L^\vee(\mathbb{A}))$, we consider the distribution

$$J(f \otimes \phi_1 \otimes \phi_2) = \iint_{[\mathrm{U}_V^0]^2} K_f(h_1, h_2)\, \theta^\vee(h_1, \phi_1)\, \theta(h_2, \phi_2)\, \mathrm{d}h_1 \mathrm{d}h_2, \tag{2.4}$$

where

$$K_f(g_1, g_2) = \sum_{\gamma \in \mathrm{U}_V(F)} f(g_1^{-1} \gamma g_2), \quad g_1, g_2 \in \mathrm{U}_V(\mathbb{A})$$

is the usual automorphic kernel function on the group U_V. The idea of considering this distribution is clear: in the ideal situation (e.g. V is anisotropic), the sum and integrals are absolutely convergent and it decomposes as

$$\sum_{\pi} \sum_{\varphi \in \pi} \mathcal{P}(\pi(f)\varphi, \phi_1) \overline{\mathcal{P}(\varphi, \phi_2)}, \tag{2.5}$$

where π runs over cuspidal automorphic representations of $\mathrm{U}_V(\mathbb{A})$ and φ runs through an orthonormal basis of π. In general, the integral is not convergent, and some truncation process is needed.

The distribution J also decomposes geometrically. The appearance of the Weil representation makes this less straightforward. To achieve the geometric decomposition, Liu [Liu14] introduced a partial Fourier transform

$$S(L^\vee(\mathbb{A})) \otimes S(L^\vee(\mathbb{A})) \mapsto S(V(\mathbb{A})), \quad \phi_1 \otimes \phi_2 \mapsto (\phi_1 \otimes \phi_2)^\ddagger,$$

which has the properties that

$$(\omega^\vee(g)\phi_1 \otimes \omega(g)\phi_2)^\ddagger(v) = (\phi_1 \otimes \phi_2)^\ddagger(g^{-1}v), \quad g \in \mathrm{U}(V)(\mathbb{A}), \quad v \in V(\mathbb{A}).$$

If we manipulate the distribution J formally, it then decomposes as follows. Consider the algebraic variety $\mathrm{U}(V) \times V$ with an action of $\mathrm{U}(V)$ given by

$$g \cdot (\delta, v) = (g \delta g^{-1}, gv).$$

Define a function $\varphi \in S(\mathrm{U}(V)(\mathbb{A}) \times V(\mathbb{A}))$ by

$$\varphi(g, v) = \int_{\mathrm{U}(V)(\mathbb{A})} f(h^{-1}, h^{-1}g)(\omega^\vee(h)\phi_1 \otimes \phi_2)^\ddagger(v).$$

Then we have

$$J(f \otimes \phi_1 \otimes \phi_2) = \sum_{(\delta, v)} \int_{\mathrm{U}(V)(\mathbb{A})} \varphi(h^{-1}ah, h^{-1}v)\mathrm{d}h,$$

where (δ, v) ranges over all the orbits of $\mathrm{U}(V)(F) \times V(F)$ under the action of $\mathrm{U}(V)(F)$. The orbital integrals appearing in this sum decomposes, and they again have infinitesimal variants which turn out to be essentially the same as the ones (2.3) introduced by Jacquet and Rallis. The above manipulation is purely formal. The sum and integrals are not absolutely convergent in general, and some truncation process is needed.

2.3. **Jacobi groups and truncation.** The Fourier–Jacobi periods (1.4) are best understood via automorphic forms on Jacobi groups, though no Jacobi groups are explicitly mentioned in the definition. It also turns out that the reformulation of the relative trace formula using this interpretation largely helps us find a good way to truncate the distribution (2.4).

Recall that we have defined the Heisenberg group $S(V)$ and the Jacobi group $J(V) = S(V) \rtimes \mathrm{U}(V)$ in Subsection 1.3. Write an irreducible cuspidal automorphic representation π of $\mathrm{U}_V(\mathbb{A})$ as $\pi_1 \boxtimes \pi_2$ where π_1, π_2 are irreducible cuspidal automorphic representations of $\mathrm{U}(V)(\mathbb{A})$. For $\varphi_1 \in \pi_1$, $\varphi_2 \in \pi_2$ and $\phi \in \mathcal{S}(L^\vee(\mathbb{A}))$, the period integral (1.4) takes the form

$$\int_{[\mathrm{U}(V)]} \varphi_1(h)\varphi_2(h)\theta^\vee(h,\phi)\mathrm{d}h. \tag{2.6}$$

The observation is that the product of the form φ_2 and $\theta^\vee(\cdot,\phi)$ gives an automorphic form on $J(V)(\mathbb{A})$. This is a function on $J(V)(\mathbb{A})$, left invariant by $J(V)(F)$, and we temporarily denote it by φ_2^J. Automorphic forms on $J(V)$ will be referred to as Jacobi forms. In the case where the underlying reductive part is SL_2, it really gives the classical Jacobi forms. We refer the readers to [BS98, Xue18] for more discussions of the classical Jacobi forms and their connection with automorphic representations. The integral (2.6) is then written as

$$\int_{[\mathrm{U}(V)]} \varphi_1(h)\varphi_2^J(h)\mathrm{d}h,$$

and can be viewed as a pairing between the restriction of a Jacobi form on $J(V)$ and an automorphic form on $\mathrm{U}(V)(\mathbb{A})$. This integral and the periods integral (2.1) take a very similar shape.

With this interpretation in mind, we look back at the distribution (2.4) and write it in terms of Jacobi groups. This will help us find a way to truncate this distribution. Put $\widetilde{\mathrm{U}}_V = \mathrm{U}(V) \times J(V)$. For the test functions $f \in \mathcal{S}(\mathrm{U}_V(\mathbb{A}))$ and $\phi_1, \phi_2 \in \mathcal{S}(L^\vee(\mathbb{A}))$ we define a function on $\widetilde{\mathrm{U}}_V(\mathbb{A})$ by

$$\tilde{f}(g_1, \tilde{g}_2) = f(g_1, g_2)\langle \omega^\vee(\tilde{g}_2)\phi_1, \overline{\phi_2}\rangle, \quad g_1 \in \mathrm{U}(V)(\mathbb{A}),\ \tilde{g}_2 \in J(V)(\mathbb{A}),$$

and the image of $\tilde{g}_2$ in $\mathrm{U}(V)(\mathbb{A})$ is g_2. Write $Z \simeq F$ for the center of $J(V)$. If $\tilde{g}_2 = ((v,z), g_2)$ where $v \in V(\mathbb{A}), z \in Z(\mathbb{A})$ and $g_2 \in \mathrm{U}(V)(\mathbb{A})$, then

$$\langle \omega^\vee(\tilde{g}_2)\phi_1, \overline{\phi_2}\rangle = \psi^{-1}(z)(\omega^\vee(g_2)\phi_1 \otimes \overline{\phi_2})^{z}(v).$$

The function $\tilde{f}$ is a Schwartz function on $\widetilde{\mathrm{U}}_V(\mathbb{A})$ in a suitable sense, cf. Subsection 4.3, and we can form a kernel function

$$K_{\tilde{f}}(g_1, g_2) = \sum_{\gamma \in \widetilde{\mathrm{U}}_V(F)/Z(F)} \tilde{f}(g_1^{-1}\gamma g_2), \quad g_1, g_2 \in \widetilde{\mathrm{U}}_V(\mathbb{A}).$$

Note that if $h_1, h_2 \in \mathrm{U}_V^0(\mathbb{A})$, by the Poisson summation formulae we have

$$K_{\tilde{f}}(h_1, h_2) = K_f(h_1, h_2)\theta^\vee(h_1, \phi_1)\overline{\theta^\vee(h_2, \phi_2)}.$$

Thus the distribution J can be written as

$$J(f \otimes \phi_1 \otimes \phi_2) = \iint_{[\mathrm{U}_V^0]^2} K_{\tilde{f}}(h_1, h_2) \mathrm{d}h_1 \mathrm{d}h_2.$$

Let $L^2([\widetilde{\mathrm{U}}_V], \psi)$ be the square integrable functions (modulo Z) on $[\widetilde{\mathrm{U}}_V]$ which satisfies $\varphi(zg) = \psi(z)\varphi(g)$ for all $z \in Z(\mathbb{A})$ and $g \in \widetilde{\mathrm{U}}_V(\mathbb{A})$. The function $\tilde{f}$ acts on this space via the right translation. A little computation gives that this right translation is given by the kernel function $K_{\tilde{f}}$. Therefore the distribution J and the one (2.2) introduced by Jacquet and Rallis take the same shape. When formulated this way, the distribution J decomposes geometrically and simplifies in the same way as the case of Jacquet and Rallis. This also give a different viewpoint of the Fourier transform ^{z} introduced in [Liu14].

With the help of this reformulation, we also get a way to truncate the distribution J. In the case of the relative trace formulae of Jacquet and Rallis, i.e. the distribution (2.2), a truncation has been introduced by [Zyd20]. It takes the form (same notation as in (2.2))

$$\iint_{[\mathrm{U}(W_n)]^2} \sum_P \epsilon_P \sum_{\gamma_1, \gamma_2 \in P_n(F) \backslash \mathrm{U}(W_n)(F)} \widehat{\tau}(\cdots) K_{f,P}(\gamma_1 h_1, \gamma_2 h_2) \mathrm{d}h_1 \mathrm{d}h_2. \tag{2.7}$$

Here P ranges over all standard parabolic subgroups of $\mathrm{U}(W_n) \times \mathrm{U}(W_{n+1})$ which are of the form $P = P_n \times P_{n+1}$ such that $P_{n+1} \cap \mathrm{U}(W_n) = P_n$, the kernel $K_{f,P}$ is a variant of the kernel K_f for the parabolic subgroup P, ϵ_P is a sign, and $\widehat{\tau}(\cdots)$ is a characteristic function which is not relevant to our discussion here.

By studying the analogy between the distribution J and (2.2), we arrive at the following truncation for J. We first introduce "parabolic subgroups" for the Jacobi group $J(V)$, and hence for the group $\widetilde{\mathrm{U}}_V$. Note that there is a definition of a parabolic subgroups for general linear algebraic groups, but our "parabolic subgroups" are not the same as them. We call them the D-parabolic subgroups, where "D" stands for "dynamical". The truncation of J then takes the form

$$\iint_{[\mathrm{U}_V^0]^2} \sum_P \epsilon_P \sum_{\gamma_1, \gamma_2 \in P_1(F) \backslash \mathrm{U}_V^0(F)} \widehat{\tau}(\cdots) K_{\tilde{f},P}(\gamma_1 h_1, \gamma_2 h_2) \mathrm{d}h_1 \mathrm{d}h_2. \tag{2.8}$$

Here P ranges over all standard parabolic subgroups of $\widetilde{\mathrm{U}}_V$ of the form $P = P_1 \times P_J$ such that $P_J \cap \mathrm{U}(V) = P_1$, and $K_{\tilde{f},P}$ is a variant of $K_{\tilde{f}}$ for P, ϵ_P is a sign, and $\widehat{\tau}(\cdots)$ is some characteristic function. One key ingredient in proving the convergence of (2.8) is the "approximation by constant term for Jacobi groups", which we prove in Theorem 4.25 (for more general groups).

Both the truncated distributions (2.7) and (2.8) can be studied via the method of descent. This method allows us to connect these distributions to infinitesimal variants of them. Miraculously, like what has been observe for orbital integrals, the infinitesimal variants of these distributions are essentially the same. This allows us to transport many known deep results on the geometric side of (2.7) to the geometric side of (2.8).

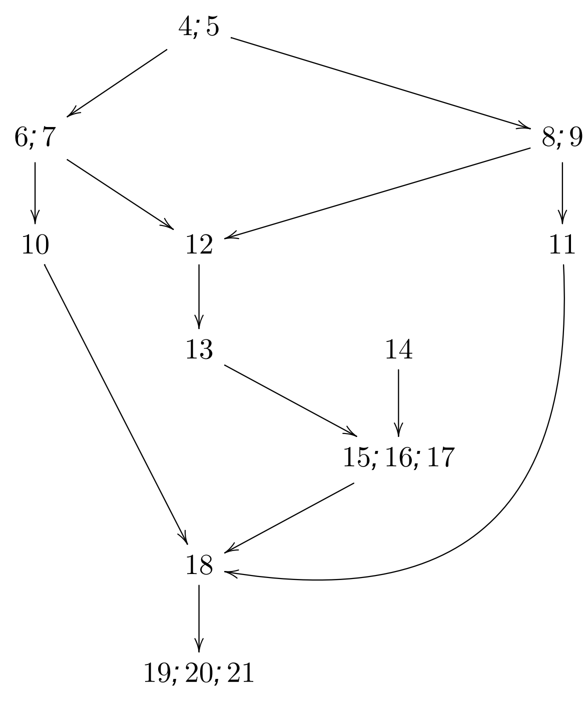
The above discussion was essentially our initial attempt in defining the truncation for J. It later turned out that the terms $K_{\tilde{f},P}$ can be expressed in a way that does not mention explicitly the

Jacobi groups, but only some partial theta functions on the unitary groups. This is better, as in the course of establishing the spectral decomposition for J, this new formulation avoids the discussion of the Langlands decomposition for Jacobi groups. This alternative expression is eventually the one that we use in this paper.

We remark that to prove the main theorems in this paper, we need to compare the distribution J with another distribution I on general linear groups. Similar considerations also apply to that distribution.

2.4. **Organization of this paper.** This paper is divided into several parts. In Part 2, we introduce the relative trace formulae on both unitary groups and general linear groups, and work out their coarse spectral and geometric expansions. In Part 3, spectral terms in the relative trace formulae satisfying some "regularity" conditions are computed. In Part 4, we carry out the comparison of the relative trace formulae, both globally and locally. In particular we establish the fundamental lemma, the transfer, the spectral characterization of the transfer and the singular transfer. Finally in Part 5, we assemble all these results to prove the main theorems.

The dependence of the sections is summarized in the following Leitfaden.



3. Notation and conventions

This section contains some notation which will be used throughout the paper. Specific sets of notation will be fixed in each part.

3.1. **General notation.** If R is a ring, R^n will denote the set of n dimensional column vectors with coefficients in R, and R_n the set of row vectors.

If f and g are functions on some space X, we write $f \ll g$ if there is a constant C such that $f(x) \leq Cg(x)$ for all $x \in X$. We say f and g are equivalent, denoted by $f \sim g$, if $f \ll g$ and $g \ll f$.

Let X be a measurable space. We denote by $\langle \cdot, \cdot \rangle_{L^2}$ the L^2-inner product and $\| \cdot \|_{L^2}$ the L^2-norm. We sometimes also write $\langle \cdot, \cdot \rangle_X$ to emphasize the space X.

If G is a group and f is a complex valued function on G. We put $f^\vee(g) = f(g^{-1})$ and $f^*(g) = \overline{f(g^{-1})}$.

When G is a group and $\mathcal{F}$ is a space of functions on G which is invariant by right (resp. left) translation, we denote by R (resp. L) the corresponding representation of G on $\mathcal{F}$. If G is a Lie group and the representation is differentiable, we will also denote by the same letter the induced action of the Lie algebra or of its associated enveloping algebra. If G is a topological group equipped with a bi-invariant Haar measure, we denote by $*$ the convolution product of functions on G (whenever it is well-defined).

Most of the topological vector spaces in this paper will be Banach, Hilbert, Fréchet, or LF spaces. By an LF space, we mean a Hausdorff space which can be expressed as a countable direct limit of Fréchet spaces. Note that it is not necessarily complete. If $V = \varinjlim_n V_n$ is an LF space, we say that it is a strict LF space if the maps $V_n \to V_{n+1}$ are all closed embeddings. Strict LF spaces are complete, cf. [Trè67, Theorem 13.1] (note the LF spaces in [Trè67] are by definition what we call strict LF spaces). The uniform boundedness principle and the closed graph theorem hold for LF spaces, cf. [BPCZ22, Appendix A]. We use the notation $\widehat{\otimes}$ to denote the projective completed tensor product of two locally convex topological vector spaces.

3.2. **Fields.** We fix a quadratic extension of number fields E/F unless otherwise specified. We denote by c the nontrivial Galois conjugation in $\mathrm{Gal}(E/F)$. Let E^- be the purely imaginary elements in E, i.e. $E^- = \{x \in E \mid x^{\mathrm{c}} = -x\}$, and fix a nonzero $\tau \in E^-$. We denote by $\mathbb{A}$ and $\mathbb{A}_E$ the ring of adeles of F and E respectively. We write $\mathbb{A}_f$ the ring of finite adeles of $\mathbb{A}$. If $\mathtt{S}$ is a finite set of places of F, we set $F_{\mathtt{S}} := \prod_{v \in \mathtt{S}} F_v$. When $\mathtt{S} = V_{F,\infty}$ is the set of Archimedean places of F, we also write $F_\infty := F_{V_{F,\infty}}$.

We denote by $|\cdot|$ and $|\cdot|_E$ the normalized absolute values on $\mathbb{A}^\times$ and $\mathbb{A}_E^\times$ respectively. They satisfy $|x|_E = |xx^{\mathrm{c}}|$ for all $x \in \mathbb{A}_E^\times$, and in particular $|x|_E = |x|^2$ if $x \in \mathbb{A}^\times$.

We fix a nontrivial additive character ψ of $F \backslash \mathbb{A}$, and let ψ^E and ψ_E be the nontrivial additive characters of $\mathbb{A}_E$ given respectively by

$$\psi^E(x) = \psi(\mathrm{Tr}_{E/F}(\tau x)), \quad \psi_E(x) = \psi(\mathrm{Tr}_{E/F}\, x).$$

Let η be the quadratic character of $F^\times \backslash \mathbb{A}^\times$ associated to the extension E/F by global class field theory. We let μ be an automorphic character of $\mathbb{A}_E^\times$ whose restriction to $\mathbb{A}^\times$ is η.

3.3. **Groups.** We denote by $\mathbb{G}_a$ and $\mathbb{G}_m$ the additive group and the multiplicative group over F respectively. If F'/F is a field extension, we denote by $G_{F'}$ the extension of scalars $G \times_F F'$. We denote by $\mathfrak{g} = \mathrm{Lie}(G)$ the Lie algebra of G (over F). We write G_∞ for $G(F_\infty)$, $\mathfrak{g}_\infty$ for $\mathrm{Lie}(G_\infty) \otimes_{\mathbb{R}} \mathbb{C}$,

$U(\mathfrak{g}_\infty)$ for the universal enveloping algebra of $\mathfrak{g}_\infty$ and $Z(\mathfrak{g}_\infty)$ for its center. If T is a torus over F, we let T_1 be the maximal split torus of $\mathrm{Res}_{F/\mathbb{Q}} T$ and T^∞ be the neutral component of $T_1(\mathbb{R})$.

3.3.1. *Hermitian spaces and unitary groups.* Let V be a vector space over E. We take the convention that a $\mathtt{c}$-Hermitian or skew $\mathtt{c}$-Hermitian form q_V on V is linear in the first variable and anti-linear in the second variable. A vector space with a nondegenerate $\mathtt{c}$-Hermitian or skew $\mathtt{c}$-Hermitian form is called a $\mathtt{c}$-Hermitian or skew $\mathtt{c}$-Hermitian space (or simply Hermitian or skew-Hermitian). For fixed n, we denote by $\mathcal{H}$ the set of all isomorphism classes of nondegenerate skew-Hermitian spaces over E of dimension n. If v is a place of F, we denote by $\mathcal{H}_v$ the set of all isomorphism classes of nondegenerate skew-Hermitian spaces over $E_v = E \otimes_F F_v$ of dimension n. More generally, for a finite set $\mathtt{S}$ of places, we denote by $\mathcal{H}_{\mathtt{S}}$ the set of isomorphism classes of nondegenerate skew-Hermitian spaces over $E_{\mathtt{S}} = E \otimes_F F_{\mathtt{S}}$ of dimension n. We also denote by $\mathcal{H}^{\mathtt{S}}$ the set of $V \in \mathcal{H}$ such that $V \otimes_E E_v$ has a self-dual lattice for all non-Archimedean $v \notin \mathtt{S}$.

If V is a Hermitian or skew-Hermitian space of dimension n, we choose a basis $v_1, \ldots, v_n$ and put

$$\operatorname{disc} V = \det(q_V(v_i, v_j))_{1 \le i,j \le n} \in E^\times.$$

If V is Hermitian then $\operatorname{disc} V \in F^\times$, while if V is skew-Hermitian then $\tau^n \operatorname{disc} V \in F^\times$. The image of $\operatorname{disc} V$ (when V is Hermitian) or $\tau^n \operatorname{disc} V$ (when V is skew-Hermitian) in $F^\times/\mathrm{Nm}_{E/F}(E^\times)$ is independent of the choice of the basis.

3.4. **Measures.** For every place v of F, let $\mathrm{d}_{\psi_v} x_v$ be the unique Haar measure on F_v which is autodual with respect to ψ_v the local component of the additive character ψ of $\mathbb{A}$.

Let G be a connected linear group over F. The choice of a right-invariant rational volume form ω on G together with the measure $\mathrm{d}_{\psi_v} x_v$ determine a right invariant Haar measure $\mathrm{d}_{\psi_v} g_v$ on each $\mathrm{G}(F_v)$ ([Wei82]). By [Gro97], there is an Artin–Tate L-function $L_{\mathrm{G}}(s) = \prod_v L_{\mathrm{G},v}(s)$, and more generally for $\mathtt{S}$ a finite set of places its partial counterpart $L_{\mathrm{G}}^{\mathtt{S}}(s) = \prod_{v \notin \mathtt{S}} L_{\mathrm{G},v}(s)$. Define Δ_{G}^* and $\Delta_{\mathrm{G}}^{\mathtt{S},*}$ to be the leading coefficient of the Laurent expansion at $s = 0$ of $L_{\mathrm{G}}(s)$ and $L_{\mathrm{G}}^{\mathtt{S}}(s)$ respectively. For each place v, set $\Delta_{\mathrm{G},v} := L_{\mathrm{G},v}(0)$. We equip $\mathrm{G}(\mathbb{A})$ with the Tamagawa measure $\mathrm{d}g$ defined as $\mathrm{d}g = \mathrm{d}g_{\mathtt{S}} \times \mathrm{d}g^{\mathtt{S}}$ where $\mathrm{d}g_{\mathtt{S}} = \prod_{v \in \mathtt{S}} \mathrm{d}_{\psi_v} g_v$ and $\mathrm{d}g^{\mathtt{S}} = (\Delta_{\mathrm{G}}^{\mathtt{S},*})^{-1} \prod_{v \notin \mathtt{S}} \Delta_{\mathrm{G},v} \mathrm{d}_{\psi_v} g_v$. Note that for any model of G over $\mathcal{O}_F^{\mathtt{S}}$ we have for almost all v

$$\operatorname{vol}(\mathrm{G}(\mathcal{O}_v), \mathrm{d}_{\psi_v} g_v) = \Delta_{\mathrm{G},v}^{-1}. \tag{3.1}$$

Although the $\mathrm{d}_{\psi_v} g_v$ depend on various choices, the Tamagawa measure $\mathrm{d}g$ does not. Note that if G is a unipotent group, the measure we just picked satisfies $\operatorname{vol}([\mathrm{G}]) = 1$.

If $\mathrm{G} = \mathrm{GL}_n$, we will take the form

$$\omega = (\det g)^{-n} \bigwedge_{1 \le i,j \le n} \mathrm{d}g_{i,j},$$

so that (3.1) is satisfied for every non-Archimedean place v of F where ψ_v is unramified.

Part 2. Coarse relative trace formulae

4. Preliminaries

4.1. **Groups, parabolic subgroups.** Let G be a connected linear algebraic group over a number field F. We introduce a class of subgroups of G which will play the role of parabolic subgroups for reductive groups. The group G we have in mind is either a reductive group, or a Jacobi group which will be introduced in Section 5.

By a theorem of Mostow [Con14, Proposition 5.4.1], the group G can be written in the form $U \rtimes H$, where H is reductive and U is the unipotent radical of G. The group H is then called a Levi subgroup of G. We fix such a decomposition, choose a maximal split torus A_0 of H and choose $P_0 \subset H$ a minimal parabolic subgroup that contains A_0. We say that a parabolic subgroup of H is semistandard if it contains A_0, and say that it is standard if it contains P_0. Note that A_0 is also a maximal split torus of G. Denote by $X^*(A_0)$ the group of rational characters of A_0, and by $X_*(A_0) = \mathrm{Hom}_{\mathbb{Z}}(X^*(A_0), \mathbb{Z})$ the groups of rational cocharacters of A_0. Denote by $\langle\cdot,\cdot\rangle : X^*(A_0) \times X_*(A_0) \to \mathbb{Z}$ the canonical pairing between these groups.

The torus A_0 has an adjoint action on the Lie algebra $\mathfrak{g}$ of G, thus a root-space decomposition

$$\mathfrak{g} = \bigoplus_{\alpha \in \Phi_G \subset X^*(A_0)} \mathfrak{g}_\alpha,$$

where $\mathfrak{g}_\alpha$ satisfies

$$\mathfrak{g}_\alpha = \{x \in \mathfrak{g} \mid \mathrm{Ad}(a)x = \alpha(a)(x) \text{ for all } a \in A_0\}.$$

The set Φ_G consists of those $\alpha \in X^*(A_0)$ such that $\mathfrak{g}_\alpha \neq 0$. For $\alpha \in \Phi_G$, $\mathfrak{g}_\alpha$ is called a root space of $\mathfrak{g}$. We say that G has the property (Symmetric Roots), or (SR) for short, if the following condition holds:

$$\text{(SR)} \qquad \alpha \in \Phi_G \iff -\alpha \in \Phi_G.$$

As maximal split tori in H are conjugate to each other, the condition (SR) is independent of the choice of A_0. Moreover, the Levi subgroups of G are conjugate so that (SR) also does not depend on the choice of H. Note that (SR) holds for all connected reductive groups.

A subsemigroup of $X^*(A_0)$ is by definition a subset Γ of $X^*(A_0)$ closed under addition. We recall the following root subgroup construction from [CGP10, Proposition 3.3.6, Theorem 3.3.11].

Proposition 4.1. *For any subsemigroup $\Gamma \subset X^*(A_0)$, there exists a unique connected F-subgroup $H_\Gamma(G)$ of G such that*

$$\mathrm{Lie}(H_\Gamma(G)) = \bigoplus_{\alpha \in \Gamma \cap \Phi_G} \mathfrak{g}_\alpha.$$

Moreover, we have the following properties.

(1) *$H_\Gamma(G)$ is stable under conjugation by A_0.*
(2) *If $0 \notin \Gamma$, then $H_\Gamma(G)$ is unipotent.*

Moreover, assume that G is solvable and suppose that there is a disjoint union decomposition $\Phi_G = \sqcup_{i=1}^n \Phi_i$ such that Φ_i is disjoint from the semigroup Γ_j generated by Φ_j whenever $i \neq j$. Then the map induced by group multiplication

$$H_{\Gamma_1}(G) \times \cdots \times H_{\Gamma_n}(G) \to G$$

is an isomorphism of F-schemes.

Remark 4.2. Since $\mathrm{char}(F) = 0$, a connected subgroup is determined by its Lie algebra. Therefore, we do not need the A_0-stable condition in [CGP10, Proposition 3.3.6] but have it as a property instead.

Let λ be a cocharacter of A_0. It determines three subsemigroups of $X^*(A_0)$ defined by the equations $\{\langle \alpha, \lambda \rangle \geq 0\}$, $\{\langle \alpha, \lambda \rangle > 0\}$ and $\{\langle \alpha, \lambda \rangle = 0\}$ respectively. The resulting root subgroups are denoted by $P(\lambda)$, $U(\lambda)$ and $Z(\lambda)$ respectively. They are the dynamical subgroups described in [Spr09, Section 13.4]. We have the semidirect product decomposition

$$P(\lambda) = Z(\lambda) \ltimes U(\lambda). \tag{4.1}$$

Lemma 4.3. *If G has the property (SR), then the decomposition (4.1) only depends on $P(\lambda)$ and not on λ.*

Proof. $P(\lambda)$ is determined by the set of roots

$$S := \{\alpha \in \Phi_G \mid \langle \alpha, \lambda \rangle \geq 0\}.$$

If G has the property (SR), then for $\alpha \in S$, we have $\langle \alpha, \lambda \rangle > 0 \iff -\alpha \notin S$. This characterization is independent of the choice λ, which concludes the proof.

Remark 4.4. Lemma 4.3 does not hold in general without the condition (SR). The mirabolic subgroup in GL_3 is a counterexample.

From now on we will always assume that G has the property (SR).

We say that a subgroup of the form $P(\lambda)$ for so $\lambda \in X_*(A_0)$ is a (semistandard) ***D-parabolic subgroup***, and we call the decomposition (4.1) the ***D-Levi decomposition*** of $P(\lambda)$. When $P = P(\lambda)$, we denote $M_P := Z(\lambda)$, $N_P := U(\lambda)$, $\mathfrak{m}_P := \mathrm{Lie}(M_P)$ and $\mathfrak{n}_P := \mathrm{Lie}(N_P)$. By Proposition 4.1 (2), N_P is always unipotent. When there is no confusion, we will omit the subscript P in M_P and N_P, and whenever we write the equality $P = MN$ we will always mean that it is the D-Levi decomposition of P in the above sense. The D-parabolic subgroup P is called ***standard*** if λ is a dominant cocharacter (with respect to P_0), or equivalently if $P(\lambda) \cap H$ is a standard parabolic subgroup of H.

Note that for any D-parabolic subgroup P of G, M_P also has the property (SR). Moreover, when G is reductive a D-parabolic subgroups is the same as a semistandard parabolic subgroup, and the D-Levi decomposition is the same as the semistandard Levi decomposition (that is, $A_0 \subset M_P$).

We denote by Ψ_P the set of roots of the adjoint action of A_0 on $\mathfrak{n}_P$. If $P = P(\lambda)$ for $\lambda \in X_*(A_0)$, then $\Psi_P = \{\alpha \in \Phi_G \mid \langle \alpha, \lambda \rangle > 0\}$.

Lemma 4.5. *Let P, Q be two D-parabolic subgroups of G. If $P \subset Q$ then $N_Q \subset N_P$.*

Proof. It suffices to check that $\mathfrak{n}_Q$ is a subspace of $\mathfrak{n}_P$. Write $P = P(\lambda)$ and $Q = P(\mu)$. Let $\alpha \in X^*(A_0)$ with $\langle \alpha, \mu \rangle > 0$. Assume for contradiction that $\langle \alpha, \lambda \rangle \leq 0$. Then $\langle -\alpha, \lambda \rangle \geq 0$ and $-\alpha \in \mathrm{Lie}(P) \subset \mathrm{Lie}(Q)$. This contradicts $\langle -\alpha, \mu \rangle < 0$. □

Let $P = P(\lambda)$ and $Q = P(\mu)$ with $P \subset Q$. Since N_Q is a normal subgroup of Q, it is a normal subgroup of N_P. Set $\Psi_P^Q = \Psi_P \setminus \Psi_Q$. Let N_P^Q be the subgroup corresponding to the semigroup $\{\alpha \in \Phi_G \mid \langle \alpha, \mu \rangle = 0, \langle \alpha, \lambda \rangle > 0\}$. By the last assertion of Proposition 4.1, we have

$$N_P = N_Q \ltimes N_P^Q.$$

The Lie algebra $\mathfrak{n}_P^Q = \mathfrak{n}_P / \mathfrak{n}_Q$ of N_P^Q is $\bigoplus_{\alpha \in \Psi_P^Q} \mathfrak{g}_\alpha$. In particular the group N_P^Q only depends on P and Q and not on the choices of λ and μ.

The following lemma provides a filtration on N_P^Q which will be useful in the proof of Theorem 4.25.

Lemma 4.6. *There exists an increasing filtration of unipotent subgroups*

$$N_0 = \{0\} \subset N_1 \subset \cdots \subset N_k = N_P^Q$$

of N_P^Q such that the following properties hold.

- *Each N_i is normal in N_P^Q.*
- *Put $\mathfrak{n}_i = \mathrm{Lie}(N_i)$. There is a complementary subspace $\mathfrak{n}_{i+1}^i$ of $\mathfrak{n}_i$ in $\mathfrak{n}_{i+1}$ which is contained in a root space. In particular, the action of A_0 on the Lie algebra of each quotient N_{i+1}/N_i $(0 \leq i \leq k-1)$ is by a character $\chi_i \in X^*(A_0)$.*
- *N_{i+1}/N_i is isomorphic to a product of $\mathbb{G}_a$'s.*

Proof. Assume that $P = P(\lambda)$ and $Q = P(\mu)$. Take a cocharacter ν of A_0 such that the numbers $\langle \alpha, \nu \rangle$, as α runs through Ψ_P^Q, are positive and distinct. For each $n > 0$, let $\Gamma_{\geq n}$ be the subsemigroup of $X^*(A_0)$ defined by $\{\alpha \mid \langle \alpha, \nu \rangle \geq n, \langle \alpha, \mu \rangle = 0, \langle \alpha, \lambda \rangle > 0\}$. For n_0 large enough, the filtration $\{0\} = H_{\Gamma_{\geq n_0}} \subset \ldots \subset H_{\Gamma_{\geq 1}} = N_P^Q$ satisfies the first two conditions. One then further refines this filtration to make it satisfy the third (e.g. take the derived series of each $H_{\Gamma_{\geq n}}/H_{\Gamma_{\geq (n+1)}}$). □

Let P be a D-parabolic subgroup. We put

$$[G]_P = N_P(\mathbb{A}) M_P(F) \backslash G(\mathbb{A}).$$

When $P = G$ we simply write $[G] = [G]_G$. There is a natural map $P(F) \backslash G(\mathbb{A}) \to [G]_P$ whose fibers are $N(F) \backslash N(\mathbb{A})$-torsors and hence are compact.

We write A_G for the maximal central split torus of G. More generally, for a D-parabolic subgroup P of G, we write $A_P = A_{M_P}$ for the maximal central split torus of M_P.

4.2. Heights and weights.

4.2.1. *Heights on adelic points of algebraic groups.* Let G be a connected linear algebraic group over F. We fix an embedding $\iota : G \to \mathrm{GL}_N$ for some $N > 0$ and define a height function on $G(\mathbb{A})$ by

$$\|g\| = \prod_v \max_{1 \leq i,j \leq N} \{|\iota(g)_{ij}|_v, |\iota(g)^{-1}_{ij}|_v\}, \tag{4.2}$$

where the product runs over all places of F. Note that for another choice of embedding ι' yielding a height $\|\,\|'$, there exists $r_0 > 0$ such that $\|g\|^{1/r_0} \ll \|g\|' \ll \|g\|^{r_0}$ for $g \in G(\mathbb{A})$. The equivalence class of $\|\,\|$ (in the preceding sense) is therefore independent of the choice of ι. Note that if $G = F_n$ or F^n is a vector space, then we may take the height function given, for $g = (x_1, \dots, x_n) \in G(\mathbb{A})$, by

$$\|g\| = \prod_v \max\{1, |x_1|_v, \dots, |x_n|_v\}. \tag{4.3}$$

The height function $\|\ \|$ on $G(\mathbb{A})$ induces a height function $\|\ \|_\infty$ on $G(F_\infty)$ by the embedding $G(F_\infty) \hookrightarrow G(\mathbb{A})$. It is explicitly given by

$$\|g\|_\infty = \prod_{v|\infty} \max_{1 \leq i,j \leq N} \{|\iota(g)_{ij}|_v, |\iota(g)^{-1}_{ij}|_v\},$$

where the product runs over all Archimedean places of F. This is called an algebraic scale on $G(F_\infty)$ in [BK14].

4.2.2. *Heights modulo a central unipotent subgroup.* In some circumstances we will need to work with the group G modulo a central unipotent subgroup Z (see Subsection 4.3). In this setting, we will equip $G(\mathbb{A})$ with the pull-back of the height function $\|\ \|$ on the group $(G/Z)(\mathbb{A})$, defined in (4.2), by the projection $G \to G/Z$. The resulting function will still be denoted $\|\ \|$ and be called a height function on $G(\mathbb{A})$. We take the convention that, whenever G is equipped with a fixed central unipotent subgroup Z, the notation $\|\ \|$ always designates the function constructed this way. If necessary, the height function on $G(\mathbb{A})$ defined in (4.2) will be denoted by $\|\ \|'$.

Note that by the equivalence of unipotent groups and nilpotent Lie algebra, Z is necessarily isomorphic to product of copies of $\mathbb{G}_a$. Since $H^1_{\mathrm{fppf}}(R, Z) = 0$ for any F-algebra R, we have $(G/Z)(R) = G(R)/Z(R)$.

The following two lemmas and Remark 4.9, where we assume the above setting so that G is equipped with a fixed unipotent central subgroup Z, show that the distinction between $\|\ \|$ and $\|\ \|'$ will be mostly inconsequential on adelic quotients.

Lemma 4.7. *There exist $r_1, r_2 > 0$, such that*

$$\|g\|^{r_1} \ll \inf_{z \in Z(\mathbb{A})} \|zg\|' \ll \|g\|^{r_2}, \quad g \in G(\mathbb{A}).$$

Proof. By [BP21a, Proposition A.1.1(ii)], the first inequality holds, so that we now prove the second. The projection map $G \to G/Z$ is a Z torsor in the fppf topology. Since G/Z is affine we have $H^1_{\mathrm{fppf}}(G/Z, Z) = 0$, and thus the torsor is a trivial torsor. It follows that there exists a section $s : G/Z \to G$. By [BP21a, Proposition A.1.1(ii)] again, there exists $r > 0$ such that $\|s(g)\|' \ll \|g\|^r$ for all $g \in (G/Z)(\mathbb{A})$, which implies the second inequality.

Let P be a D-parabolic subgroup of G. Note that it contains Z. We define a height function on $[G]_P$ by

$$\|g\|_P = \inf_{\gamma \in P(F)} \|\gamma g\|, \quad g \in [G]_P. \tag{4.4}$$

Lemma 4.8. *There exist $r_1, r_2 > 0$ such that*

$$\|g\|_P^{r_1} \ll \inf_{\gamma \in P(F)} \|\gamma g\|' \ll \|g\|_P^{r_2}, \quad g \in [G]_P.$$

Proof. This directly follows from 4.7, since $Z(F) \subset P(F)$ and $Z(F)\backslash Z(\mathbb{A})$ is compact.

Remark 4.9. Lemma 4.8 implies that, if we temporarily put

$$\|x\|'_P = \inf_{\gamma \in P(F)} \|\gamma x\|',$$

there exists $r_0 > 0$ such that $\|g\|_P^{1/r_0} \ll \|g\|'_P \ll \|g\|_P^{r_0}$ for all $g \in [G]_P$. In particular, since the spaces of functions that we will define in Subsection 4.3 are mostly insensitive to raising the heights to a power, we may interchange the two constructions.

Remark 4.10. Set $P_H = P \cap H$, which is parabolic subgroup of H. Using the projection $G \to H$ and the embedding $H \hookrightarrow G$, by [BP21a, Proposition A.1.1] we see that there exist $r_1, r_2 > 0$ such that for $x \in H(\mathbb{A})$, we have

$$\|x\|_{P_H}^{r_1} \ll \|x\|_P \ll \|x\|_{P_H}^{r_2}.$$

4.2.3. *Weights.* We keep the setting of Subsection 4.2.2, so that G is equipped with a central unipotent subgroup Z and that P is a D-parabolic subgroup.

By a weight on $[G]_P$ we mean a positive measurable function on $[G]_P$ such that there exist a positive number N and a constant C such that

$$w(xg) \le C w(x) \|g\|_P^N, \quad x \in [G]_P,\ g \in G(\mathbb{A}). \tag{4.5}$$

For example, $\|\cdot\|_P$ is a weight function on $[G]_P$.

We say that a function w on $P(F)\backslash G(\mathbb{A})$ is a weight on $P(F)\backslash G(\mathbb{A})$ if it is the composition of a weight on $[G]_P$ with the map $P(F)\backslash G(\mathbb{A}) \to [G]_P$. In particular $\|\cdot\|_P$ gives a weight on $P(F)\backslash G(\mathbb{A})$, and we say that it is a height function on $P(F)\backslash G(\mathbb{A})$.

Remark 4.11. The definition of the weight in [BPCZ22] differs slightly from ours. The definition there requires that for all compact subgroups J of $G(\mathbb{A})$ we have

$$w(x) \ll w(xk), \quad \text{for all } x \in [G]_P,\ k \in J.$$

If G is reductive, which is the case considered in [BPCZ22], the two definitions are equivalent by [BPCZ22, Lemma 2.4.3.1]. In general our definition imposes a stronger condition. Indeed exponential functions on affine spaces satisfy the condition in [BPCZ22] while our condition rules them out.

4.3. **Space of functions.** Let G be a connected linear algebraic group over F.

4.3.1. *Representations.* Following [BK14, BPCZ22] we introduce certain nice categories of representations. We first consider representations of $G(F_\infty)$. A Fréchet representation V of $G(F_\infty)$ is called an F-representation if its topology is induced by a countable family of $G(F_\infty)$-continuous $\|\ \|_\infty$-bounded semi-norms. Here a semi-norm ν on V is called $G(F_\infty)$-continuous if the map

$$G(F_\infty)\times(V,\nu)\to(V,\nu)$$

is continuous, where (V,ν) is the vector space V endowed with the topology induced from ν. The semi-norm ν is called $\|\ \|_\infty$-bounded if there is a positive number A such that

$$\sup_{v\in V,\ \nu(v)\neq 0}\frac{\nu(g\cdot v)}{\nu(v)}\le\|g\|_\infty^A$$

for all $g\in G(F_\infty)$. By [BK14, Lemma 2.10], the Fréchet representation V is an F-representation if and only if it is of moderate growth, i.e. if for any continuous semi-norm ν on V there exists another continuous semi-norm ν' on V and a positive number A such that

$$\nu(g\cdot v)\le\|g\|_\infty^A\,\nu'(v)$$

for all $g\in G(F_\infty)$ and $v\in V$. An F-representation V of $G(F_\infty)$ is called smooth, or is said to be an SF-representation if for each $v\in V$ the map

$$G(F_\infty)\to V,\quad g\mapsto g\cdot v$$

is smooth, and if for every $X\in U(\mathfrak{g}_\infty)$ the resulting map

$$V\to V,\quad v\mapsto X\cdot v$$

is continuous.

Remark 4.12. The $\|\ \|_\infty$-bounded condition is not included in [BPCZ22, Section 2.5.3]. This is because only reductive groups are considered there, where $\|\ \|_\infty$ is equivalent to the "maximal scale", cf. [BK14, Section 2.1], and thus the condition of $\|\ \|_\infty$-boundedness is automatic. We however need to work with representations of nonreductive groups, and $\|\ \|_\infty$-boundedness is crucial.

We now consider representations of $G(\mathbb{A})$. An SLF-representation of $G(\mathbb{A})$ is a vector space V equipped with a $G(\mathbb{A})$-action, with the following properties.

- For each open compact subgroup J of $G(\mathbb{A}_f)$ the space V^J is an SF-representation of $G(F_\infty)$.

We have

$$V = \bigcup_{J \subset G(\mathbb{A}_f)} V^J,$$

where J runs over all open compact subgroups of $G(\mathbb{A}_f)$.

If $J' \subset J$ then the natural map $V^J \to V^{J'}$ is a closed embedding.

4.3.2. *Smooth functions and Schwartz space.* We now let $Z \subset G$ be a central unipotent subgroup.

Let $\chi : Z(F)\backslash Z(\mathbb{A}) \to \mathbb{C}^\times$ be a character. We define $C^\infty(G(\mathbb{A}), \chi)$ to be the vector space of functions $f : G(\mathbb{A}) \to \mathbb{C}$ such that

- f is right invariant under a compact open subgroup $J \subset G(\mathbb{A}_f)$,
- for all $g_f \in G(\mathbb{A}_f)$, the function $g_\infty \mapsto f(g_f g_\infty)$ is a smooth function on the Lie group $G(F_\infty)$,
- $f(zg) = \chi(z)f(g)$ for all $z \in Z(\mathbb{A})$ and $g \in G(\mathbb{A})$.

Remark that we do not require f to be left invariant under some compact open subgroup of $G(\mathbb{A})$. We denote by $C_c^\infty(G(\mathbb{A}), \chi)$ the subspace of compactly supported functions modulo $Z(\mathbb{A})$. If χ is trivial we omit it from the notation in all function spaces.

Choose a height function $\|\cdot\|$ on $G(\mathbb{A})$ which is the pullback of a height function on $(G/Z)(\mathbb{A})$, as described in Subsection 4.2.2. For a compact open subset C of $G(\mathbb{A}_f)$ and a compact open subgroup J of $G(\mathbb{A}_f)$, we denote by $\mathcal{S}(G(\mathbb{A}), C, J, \chi)$ the subspace of $C^\infty(G(\mathbb{A}), \chi)$ consisting of functions f that are bi-invariant under J, supported in $G(F_\infty) \times CZ(\mathbb{A})$ and such that for all $N > 0$ and $X, Y \in U(\mathfrak{g}_\infty)$ the semi-norm

$$\|f\|_{X,Y,N} := \sup_{g \in G(\mathbb{A})} \|g\|^N |\mathrm{R}(X)\mathrm{L}(Y)f(g)|$$

is finite. This family of semi-norms gives $C^\infty(G(\mathbb{A}), \chi)$ the structure of a Fréchet space. Define the space of Schwartz functions $\mathcal{S}(G(\mathbb{A}), \chi)$ to be the union of all such $\mathcal{S}(G(\mathbb{A}), C, J, \chi)$ as C and J vary, endowed with the natural topology of an LF space.

Remark 4.13. Note that $\mathrm{L}(Y)$ and $\mathrm{R}(Y)$ differ by the adjoint action, which is an algebraic representation on G. Therefore, the topology of $\mathcal{S}(G(\mathbb{A}), C, J, \chi)$ is also generated by the semi-norms $\|\cdot\|_{X,N}$ where

$$\|f\|_{X,N} := \sup_{g \in G(\mathbb{A})} \|g\|^N |\mathrm{R}(X)f(g)|.$$

4.3.3. *The subgroup* Γ. Let Γ be a subgroup of $G(\mathbb{A})$ such that its intersection with $Z(\mathbb{A})$ equals $Z(F)$. We let $C^\infty(\Gamma\backslash G(\mathbb{A}), \chi)$ be the subspace of $C^\infty(G(\mathbb{A}), \chi)$ of left Γ-invariant functions. For any open compact subgroup $J \subset G(\mathbb{A}_f)$, we denote by $C^\infty(\Gamma\backslash G(\mathbb{A}), \chi)^J$ its subspace of right J-invariant functions.

We will say that Γ satisfies the condition (SL) if the following holds: Γ *contains $P(F)$ for some D-parabolic subgroup P of G with the property that for any open compact subgroup $J \subset G(\mathbb{A}_f)$, there exists an open compact subset $C \subset G(\mathbb{A}_f)$ such that the support of any $f \in C^\infty(P(F)\backslash G(\mathbb{A}), \chi)^J$*

is contained in $\Gamma(C \cap G(F_\infty))$. We will assume that Γ satisfies this condition for the rest of this section.

Let C be a set of representatives of $P(F)\backslash G(\mathbb{A}_f)/J$. The (SL) condition implies that there is a finite subset C_0 of C such that $C^\infty(P(F)\backslash G(\mathbb{A}); \chi)^J$ can be identified with a space of smooth functions on $\bigsqcup_{\gamma \in C_0}(P(F) \cap \gamma J \gamma^{-1})\backslash G(F_\infty)$, and $C^\infty(\Gamma\backslash G(\mathbb{A}); \chi)^J$ is a subspace of it. If we didn't have this condition later when we define our various spaces of functions on $\Gamma\backslash G(\mathbb{A})$, we would need to work with Schwartz functions or functions of uniform moderate growth on manifolds with infinitely many connected components, in which case no suitable formalism is available.

Note that this is not merely a condition on Γ, but that it also depends on χ. If G is reductive and $P \subset Q$ are parabolic subgroups, then $\Gamma = P(F)$, $M_P(F)N_Q(\mathbb{A})$, or $M_P(F)N_P(\mathbb{A})$ satisfy condition (SL). Other examples include the analogue case of Jacobi groups and their D-parabolic subgroups (and nontrivial χ), which will be explained in Subsection 5.1.

We slightly extend the definition of heights and weights to the case of $\Gamma\backslash G(\mathbb{A})$ where Γ is as above. We define a height $\|\cdot\|_\Gamma$ on $\Gamma\backslash G(\mathbb{A})$ by

$$\|g\|_\Gamma := \inf_{\gamma \in \Gamma} \|\gamma g\|.$$

Note that $\|g\|_\Gamma \geq 1$ for all g. We define a weight on $\Gamma\backslash G(\mathbb{A})$ to be a positive measurable function on $\Gamma\backslash G(\mathbb{A})$ such that there exist a positive number N and a constant C such that

$$w(xg) \leq C w(x) \|g\|_P^N, \quad \text{for all } x \in [G]_P \text{ and } g \in G(\mathbb{A}).$$

When $\Gamma = P(F)$ or $M(F)N(\mathbb{A})$, these definitions coincide with the definition of weight on $[G]_P$ given in (4.5).

4.3.4. *Spaces of Schwartz functions.* We keep Γ to be as in Subsection 4.3.3, so that it satisfies condition (SL).

Let $S^0(\Gamma\backslash G(\mathbb{A}); \chi)$ be the space of measurable functions on φ on $\Gamma\backslash G(\mathbb{A})$ such that $\varphi(zx) = \chi(z)\varphi(x)$ for almost all $(z, x) \in [Z] \times \Gamma\backslash G(\mathbb{A})$, and such that for any $N > 0$

$$\|\varphi\|_{\infty,N} := \sup_{x \in \Gamma\backslash G(\mathbb{A})} \|x\|_\Gamma^N |\varphi(x)| < \infty.$$

It is equipped with the family of seminorms $\|\cdot\|_{\infty,N}$ which gives it the structure of Fréchet space. Let $S^{00}(\Gamma\backslash G(\mathbb{A}); \chi)$ be the closed subspace of $S^0(\Gamma\backslash G(\mathbb{A}); \chi)$ consisting of continuous functions.

Let $S(\Gamma\backslash G(\mathbb{A}); \chi)$ be the space of *Schwartz functions on* $\Gamma\backslash G(\mathbb{A})$. It is the subspace of $C^\infty(\Gamma\backslash G(\mathbb{A}); \chi)$ consisting of φ satisfying that for all $X \in U(\mathfrak{g}_\infty)$ and integers $N > 0$,

$$\|\varphi\|_{X,N,\infty} := \sup_{g \in \Gamma\backslash G(\mathbb{A})} \|g\|_\Gamma^N |\mathrm{R}(X)\varphi(g)| < \infty.$$

For each compact open subgroup $J \subset G(\mathbb{A}_f)$, $S(\Gamma\backslash G(\mathbb{A}); \chi)^J$ is a Fréchet space under the seminorms $\|\cdot\|_{X,N,\infty}$, and $S(\Gamma\backslash G(\mathbb{A}); \chi)$ is an SLF representation of $G(\mathbb{A})$ for the right translation R.

Let w be a weight on $\Gamma\backslash G(\mathbb{A})$. For each $N \geq 0$, let $\mathcal{S}_{w,N}(\Gamma\backslash G(\mathbb{A});\chi)$ be the space of smooth functions $\varphi \in C^\infty(\Gamma\backslash G(\mathbb{A});\chi)$ satisfying

$$\|\varphi\|_{\infty,N,w^r,X} := \sup_{g\in\Gamma\backslash G(\mathbb{A})} \|g\|_\Gamma^N w(g)^r |\mathrm{R}(X)\varphi(g)| < \infty$$

for all $r \geq 0$ and $X \in \mathcal{U}(\mathfrak{g}_\infty)$. For each open compact subgroup J of $G(\mathbb{A}_f)$, the space $\mathcal{S}_{w,N}(\Gamma\backslash G(\mathbb{A});\chi)^J$ is a Fréchet space, and hence $\mathcal{S}_{w,N}(\Gamma\backslash G(\mathbb{A});\chi)$ is a strict LF space. Put

$$\mathcal{S}_w(\Gamma\backslash G(\mathbb{A});\chi) = \bigcup_{N\geq 0} \mathcal{S}_{w,N}(\Gamma\backslash G(\mathbb{A});\chi),$$

which is naturally a non-strict LF space. This is the χ-equivariant *weighted Schwartz space*. In particular, when $w = \|\cdot\|_\Gamma$, we recover the definition of $\mathcal{S}(\Gamma\backslash G(\mathbb{A});\chi)$.

4.3.5. *Space of functions of uniform moderate growth.* We retain the subgroup Γ from the previous subsection. Fix a weight w on $\Gamma\backslash G(\mathbb{A})$, and denote by $\mathcal{T}_w^0(\Gamma\backslash G(\mathbb{A});\chi)$ the space of complex Radon measures φ on $\Gamma\backslash G(\mathbb{A})$ with the properties that

$$\int_{\Gamma\backslash G(\mathbb{A})} f(g)\,\varphi(g) = \chi(z)\int_{\Gamma\backslash G(\mathbb{A})} f(zg)\,\varphi(g)$$

for any integrable function f on $\Gamma\backslash G(\mathbb{A})$ and $z \in [Z]$, and moreover that

$$\|\varphi\|_{1,w} := \int_{\Gamma\backslash G(\mathbb{A})} w(g)|\varphi(g)| < \infty.$$

The space $\mathcal{T}_w^0(\Gamma\backslash G(\mathbb{A});\chi)$ is naturally a Banach space with the norm $\|\cdot\|_{1,w}$. We write $\mathcal{T}_N^0(\Gamma\backslash G(\mathbb{A});\chi)$ for the space $\mathcal{T}^0_{\|\cdot\|^{-N}}(\Gamma\backslash G(\mathbb{A});\chi)$, and set

$$\mathcal{T}^0(\Gamma\backslash G(\mathbb{A});\chi) := \bigcup_{N>0} \mathcal{T}_N^0(\Gamma\backslash G(\mathbb{A});\chi),$$

which is naturally an LF space.

We set

$$\mathcal{T}(\Gamma\backslash G(\mathbb{A});\chi) := \mathcal{S}_1(\Gamma\backslash G(\mathbb{A});\chi) = \bigcup_{N>0} \mathcal{T}_N(\Gamma\backslash G(\mathbb{A});\chi),$$

where $\mathcal{T}_N(\Gamma\backslash G(\mathbb{A});\chi) := \mathcal{S}_{1,N}(\Gamma\backslash G(\mathbb{A});\chi)$. It is the *space of χ-equivariant functions of uniform moderate growth*. It is equipped with the corresponding LF topology.

There is a natural pairing

$$\mathcal{S}^{00}(\Gamma\backslash G(\mathbb{A});\chi) \times \mathcal{T}^0(\Gamma\backslash G(\mathbb{A});\chi^{-1}) \to \mathbb{C}, \quad (\varphi,\varphi') \mapsto \langle\varphi,\varphi'\rangle = \int_{\Gamma\backslash G(\mathbb{A})} \overline{\varphi(x)}\,\varphi'(x). \tag{4.6}$$

It identifies $\mathcal{T}^0(\Gamma\backslash G(\mathbb{A});\chi^{-1})$ with the topological dual of $\mathcal{S}^{00}(\Gamma\backslash G(\mathbb{A});\chi)$.

If $\varphi \in \mathcal{T}^0(\Gamma\backslash G(\mathbb{A});\chi)$ and $f \in \mathcal{S}(G(\mathbb{A});\chi^{-1})$, we define a function $\mathrm{R}(f)\varphi$ on $G(\mathbb{A})$ by

$$\mathrm{R}(f)\varphi(x) = \int_{Z(\mathbb{A})\backslash G(\mathbb{A})} f(x^{-1}y)\,\varphi(y).$$

Lemma 4.14. *Let $f \in \mathcal{S}(G(\mathbb{A});\chi^{-1})$, and $\varphi \in \mathcal{T}^0(\Gamma\backslash G(\mathbb{A});\chi)$. Let w be a weight on $\Gamma\backslash G(\mathbb{A})$. If w is bounded on the support of φ, then $\mathrm{R}(f)\varphi \in \mathcal{S}_w(\Gamma\backslash G(\mathbb{A});\chi)$.*

Proof. The proof is the same as [BPCZ22, Lemma 2.5.1.1].

4.3.6. *Spaces of weighted L^2-functions on reductive groups.* We now assume that G is reductive and let P be a standard parabolic subgroup of G. By taking $\Gamma = M_P(F)N_P(\mathbb{A})$ and $Z = 1$, we get various spaces of functions on $[G]_P$, including $\mathcal{S}([G]_P), \mathcal{T}([G]_P)$, etc. These spaces are related to smooth vectors in weighted L^2 spaces as we now explain.

The space $[G]_P = M_P(F)N_P(\mathbb{A})\backslash G(\mathbb{A})$ is equipped with the quotient of the Tamagawa measure on $G(\mathbb{A})$ by the product of the counting measure on $M_P(\mathbb{A})$ with the Tamagawa measure on $N_P(\mathbb{A})$ (see Subsection 3.4). We denote by $L^2([G]_P)$ the space of L^2 functions on $[G]_P$. It is a Hilbert space when equipped with the scalar product

$$\langle \varphi_1, \varphi_2 \rangle_P = \int_{[G]_P} \varphi_1(g)\overline{\varphi_2(g)}\mathrm{d}g.$$

More generally, if w is a weight on $[G]_P$, we write $L^2_w([G]_P)$ for the Hilbert space of square-integrable functions with respect to the measure $w(g)\mathrm{d}g$. We denote the resulting norm $\|\cdot\|_{w,P,L^2}$. If $N \in \mathbb{R}$, we will simply write $L^2_N([G]_P)$ for $L^2_{\|\cdot\|_P^N}([G]_P)$. It is equipped with a continuous (non-unitary) representation R of $G(\mathbb{A})$ by right-translation.

The subspace $L^2_w([G]_P)^\infty$ of smooth vectors consist of smooth functions $\varphi : [G]_P \to \mathbb{C}$ such that for every $X \in \mathcal{U}(\mathfrak{g}_\infty)$ we have $\mathrm{R}(X)\varphi \in L^2_w([G]_P)$. It is equipped with the family of semi-norms $\|R(X)f\|_{w,P,L^2}$. For every open compact subgroup $J \subset G(\mathbb{A}_f)$, the subspace $(L^2_w([G]_P)^\infty)^J$ is a Frechet space, and we equip $L^2_w([G]_P)^\infty = \bigcup_J (L^2_w([G]_P)^\infty)^J$ with the corresponding structure of LF-space.

By the Sobolev inequality ([BPCZ22, (2.5.5.4)]) and the open mapping theorem, we have an equality of SLF representations (where the right-hand side is equipped with the locally convex projective limit topology)

$$\mathcal{S}([G]_P) = \bigcap_{N>0} L^2_N([G]_P)^\infty.$$

Moreover, for every weight w on $[G]_P$, $\mathcal{S}([G]_P)$ is dense in $L^2_w([G]_P)^\infty$.

By another use of the Sobolev inequality, we also have the equality of LF spaces

$$\mathcal{T}([G]_P) = \bigcup_{N>0} L^2_{-N}([G]_P)^\infty.$$

It follows that $\mathcal{S}([G]_P)$ is dense in $\mathcal{T}([G]_P)$ (but it is not dense in any $\mathcal{T}_N([G]_P)$ in general). Finally, for every $N > 0$ there exist $N' > 0$ and a continuous inclusion $L^2_{-N}([G]_P)^\infty \subset \mathcal{T}_{N'}([G]_P)$.

4.4. **Pseudo-Eisenstein series and constant terms.** Let G be a connected linear algebraic group over F that satisfies (SR). Recall that in Subsection 4.1 we have fixed a Levi decomposition $G = U \rtimes H$, a maximal split torus $A_0 \subset H$ and a minimal parabolic subgroup P_0 of H which contains A_0. Let P, Q be two semistandard D-parabolic subgroups of G with $P \subset Q$. We keep Z to be a central unipotent subgroup of G, and χ a character of $[Z]$. Note that $M_P(F)N_P(\mathbb{A}) \cap Z(\mathbb{A}) = Z(F)$. We may therefore take $\Gamma = M_P(F)N_P(\mathbb{A})$ or $\Gamma = P(F)$ in the definitions of the various function spaces of Subsection 4.3. In particular, we get spaces of χ-equivariant functions on $P(F)\backslash G(\mathbb{A})$ or $[G]_P$. The same holds for Q.

Like in the case of reductive groups, we have the construction of pseudo-Eisenstein series and the constant terms, which we now explain. For $\varphi \in \mathcal{S}^0(P(F)\backslash G(\mathbb{A}), \chi)$, we define a pseudo-Eisenstein series

$$E_P^Q \varphi(g) = \sum_{\gamma \in P(F)\backslash Q(F)} \varphi(\gamma g).$$

Lemma 4.15. *The pseudo-Eisenstein series $\varphi \mapsto E_P^Q \varphi$ defines a continuous linear map*

$$E_P^Q : \mathcal{S}^?(P(F)\backslash G(\mathbb{A}), \chi) \to \mathcal{S}^?(Q(F)\backslash G(\mathbb{A}), \chi),$$

where $? = 0$, 00, or empty.

Note that by restriction we also obtain a continuous linear map $E_P^Q : \mathcal{S}^?([G]_P, \chi) \to \mathcal{S}^?([G]_Q, \chi)$ for $? = 0$, 00, or empty.

Proof. It is more convenient to use the height function $\|\cdot\|'$ on $G(\mathbb{A})$ instead of the one pulled back from $G/Z(\mathbb{A})$ (see Remark 4.9). So $\|\cdot\|$ in the proof below will denote the height function on $G(\mathbb{A})$. Let $d_1, d_2 > 0$. For every $\gamma \in Q(F)$ we have $\|\gamma\|_P^{-d_2} \ll \sum_{\delta \in P(F)\gamma} \|\delta\|^{-d_2}$. Moreover, it follows from [BP21b, Theorem A.1.1(1)] that there exists $c > 0$ such that $\|xy\|^c \ll \|x\|\|y\|$. Therefore for every $\varphi \in \mathcal{S}^0(P(F)\backslash G(\mathbb{A}), \chi)$ and $g \in G(\mathbb{A})$ we have

$$\begin{aligned}\sum_{\gamma \in P(F)\backslash Q(F)} |\varphi(\gamma g)| &\ll \sum_{\gamma \in P(F)\backslash Q(F)} \|\gamma g\|_P^{-d_1-d_2} \ll \|g\|_Q^{-d_2} \sum_{\gamma \in P(F)\backslash Q(F)} \|\gamma g\|_P^{-d_1} \\ &\ll \|g\|_Q^{-d_2} \sum_{\gamma \in Q(F)} \|\gamma g\|^{-d_1} \\ &\ll \|g\|_Q^{-d_2} \|g\|^{d_1} \sum_{\gamma \in Q(F)} \|\gamma\|^{-cd_1}.\end{aligned}$$

By [BP21b, Proposition A.1.1 (v)], the sum $\sum_{\gamma \in Q(F)} \|\gamma\|^{-cd_1}$ is finite for d_1 large enough. As the LHS only depends on the class of g in $Q(F)\backslash G(\mathbb{A})$, we see that

$$\sum_{\gamma \in P(F)\backslash Q(F)} |\varphi(\gamma g)| \ll \|\varphi\|_{d_1+d_2,1} \|g\|_Q^{-d_2+d_1}, \quad \varphi \in \mathcal{S}^0(P(F)\backslash G(\mathbb{A}), \chi).$$

By taking d_2 large enough, this shows that E_P^Q sends $\mathcal{S}^0(P(F)\backslash G(\mathbb{A}), \chi)$ to $\mathcal{S}^0(Q(F)\backslash G(\mathbb{A}), \chi)$, and that it is continuous. By a similar estimate we see that E_P^Q restricts a to continuous map from $\mathcal{S}^?(P(F)\backslash G(\mathbb{A}), \chi)$ to $\mathcal{S}^?(Q(F)\backslash G(\mathbb{A}), \chi)$, where $? = 00$ or empty.

Remark 4.16. If G is reductive and $\varphi \in \mathcal{S}^0([G]_P)$ then we have $E_P^Q \varphi \in \mathcal{S}^0([G]_Q)$. This is the usual definition of pseudo-Eisenstein series in [BPCZ22, Section 2.5.13].

We now define the constant terms. If φ is a Radon measure on $P(F)\backslash G(\mathbb{A})$, we define its constant term φ_P to be its pushforward to $[G]_P$ along the projection map $P(F)\backslash G(\mathbb{A}) \to [G]_P$. Since the fibers of $P(F)\backslash G(\mathbb{A}) \to [G]_P$ are compact, if $\varphi \in \mathcal{T}'(P(F)\backslash G(\mathbb{A}), \Lambda)$, then $\varphi_P \in \mathcal{T}'([G]_P, \Lambda)$. Note that this map is the transpose of the pullback of functions $\mathcal{S}''([G]_P, \Lambda^{-1}) \to \mathcal{S}''(P(F)\backslash G(\mathbb{A}), \Lambda^{-1})$ under the pairing (4.6), where we use the compactness of the fibers again to ensure that a function on $[G]_P$ of rapid decay is of rapid decay on $P(F)\backslash G(\mathbb{A})$. On the subspace $\mathcal{T}(P(F)\backslash G(\mathbb{A}), \Lambda)$, the constant term is given by

$$\varphi_P(x) = \int_{[N_P]} \varphi(nx)\mathrm{d}n,$$

and $\varphi_P \in \mathcal{T}([G]_P, \Lambda)$. Moreover $\varphi \mapsto \varphi_P$ induces continuous linear maps $\mathcal{T}(P(F)\backslash G(\mathbb{A}), \Lambda) \to \mathcal{T}([G]_P, \Lambda)$ and $\mathcal{T}_N(P(F)\backslash G(\mathbb{A}), \Lambda) \to \mathcal{T}_N([G]_P, \Lambda)$ for all N.

By Lemma 4.15, the pseudo-Eisenstein series E_P^Q gives a continuous linear map $\mathcal{S}''(P(F)\backslash G(\mathbb{A}), \Lambda^{-1}) \to \mathcal{S}''(Q(F)\backslash G(\mathbb{A}), \Lambda^{-1})$. Its transpose gives a continuous linear map

$$\mathcal{T}'(Q(F)\backslash G(\mathbb{A}), \Lambda) \to \mathcal{T}'(P(F)\backslash G(\mathbb{A}), \Lambda).$$

Therefore we have a series of continuous linear maps

$$\mathcal{T}'([G]_Q, \Lambda) \to \mathcal{T}'(Q(F)\backslash G(\mathbb{A}), \Lambda) \to \mathcal{T}'(P(F)\backslash G(\mathbb{A}), \Lambda) \to \mathcal{T}'([G]_P, \Lambda). \tag{4.7}$$

We denote the composition also by $\varphi \mapsto \varphi_P$ and again call it the constant term. If G is reductive, this is the definition of the constant term given in [BPCZ22, Section 2.5.13]. The composition in (4.7) restricts to a continuous map $\mathcal{T}([G]_Q, \Lambda) \to \mathcal{T}([G]_P, \Lambda)$.

By construction we have

$$\langle \varphi, E_P^Q \varphi' \rangle = \langle \varphi_P, \varphi' \rangle, \quad \varphi \in \mathcal{S}'([G]_Q, \Lambda^{-1}),\ \varphi' \in \mathcal{T}'([G]_P, \Lambda), \tag{4.8}$$

where $\langle \cdot, \cdot \rangle$ stands for the pairing between $\mathcal{S}''$ and $\mathcal{T}'$ defined in (4.6).

4.5. **Reductive groups, reduction theory.** In this section, we let G be a connected reductive group over F.

4.5.1. *Main notations.* Let $X^*(G)$ be the group of rational characters of G. Put $\mathfrak{a}_G = \mathrm{Hom}_{\mathbb{Z}}(X^*(G), \mathbb{R})$ and $\mathfrak{a}_G^* = X^*(G) \otimes_{\mathbb{Z}} \mathbb{R}$. We have a canonical pairing

$$\langle \cdot, \cdot \rangle : \mathfrak{a}_G^* \times \mathfrak{a}_G \to \mathbb{R}.$$

Let $H_G : G(\mathbb{A}) \to \mathfrak{a}_G$ the Harish-Chandra map, with the defining property that for any $\chi \in X^*(G)$, we have

$$\log|\chi(g)| = \langle \chi, H_G(g) \rangle. \tag{4.9}$$

Let $P = MN$ be a parabolic subgroup and K be a good maximal compact subgroup of $G(\mathbb{A})$, i.e. such that we have the Iwasawa $G(\mathbb{A}) = P(\mathbb{A})K$. We put $\mathfrak{a}_P = \mathfrak{a}_M$ and $\mathfrak{a}_P^* = \mathfrak{a}_M^*$. For any $mnk \in G(\mathbb{A})$ with $m \in M(\mathbb{A})$, $n \in N(\mathbb{A})$ and $k \in K$, set

$$H_P(mnk) = H_M(m) \in \mathfrak{a}_P,$$

where H_M is the Harish-Chandra map defined in (4.9) with respecto to the reductive group M. We define $G(\mathbb{A})^1$ to be the kernel of H_G, and set $[G]_P^1 = M(F)N(\mathbb{A})\backslash G(\mathbb{A})^1$. We extend this definition by putting $P(\mathbb{A})^1 = M(\mathbb{A})^1 N(\mathbb{A})$.

Recall that we have fixed a maximal split torus A_0 of G and a minimal parabolic $P_0 = M_0 N_0$. We put $\mathfrak{a}_0 = \mathfrak{a}_{P_0}$, $\mathfrak{a}_0^* = \mathfrak{a}_{P_0}^*$ and $H_0 = H_{P_0}$ to shorten notation.

Let $P = MN$ be a parabolic subgroup. We denote by $\delta_P : M(\mathbb{A}) \to \mathbb{C}^\times$ the modulus character, and by ρ_P the half sum of roots of A_P in N. Then $\rho_P \in \mathfrak{a}_P^*$ and satisfies

$$\delta_P(m) = e^{\langle \rho_P, H_M(m) \rangle}, \quad m \in M(\mathbb{A}).$$

Let $P \subset Q$ be parabolic subgroups. Then we have $A_Q \subset A_P$. The restriction $X^*(Q) \to X^*(P)$ induces the maps $\mathfrak{a}_Q^* \to \mathfrak{a}_P^*$ (which is an injection) and $\mathfrak{a}_P \to \mathfrak{a}_Q$, whose kernel is denoted by $\mathfrak{a}_P^Q$. The restriction $X^*(A_P) \to X^*(A_Q)$ induces the maps $\mathfrak{a}_Q \to \mathfrak{a}_P$ (which is an injection) and $\mathfrak{a}_P^* \to \mathfrak{a}_Q^*$, whose kernel is denoted by $\mathfrak{a}_P^{Q,*}$. We have canonical decompositions

$$\mathfrak{a}_P = \mathfrak{a}_Q \oplus \mathfrak{a}_P^Q, \quad \mathfrak{a}_P^* = \mathfrak{a}_Q^* \oplus \mathfrak{a}_P^{Q,*}.$$

Define $\Delta_P^Q \subset \mathfrak{a}_P^{Q,*}$ to be the set of simple roots of A_P in $M_Q \cap P$. We have the set of coroots $\Delta_P^{Q,\vee} \subset \mathfrak{a}_P^Q$. By duality, we also have the set of simple weights $\widehat{\Delta}_P^Q \subset \mathfrak{a}_P^{Q,*}$. The sets Δ_P^Q and $\widehat{\Delta}_P^Q$ define open cones in $\mathfrak{a}_0$ whose characteristic functions are denoted by τ_P^Q and $\widehat{\tau}_P^Q$ respectively. If $P = P_0$ then we replace the subscript P_0 in the notation by 0. If $Q = G$ then we omit the superscript G.

Let W be the Weyl group of (G, A_0), that is the quotient of the normalizer of A_0 in $G(F)$ by M_0 the Levi subgroup of P_0. For $P = M_P N_P$ and $Q = M_Q N_Q$ two standards parabolic subgroups of G, denote by $W(P, Q)$ the set $w \in W$ such that $w\Delta_0^P = \Delta_0^Q$. In particular, for $w \in W(P, Q)$ we have $wM_P = M_Q$.

4.5.2. *Haar measures.* We equip $\mathfrak{a}_P$ with the Haar measure giving the lattice $\mathrm{Hom}(X^*(P), \mathbb{Z})$ covolume 1, and $i\mathfrak{a}_P^*$ with the dual measure. If $Q \supset P$ is another parabolic subgroup, then $\mathfrak{a}_P^Q = \mathfrak{a}_P/\mathfrak{a}_Q$ and $i\mathfrak{a}_P^{Q,*} = i\mathfrak{a}_P^*/\mathfrak{a}_Q^*$ are equipped with the quotient Haar measures.

4.5.3. *Truncation parameters.* The fixed minimal parabolic subgroup P_0 determine a positive chamber in $\mathfrak{a}_0$. By “$T \in \mathfrak{a}_0$ is sufficiently positive” we mean that “for T such that $\inf_{\alpha \in \Delta_0} \alpha(T) \geq \max\{\epsilon\|T\|, C\}$”, where $\|\cdot\|$ is an arbitrary norm on the real vector space $\mathfrak{a}_0$, $C > 0$ is a large enough constant and $\epsilon > 0$ is an arbitrary (but in practice small enough) constant. We say that $T \in \mathfrak{a}_0$ is sufficiently negative if $-T$ is sufficiently positive.

For $T \in \mathfrak{a}_0$ and a standard parabolic subgroup P, we write T_P for the image of T under the projection map $\mathfrak{a}_0 \to \mathfrak{a}_P$. If P is more generally semistandard, we write T_P for the image of $w \cdot T$ under the projection map $\mathfrak{a}_0 \to \mathfrak{a}_P$, where w is any element in the Weyl group satisfies $wP_0w^{-1} \subset P$.

4.5.4. *Reduction theory.* Let $\omega_0 \subset P_0(\mathbb{A})^1$ be a compact subset such that $P_0(\mathbb{A})^1 = \omega_0 P_0(F)$. Let $P = MN$ be a standard parabolic subgroup. By a Siegel domain $\mathfrak{s}^P$ of $[G]_P$ we mean a subset of $[G]_P$ of the form

$$\mathfrak{s}^P = \omega_0 \{ a \in A_0^\infty \mid \langle \alpha, H_0(a) - T_- \rangle \geq 0, \ \alpha \in \Delta_0^P \} K,$$

where $T_- \in \mathfrak{a}_0$, and such that $G(\mathbb{A}) = M(F)N(\mathbb{A})\mathfrak{s}^P$. We assume that for different parabolic subgroups of G, their Siegel domains are defined by the same T_-. In particular if $P \subset Q$ then $\mathfrak{s}^P \subset \mathfrak{s}^Q$.

For $T_-, T \in \mathfrak{a}_0$, we define

$$A_0^{P,\infty}(T_-, T) = \left\{ a \in A_0^\infty \mid \langle \alpha, H_0(a) \rangle \geq \langle \alpha, T_- \rangle, \forall \alpha \in \Delta_0^P \text{ and } \langle \varpi, H_0(a) \rangle \leq \langle \varpi, T \rangle, \forall \varpi \in \hat{\Delta}_0^P \right\}.$$

We put $\mathfrak{s}^P(T) = \mathfrak{s}^P(T_-, T, \omega_0, K) = \omega_0 A_0^{P,\infty}(T_-, T)K$ and call it the truncated Siegel set. Let $F^P(\cdot, T)$ be the characteristic function on $[G]_P$ of the set $M_P(F)N_P(\mathbb{A})\mathfrak{s}^P(T)$. For T sufficiently positive and T_- sufficiently negative, we have the Langlands partition formula [Art78, Lemma 6.4]:

$$\sum_{P_0 \subset Q \subset P} \sum_{\delta \in Q(F) \backslash P(F)} F^Q(\delta x, T) \tau_Q^P(H_0(\delta x) - T) = 1. \tag{4.10}$$

Let $\lambda \in \mathfrak{a}_0^*$ and P be a parabolic subgroup. A weight $d_{P,\lambda}$ on $[G]_P$ is introduced in [BPCZ22, Section 2.4.3]. If $g \in \mathfrak{s}^P$, then we have

$$d_{P,\lambda}(g) \sim e^{\langle \lambda, H_P(g) \rangle}.$$

If $P \subset Q$ are parabolic subgroups, two weights d_P^Q and d_Q^P on $[G]_P$ and $[G]_Q$ respectively are introduced in [BPCZ22, Section 2.4.4] and are given by

$$d_P^Q(g) = \min_{\lambda \in \Psi_P^Q} d_{P,\lambda}(g), \quad d_Q^P(g) = \min_{\lambda \in \Psi_P^Q} d_{Q,\lambda}(g),$$

where $g \in [G]_P$ and $[G]_Q$ respectively. Consider the projections

$$[G]_P \xleftarrow{\pi_P^Q} P(F)N_Q(\mathbb{A}) \backslash G(\mathbb{A}) \xrightarrow{\pi_Q^P} [G]_Q. \tag{4.11}$$

For $C > 0$ define

$$\omega_P^Q[> C] = \{ g \in P(F)N_Q(\mathbb{A}) \backslash G(\mathbb{A}) \mid d_P^Q(\pi_P^Q(g)) > C \}.$$

We recall [BPCZ22, Lemma 2.4.4.1] which summarizes some classical results from reduction theory.

Lemma 4.17. *We have the following assertions.*

(1) *There is an $\epsilon > 0$ such that π_Q^P maps $\omega_P^Q[> \epsilon]$ onto $[G]_Q$.*

(2) *For any $\epsilon > 0$, we have*

$$d_P^Q(g) \sim d_Q^P(g), \quad \|g\|_P \sim \|g\|_Q$$

for all $g \in \omega_P^Q[> \epsilon]$.

(3) *For all $\epsilon > 0$, the restriction of π_Q^P to $\omega_P^Q[> \epsilon]$ has uniformly bounded fibers.*

(4) *For all* $\epsilon > 0$, *there is a* $C > 0$ *such that if* $(g_1, g_2) \in \omega_P^Q[>\epsilon] \times \omega_P^Q[>C]$ *and* $\pi_Q^P(g_1) = \pi_Q^P(g_2)$, *then* $g_1 = g_2$.

(5) *The map* π_Q^P *is a local homeomorphism that locally preserves the measures. The map* π_P^Q *is proper and the pushforward of the invariant measure on* $P(F)N_Q(\mathbb{A})\backslash G(\mathbb{A})$ *by it is the invariant measure on* $[G]_P$.

4.6. **Automorphic forms and Eisenstein series.** We assume that G is reductive and connected in this subsection.

4.6.1. *Spaces of automorphic forms.* Let $P = MN$ be a standard parabolic subgroup of G. Equip A_P^∞ with the Haar measure da such that the isomorphism $H_P : A_P^\infty \to \mathfrak{a}_P$ is measure preserving (see Subsection 4.5.2). Set $[G]_{P,0} = A_P^\infty \backslash [G]_P$. It is equipped with the quotient of the Tamagawa invariant measure on $[G]_P$ (see Subsection 4.3.6) by da.

We define the space of automorphic forms $\mathcal{A}_P(G)$ to be the subspace of $\mathcal{Z}(\mathfrak{g}_\infty)$-finite functions in $\mathcal{T}([G]_P)$. We define $\mathcal{A}_{P,\mathrm{cusp}}(G)$ (resp. $\mathcal{A}_{P,\mathrm{disc}}(G)$) to be the subspace of cuspidal automorphic forms (resp. discrete automorphic forms), i.e. functions $\varphi \in \mathcal{A}_P(G)$ such that $\varphi_Q = 0$ for all $Q \subsetneq P$ (resp. such that $|\varphi| \in L^2([G]_{P,0})$). We will simply drop the subscripts P when $P = G$.

A cuspidal (resp. discrete) automorphic representation π of $M(\mathbb{A})$ is a topologically irreducible subrepresentation of $\mathcal{A}_{\mathrm{cusp}}(M)$ (resp. of $\mathcal{A}_{\mathrm{disc}}(M)$). Let π be a cuspidal (resp. discrete) automorphic representation of $M(\mathbb{A})$. We define $\mathcal{A}_{\pi,\mathrm{cusp}}(M)$ (resp. $\mathcal{A}_{\pi,\mathrm{disc}}(M)$) to be the π-isotypic component of $\mathcal{A}_{\mathrm{cusp}}(M)$ (resp. $\mathcal{A}_{\mathrm{disc}}(M)$), and set

$$\Pi = \mathrm{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})} \pi, \quad \mathcal{A}_{P,\pi,\mathrm{cusp}}(G) = \mathrm{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})} \mathcal{A}_{\pi,\mathrm{cusp}}(M), \quad \text{resp. } \mathcal{A}_{P,\pi,\mathrm{disc}}(G) = \mathrm{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})} \mathcal{A}_{\pi,\mathrm{disc}}(M).$$

Here Ind stands for the normalized smooth induction. These spaces have a natural topology described in [BPCZ22, §2.7] which gives them the structure of SLF representations of $G(\mathbb{A})$. We identify Π (resp. $\mathcal{A}_{P,\pi,\mathrm{cusp}}(G)$, $\mathcal{A}_{P,\pi,\mathrm{disc}}(G)$) with the space of forms $\varphi \in \mathcal{A}_P(G)$ such that the function

$$m \mapsto e^{-\langle \rho_P, H_P(m)\rangle} \varphi(mg), \quad m \in [M]$$

belongs to π (resp. $\mathcal{A}_{\pi,\mathrm{cusp}}(M)$, $\mathcal{A}_{\pi,\mathrm{disc}}(G)$) for every $g \in [G]_P$.

Let $\lambda \in \mathfrak{a}_{P,\mathbb{C}}^*$. We define the twist π_λ as the space of functions of the form

$$m \mapsto e^{\langle \lambda, H_P(m)\rangle} \varphi(m), \quad m \in M(\mathbb{A}), \quad \varphi \in \pi.$$

If π is cuspidal, for $f \in \mathcal{S}(G(\mathbb{A}))$ we denote by $I(\lambda, f)$ the action of f on $\mathcal{A}_{P,\pi,\mathrm{cusp}}(G)$ obtained by transporting the action on $\mathcal{A}_{P,\pi_\lambda,\mathrm{cusp}}(G)$ through the identification

$$\mathcal{A}_{P,\pi,\mathrm{cusp}}(G) \to \mathcal{A}_{P,\pi_\lambda,\mathrm{cusp}}(G), \quad \varphi \mapsto e^{\langle \lambda, H_P(\cdot)\rangle} \varphi(\cdot).$$

In the same way, if π is discrete we also get an action on $\mathcal{A}_{P,\pi,\mathrm{disc}}(G)$ still denoted by $I(\lambda, f)$.

If the central character of π is unitary, we equip Π and $\mathcal{A}_{P,\pi,\mathrm{cusp}}(G)$ (resp. $\mathcal{A}_{P,\pi,\mathrm{disc}}(G)$) with the Petersson inner product

$$\langle \varphi_1, \varphi_2 \rangle_{\mathrm{Pet}} = \int_{[G]_{P,0}} \varphi_1(g)\overline{\varphi_2(g)}\,\mathrm{d}g. \tag{4.12}$$

For every $\varphi \in \mathcal{A}_{P,\mathrm{disc}}(G)$, $\lambda \in \mathfrak{a}_{P,\mathbb{C}}^*$, we have the Eisenstein series

$$E(g,\varphi,\lambda) = \sum_{\gamma \in P(F)\backslash G(F)} \varphi(\gamma g) e^{\langle \lambda, H_P(\gamma g)\rangle}.$$

This sum is absolutely convergent when $\Re\lambda$ is in a certain cone, and $E(g,\varphi,\lambda)$ has a meromorphic continuation to all λ which is regular on $i\mathfrak{a}_P^*$. By [Lap08, Theorem 2.2], for every discrete automorphic representation π of $M(\mathbb{A})$ and for every $\lambda \in i\mathfrak{a}_P^*$, the map $\varphi \mapsto E(\cdot,\varphi,\lambda)$ induces a continuous map $\Pi \to \mathcal{T}([G])$ that actually factors through $\mathcal{T}_N([G])$ for some $N > 0$. The resulting map $\Pi \to \mathcal{T}_N([G])$ is continuous by the closed graph theorem (see Remark 6.9).

4.6.2. *Relative characters.* Let π be an irreducible cuspidal or discrete automorphic representation of $M(\mathbb{A})$. In the following, we simply write $\mathcal{A}_{P,\pi}$ for $\mathcal{A}_{P,\pi,\mathrm{cusp}}$ or $\mathcal{A}_{P,\pi,\mathrm{disc}}$.

Recall that we have fixed a maximal compact subgroup K of $G(\mathbb{A})$. Let $\widehat{K}$ be the set of all irreducible unitary representations of K. For any $\tau \in \widehat{K}$, we denote by $\mathcal{A}_{P,\pi}(G,\tau)$ the (finite dimensional) subspace of $\mathcal{A}_{P,\pi}(G)$ spanned by the functions which translate by K according to τ. A K-basis of $\mathcal{A}_{P,\pi}(G)$ is by definition the union over all $\tau \in \widehat{K}$ of orthonormal basis $\mathcal{B}_{P,\pi,\tau}$ of $\mathcal{A}_{P,\pi}(G,\tau)$ for the Petersson inner product. The following proposition will be useful to define relative characters.

Proposition 4.18. *Let $\mathcal{B}_{P,\pi}$ be a K-basis of $\mathcal{A}_{P,\pi}(G)$.*

(1) *Let $f \in \mathcal{S}(G(\mathbb{A}))$. The sum*

$$\sum_{\varphi \in \mathcal{B}_{P,\pi}} I(\lambda, f)\varphi \otimes \overline{\varphi}$$

is absolutely convergent in $\mathcal{A}_{P,\pi}(G) \widehat{\otimes} \overline{\mathcal{A}_{P,\pi}}(G)$. The convergence is uniform for λ when $\Re\lambda$ lies in any fixed compact subset. Moreover, for any continuous sesquilinear form B on $\mathcal{A}_{P,\pi}(G) \times \mathcal{A}_{P,\pi}(G)$, the map

$$f \mapsto \sum_{\varphi \in \mathcal{B}_{P,\pi}} B(I(\lambda,f)\varphi,\varphi)$$

is a continuous linear form on $\mathcal{S}(G(\mathbb{A}))$.

(2) *Let $f \in \mathcal{S}(G(\mathbb{A}))$. For all $\lambda \in i\mathfrak{a}_P^*$, there exists N such that the series*

$$\sum_{\varphi \in \mathcal{B}_{P,\pi}} E(\cdot, I(\lambda,f)\varphi,\lambda)\overline{E(\cdot,\varphi,\lambda)}$$

is absolutely convergent in $\mathcal{T}_N([G \times G])$.

(3) *Let $f \in \mathcal{S}([G])$. For all $\lambda \in i\mathfrak{a}_P^*$, there exists N such that the series*

$$\sum_{\varphi \in \mathcal{B}_{P,\pi}} \langle f, E(\cdot,\varphi,\lambda)\rangle E(\cdot,\varphi,\lambda) \tag{4.13}$$

is absolutely convergent in $\mathcal{T}_N([G])$, where $\langle\cdot,\cdot\rangle$ is the pairing defined in (4.6).

Moreover, these sums do not depend on the choice of $\mathcal{B}_{P,\pi}$.

Proof. The first assertion is [BPCZ22, Proposition 2.8.4.1]. The second is a consequence of (1) and the existence of the continuous map $\varphi \in \Pi \mapsto E(\varphi,\lambda) \in \mathcal{T}_N([G])$ for some N ([Lap08, Theorem 2.2]). The third is not explicitly listed as a theorem in [BPCZ22], but is contained in the discussion in the first paragraph of [BPCZ22, Section 2.9.8], and in particular in the expression (2.9.8.14). Note that in Loc. cit, the group is assumed to be a product of restrictions of general linear groups, and π is assumed to be cuspidal, but the discussion in the first paragraph does not require these assumptions.

4.7. Cuspidal data and Langlands decompositions.

4.7.1. *Cuspidal data.* We continue to assume that G is reductive in this subsection. Let $\underline{\mathfrak{X}}(G)$ be the set of pairs (M_P,π) where

- $P = M_P N_P$ is a standard parabolic subgroup of G,
- π is an (isomorphism class of a) cuspidal automorphic representation of $M_P(\mathbb{A})$ whose central character is trivial on A_P^∞.

Two elements (M_P,π) and (M_Q,τ) of $\underline{\mathfrak{X}}(G)$ are equivalent if there is a $w \in W(P,Q)$ such that $w\pi w^{-1} = \tau$. We define a cuspidal datum to be an equivalence class of such (M_P,π) and denote by $\mathfrak{X}(G)$ the set of all cuspidal data. If $\chi \in \mathfrak{X}(G)$ is represented by (M_P,π) we define $\overline{\chi}$ to be the cuspidal datum represented by $(M_P,\overline{\pi})$. Note that the natural inclusion $\underline{\mathfrak{X}}(M) \subset \mathfrak{X}(G)$ descends to a finite-to-one map $\mathfrak{X}(M) \to \mathfrak{X}(G)$.

4.7.2. *Coarse Langlands decomposition.* For $(M_P,\pi) \in \underline{\mathfrak{X}}(G)$, let $\mathcal{S}_\pi([G]_P)$ be the space of $\varphi \in \mathcal{S}([G]_P)$ such that

$$\varphi_\lambda(x) := \int_{A_P^\infty} e^{-\langle \lambda+\rho_P, H_P(a)\rangle} \varphi(ax)\,da, \quad x \in [G]_P,$$

belongs to $\mathcal{A}_{P,\pi,\mathrm{cusp}}(G)$ for every $\lambda \in \mathfrak{a}_{P,\mathbb{C}}^*$.

Let $P \subset G$ be a standard parabolic subgroup, $\chi \in \mathfrak{X}(G)$ be a cuspidal datum and $\{(M_{Q_i},\pi_i) \mid i \in I\}$ be the inverse image of χ in $\underline{\mathfrak{X}}(M_P)$. Denote by $L^2_\chi([G]_P)$ the closure in $L^2([G]_P)$ of the subspace

$$\sum_{i \in I} E_{Q_i}^P(\mathcal{S}_{\pi_i}([G]_{Q_i})). \tag{4.14}$$

We define similarly $L^2_\chi([G]_{P,0}) \subset L^2([G]_{P,0})$. The coarse Langlands decomposition ([MW95, Proposition II.2.4]) states that we have decompositions in orthogonal direct sums

$$L^2([G]_P) = \bigoplus_{\chi \in \mathfrak{X}(G)} L^2_\chi([G]_P) \quad \text{and} \quad L^2([G]_{P,0}) = \bigoplus_{\chi \in \mathfrak{X}(G)} L^2_\chi([G]_{P,0}).$$

For any subset $\mathfrak{X} \subset \mathfrak{X}(G)$, set

$$L^2_{\mathfrak{X}}([G]_P) = \widehat{\bigoplus_{\chi \in \mathfrak{X}}} L^2_\chi([G]_P),$$

and define

$$\mathcal{S}_{\mathfrak{X}}([G]_P) = \mathcal{S}([G]_P) \cap L^2_{\mathfrak{X}}([G]_P).$$

Let w be a weight on $[G]_P$. For any $\mathcal{F} \in \{L^2_w, \mathcal{T}_N, \mathcal{T}, \mathcal{S}_{w,N}, \mathcal{S}_w\}$, define $\mathcal{F}_{\mathfrak{X}}([G]_P)$ to be the orthogonal of $\mathcal{S}_{\mathfrak{X}^c}([G]_P)$ in $\mathcal{F}([G]_P)$, where $\mathfrak{X}^c$ is the complement of $\mathfrak{X}$ in $\mathfrak{X}(G)$. By [BPCZ22, Section 2.9.4], for $\mathcal{F} \in \{L^2_w, \mathcal{T}, \mathcal{S}\}$ there are canonical projections

$$\mathcal{F}([G]_P) \to \mathcal{F}_{\mathfrak{X}}([G]_P), \quad \varphi \mapsto \varphi_{\mathfrak{X}}. \tag{4.15}$$

The following proposition is contained in [BPCZ22, Theorem 2.9.4.1].

Proposition 4.19. *There exists an integer N_0, such that for all $\varphi \in \mathcal{T}_w([G]_P)$ (resp. $\mathcal{S}_{w,N}([G]_P)$), the family $(\varphi_\chi)_{\chi \in \mathfrak{X}(G)}$ is absolutely summable with the sum φ in $\mathcal{T}_{w,N}([G]_P)$ (resp. $\mathcal{S}_{w,N+N_0}([G]_P)$).*

Moreover, we have the following result from [BPCZ22, Section 2.9.5] (which follows from the density of $\mathcal{S}([G]_P)$ in $L^2_w([G]_P)^\infty$ stated in Subsection 4.3.6 and the continuity of the projections $\varphi \mapsto \varphi_{\mathfrak{X}}$).

Proposition 4.20. *Let $\mathfrak{X}$ be a subset of $\mathfrak{X}(G)$. The space $\mathcal{S}_{\mathfrak{X}}([G]_P)$ is dense in $L^2_{w,\mathfrak{X}}([G]_P)^\infty$.*

4.7.3. *Spectral expansion of kernel functions.* The right convolution by $f \in \mathcal{S}(G(\mathbb{A}))$ on the spaces $L^2_\chi([G]_P)$ and $L^2([G]_P)$ gives rise to integral operators whose kernel are denoted by $K_{f,P,\chi}$ and $K_{f,P}$ respectively. If $P = G$ we omit the subscript G. These kernel functions satisfy the following estimates, cf. [BPCZ22, Lemma 2.10.1.1].

Lemma 4.21. *There exists $N_0 > 0$ such that for every weight w on $[G]_P$ and every continuous seminorm $\|\cdot\|_{w,N_0}$ on $\mathcal{T}_{w,N_0}([G]_P)$, there exists a continuous seminorm $\|\cdot\|_{\mathcal{S}}$ on $\mathcal{S}(G(\mathbb{A}))$ such that for $f \in \mathcal{S}(G(\mathbb{A}))$ we have*

$$\sum_{\chi \in \mathfrak{X}(G)} \|K_{f,P,\chi}(\cdot, y)\|_{w,N_0} \ll \|f\|_{\mathcal{S}} w(y)^{-1}, \quad y \in [G]_P.$$

In particular

$$\sum_{\chi \in \mathfrak{X}(G)} |K_{f,P,\chi}(x,y)| \ll \|x\|_P^{N_0} w(x) w(y)^{-1} \|f\|_{\mathcal{S}}, \quad x, y \in [G]_P.$$

Put

$$K^0_{f,\chi}(x,y) = \int_{A_G^\infty} K_{f,\chi}(x, ay)\,\mathrm{d}a.$$

For every standard parabolic subgroup P of G, let $\mathcal{A}^0_{P,\chi,\mathrm{disc}}(G)$ be the closed subspace of $\mathcal{A}_{P,\mathrm{disc}}(G)$ spanned by left A_P^∞-invariant functions whose class belongs to $L^2_\chi([G]_{P,0})$. We have an isotypic decomposition

$$\mathcal{A}^0_{P,\chi,\mathrm{disc}}(G) = \widehat{\bigoplus_{\pi}} \mathcal{A}_{P,\pi,\mathrm{disc}}(G)$$

where the direct sum is indexed by a certain set of discrete automorphic representations π of $M_P(\mathbb{A})$, cf. [BPCZ22, Section 2.10.2]. Let $\mathcal{B}_{P,\pi}$ be a K-basis of $\mathcal{A}_{P,\pi,\mathrm{disc}}(G)$ (see Subsection 4.6.2) and set $\mathcal{B}_{P,\chi} = \bigcup \mathcal{B}_{P,\pi}$ where π ranges over the representations appearing in the above isotypic decomposition. Similarly, set $\mathcal{B}_{P,\pi,\chi} = \bigcup \mathcal{B}_{P,\pi,\chi}$. Recall that we have defined a measure $d\lambda$ on $i\mathfrak{a}_P^{G,*}$ in Subsection 4.5.2. The following spectral expansion is an extension of a result of Arthur [Art78, §4] to Schwartz functions (see [BPCZ22, Lemma 2.10.2.1]).

Lemma 4.22. *There is a continuous semi-norm $\|\cdot\|$ on $\mathcal{S}(G(\mathbb{A}))$ and an integer N such that for all $X, Y \in \mathcal{U}(\mathfrak{g}_\infty)$, all $x, y \in G(\mathbb{A})^1$, and all $f \in \mathcal{S}(G(\mathbb{A}))$ we have*

$$\sum_{\chi\in\mathfrak{X}(G)}\sum_{P_0\subset P}|\mathcal{P}(M_P)|^{-1}\int_{i\mathfrak{a}_P^{G,*}}\sum_{\varphi\in\mathcal{B}_{P,\pi,\chi}}\left|(\mathrm{R}(X)E(x,I(\lambda,f)\varphi,\lambda))(\mathrm{R}(Y)E(y,\varphi,\lambda))\right|\,d\lambda$$
$$\ll \|\mathrm{L}(X)\mathrm{R}(Y)f\|\,\|x\|_G^N\|y\|_G^N,$$

where $\mathcal{P}(M_P)$ is the set of semi-standard parabolic subgroups Q of G such that $M_Q \simeq M_P$. Moreover for all $x, y \in G(\mathbb{A})$ and all $\chi \in \mathfrak{X}(G)$ we have

$$K^0_{f,\chi}(x,y) = \sum_{P_0\subset P}|\mathcal{P}(M_P)|^{-1}\int_{i\mathfrak{a}_P^{G,*}}\sum_{\varphi\in\mathcal{B}_{P,\chi}}E(x,I(\lambda,f)\varphi,\lambda)\overline{E(y,\varphi,\lambda)}\,d\lambda. \tag{4.16}$$

Remark 4.23. In this paper, we will only use Lemma 4.22 under the assumption that the cuspidal datum χ is regular, that is that if χ is represented by (M_P,σ), the only $w \in W(P,P)$ satisfying $w\sigma \simeq \sigma$ is $w = 1$. Under this condition, the Langlands spectral decomposition [Lan76] implies that (4.16) reduces to

$$K^0_{f,\chi}(x,y) = \int_{i\mathfrak{a}_P^{G,*}}\sum_{\varphi\in\mathcal{B}_{P,\chi}}E(x,I(\lambda,f)\varphi,\lambda)\overline{E(y,\varphi,\lambda)}\,d\lambda.$$

4.8. **Approximation by constant terms.** If G is reductive, functions of uniformly moderate growth are approximated by their constant terms in a precise sense. This is the "approximation by constant terms", cf. [BPCZ22, Theorem 2.5.14.1 (1)]. We extend this to the case of possibly nonreductive groups satisfying the condition (SR).

Let $G = U \rtimes H$ be a connected algebraic group satisfying the condition (SR) as in Section 4.1. Let P be a standard D-parabolic subgroup. Put $P_H = P \cap H$, which is a parabolic subgroup of H. For $P \subset Q$, we define a weight function d_P^Q on $[H]_{P_H}$ by

$$d_P^Q(h) = \min_{\alpha\in\Psi_P^Q} d_{P_H,\alpha}(h), \quad h \in [H]_{P_H}. \tag{4.17}$$

We recall that $\Psi_P^Q = \Psi_P \setminus \Psi_Q$ is the set of roots of A_0 action on $\mathfrak{n}_P^Q = \mathfrak{n}_P/\mathfrak{n}_Q$. Note that P, Q are not parabolic subgroups of H, and the weight d_P^Q is not to be confused with $d_{P_H}^{Q_H}$ which we recall is defined by

$$d_{P_H}^{Q_H}(h) = \min_{\alpha\in\Psi_{P_H}^{Q_H}} d_{P_H,\alpha}(h), \quad h \in [H]_{P_H}.$$

However we have the following relation.

Lemma 4.24. *We have $d_P^Q \ll d_{P_H}^{Q_H}$.*

Proof. By definition, we have $N_P \cap H = N_{P_H}$ and $N_Q \cap H = N_{Q_H}$, so that $\mathfrak{n}_{P_H}/\mathfrak{n}_{Q_H}$ embeds into $\mathfrak{n}_P/\mathfrak{n}_Q$ as a subspace. It follows that $\Psi_{P_H}^{Q_H} \subset \Psi_P^Q$ which implies $d_P^Q \ll d_{P_H}^{Q_H}$.

The "approximation by constant terms" refers to the following theorem.

Theorem 4.25. *Assume that G satisfies condition (SR). Let $P \subset Q$. Let $N > 0$, $r \geq 0$ and $X \in \mathcal{U}(\mathfrak{g}_\infty)$. There exists a continuous seminorm $\|\cdot\| = \|\cdot\|_{N,X,r}$ such that for any $\varphi \in \mathcal{T}_N([G]_Q, \chi)$ and $x \in P_H(F)N_{Q_H}(\mathbb{A})\backslash H(\mathbb{A})$, we have*

$$|\mathrm{R}(X)\varphi(x) - \mathrm{R}(X)\varphi_P(x)| \ll \|x\|_P^N d_P^Q(x)^{-r}\|\varphi\|. \tag{4.18}$$

By Remark 4.10, we can also use $\|x\|_{P_H}$ in right hand side of 4.18. It is however of critical importance that the weight d_P^Q appears on the right hand side, not $d_{P_H}^{Q_H}$.

We begin with an elementary lemma.

Lemma 4.26. *Let V be a finite dimensional vector space over F and $U \subset V(\mathbb{A})$ be an open compact subgroup. Then there exists a homogeneous $X \in \mathcal{U}(\mathrm{Lie}(V(F_\infty)))$ such that for any $f \in C^\infty([V])^U$ and any sufficiently large integer r, we have*

$$\left\| f - \int_{[V]} f(v)\mathrm{d}v \right\|_{L^\infty} \ll_r \|\mathrm{R}(X^r)f\|_{L^\infty}. \tag{4.19}$$

Note that, by our convention on measures, the measure on $[V]$ is the Tamagawa measure, i.e. $\mathrm{vol}([V]) = 1$.

Proof. Let $\Lambda = V(F) \cap U$ which is a lattice in $V(F_\infty)$. Since $V(F)$ is dense in $V(\mathbb{A})$, the natural embedding induces a $V(F_\infty)$-equivariant isomorphism

$$\Lambda\backslash V(F_\infty) = V(F)\backslash V(\mathbb{A})/U.$$

Thus we are reduced to the same problem for $\Lambda\backslash V(F_\infty)$. We pick a suitable basis and identify it with $\mathbb{Z}^n\backslash\mathbb{R}^n$ for $n = \dim_{\mathbb{Q}} V$. By classical Fourier series theory, $X = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$ suffices.

Proof of Theorem 4.25. Up to replacing $\mathrm{R}(X)\varphi$ by φ, we can assume that $X = 1$. Since the constant term map $\varphi \mapsto \varphi_P$ is continuous from $\mathcal{T}_N([G]_Q, \chi)$ to $\mathcal{T}_N(P(F)N_Q(\mathbb{A})\backslash G(\mathbb{A}), \chi)$, there exists a continuous semi-norm $\|\cdot\|$ on $\mathcal{T}_N([G]_Q, \chi)$ such that for all $x \in P(F)N_Q(\mathbb{A})\backslash G(\mathbb{A})$, we have

$$|\mathrm{R}(X)\varphi(x) - \mathrm{R}(X)\varphi_P(x)| \ll \|x\|_P^N\|\varphi\|.$$

Thus we only need to consider those x such that $d_P^Q(x) > C$ for some C, hence $x \in \omega_{P_H}^{Q_H}[> C]$. For C large enough, $P_H(F)\mathfrak{s}^{Q_H} \supset \omega_{P_H}^{Q_H}[> C]$, hence we can assume $x \in \mathfrak{s}^{Q_H}$ at the beginning.

Note that

$$\varphi_P(x) = \int_{[N_P^Q]} \varphi(nx)\mathrm{d}n.$$

Take a filtration of N_P^Q

$$\{0\} = N_0 \subset N_1 \subset \cdots \subset N_k = N_P^Q$$

as in Lemma 4.6. For each $0 \le i \le k-1$, set

$$\varphi_i(x) := \int_{[N_i]} \varphi(nx)\mathrm{d}n.$$

Then $\varphi_0 = \varphi$ and $\varphi_k = \varphi_P$. It suffices to show that, for each i with $0 \le i \le k-1$, there exists a continuous semi-norm $\|\cdot\|_i$ on $\mathcal{T}_N([G]_Q, \chi)$ such that for $x \in \mathfrak{s}^{Q_H}$, we have

$$|\varphi_i(x) - \varphi_{i+1}(x)| \ll \|x\|_P^N d_P^Q(x)^{-r} \|\varphi\|_i.$$

Once we have this, the semi-norm $\|\cdot\| = \sup_i \|\cdot\|_i$ satisfies the condition.

Since N_{i+1}/N_i is a vector space, up to enlarging r (which is possible as $d_P^Q(x) > C$), by Lemma 4.26 there exists an $\bar{X} \in \mathfrak{n}_{i+1,\infty}/\mathfrak{n}_{i,\infty}$ such that we have

$$|\varphi_i(x) - \varphi_{i+1}(x)| = \left|\varphi_i(x) - \int_{[N_{i+1}/N_i]} \varphi_i(nx)\mathrm{d}n\right| \ll \left\|\mathrm{R}(\bar{X}^r)\mathrm{R}(x)\varphi_i\right\|_{L^\infty([N_{i+1}])}. \tag{4.20}$$

Take the complementary subspace $\mathfrak{n}_{i+1}^i$ of $\mathfrak{n}_i$ in $\mathfrak{n}_{i+1}$ as in the Lemma 4.6. Let $X \in \mathfrak{n}_{i+1}^i$ with image $\overline{X}$, then

$$\mathrm{R}(\overline{X}^r)\mathrm{R}(x)\varphi_i = \mathrm{R}(X^r)\mathrm{R}(x)\varphi_i = \mathrm{R}(x)\mathrm{R}\left(\mathrm{Ad}(x_\infty^{-1})(X^r)\right)\varphi_i.$$

Therefore, as $[N_{i+1}]$ is compact,

$$\left\|\mathrm{R}(\bar{X}^r)\mathrm{R}(x)\varphi_i\right\|_{L^\infty([N_{i+1}])} \ll \|x\|_P^N \sup_{y \in [G]_Q} \|y\|_P^{-N} \left|\mathrm{R}\left(\mathrm{Ad}(x_\infty^{-1})(X^r)\right)\varphi_i(y)\right|. \tag{4.21}$$

For $x \in \mathfrak{s}^{Q_H}$, x_∞ can be written in the form $x_\infty = ac$, where $a \in A_0^Q(T)$, and c lies in a compact subset of $H(F_\infty)$. Thus

$$\mathrm{Ad}(x_\infty^{-1})X = \mathrm{Ad}(c^{-1})\mathrm{Ad}(a^{-1})X = \mathrm{Ad}(c^{-1})e^{-\langle \alpha, H_0(a)\rangle}X,$$

where α is the only root of A_0 acting on $\mathfrak{n}_{i+1}^i$. Pick a basis E_i of $\mathcal{U}(\mathfrak{g}_\infty)_{\le r}$, the elements in $\mathcal{U}(\mathfrak{g}_\infty)$ of degree $\le r$. If we write

$$\mathrm{Ad}(x_\infty^{-1})X^r = \sum_i c_i(x)E_i, \tag{4.22}$$

then we deduce that $|c_i(x)| \ll d_{P_H,\alpha}(x)^{-r}$, since we recall that we have

$$d_{P_H,\alpha}(x) \ll e^{\langle \alpha, H_0(x)\rangle}.$$

when $x \in \mathfrak{s}^{P_H}$. The theorem then follows from (4.20), (4.21) and (4.22).

4.9. **A mild extension of Arthur's truncation operator.** Let G be a connected reductive group over F. Fix a minimal parabolic subgroup P_0. Let $\mathcal{F}$ be the set of standard parabolic subgroups of G. We define a space of functions $\mathcal{T}_{\mathcal{F}}(G)$ as

$$\mathcal{T}_{\mathcal{F}}(G) = \left\{ (\varphi_P) \in \prod_{P \in \mathcal{F}} \mathcal{T}(P(F)\backslash G(\mathbb{A})) \;\middle|\; \varphi_P - \varphi_Q \in \mathcal{S}_{d_P^Q}(P(F)\backslash G(\mathbb{A})) \text{ for any } P \subset Q \right\}.$$

This space embeds as a closed subspace of the LF space,

$$\prod_{P \in \mathcal{F}} \mathcal{T}(P(F)\backslash G(\mathbb{A})) \times \prod_{\substack{P \subset Q \\ P,Q \in \mathcal{F}}} \mathcal{S}_{d_P^Q}(P(F)\backslash G(\mathbb{A})),$$

and as such inherits an LF topology.

We define $S^0([G]_P^1)$ to be the Banach space of measurable functions on $[G]_P^1$ such that for any N,

$$\|f\|_{\infty,N} := \sup_{x \in [G]_P^1} \|x\|_P^N |f(x)| < \infty. \tag{4.23}$$

Proposition 4.27. *Let $\underline{\varphi} = (\varphi_P) \in \mathcal{T}_{\mathcal{F}}(G)$ be a collection of functions. For $g \in [G]_P^1$ and $T \in \mathfrak{a}_0$, define*

$$\Lambda^T \underline{\varphi}(g) = \sum_{P \in \mathcal{F}_0} \epsilon_P \sum_{\gamma \in P(F)\backslash G(F)} \hat{\tau}_P(H_P(\gamma g) - T_P)\varphi_P(\gamma g).$$

and

$$\Pi^T \underline{\varphi}(g) = F^G(g;T)\varphi_G(g).$$

Then for every $c > 0, N > 0$, there exists a continuous seminorm $\|\cdot\|_{c,N}$ on $\mathcal{T}_{\mathcal{F}}(G)$ such that

$$\|\Lambda^T \underline{\varphi} - \Pi^T \underline{\varphi}\|_{\infty,N} \leq e^{-c\|T\|}\|\underline{\varphi}\|_{c,N},$$

for $\varphi \in \mathcal{T}_{\mathcal{F}}(G)$ and $T \in \mathfrak{a}_0$ sufficiently positive, where the norm on left hand side is as defined in (4.23).

Remark 4.28. When $\varphi \in \mathcal{T}([G])$ and $\varphi_P = \varphi_P$ for all $P \in \mathcal{F}$, we have $(\varphi_Q)_P = \varphi_P$, and hence the family $\underline{\varphi} = (\varphi_P)_{P \in \mathcal{F}}$ is in $\mathcal{T}_{\mathcal{F}}(G)$ by the approximation by constant terms for G. In this case $\Lambda^T \underline{\varphi}$ is Arthur's truncation operator defined in [Art80].

Following Arthur [Art78, Section 6], for standard parabolic subgroups $P \subset Q$, we let Γ_P^Q be the characteristic functions of $H \in \mathfrak{a}_0$ that satisfies the following properties.

- $\langle \alpha, H\rangle > 0$ for all $\alpha \in \Delta_P^Q$.
- $\langle \alpha, H\rangle \leq 0$ for all $\alpha \in \Delta_P \setminus \Delta_P^Q$.
- $\langle \varpi, H\rangle > 0$ for all $\varpi \in \hat{\Delta}_Q$.

We will need the following lemma in the proof of Proposition 4.27.

Lemma 4.29. *Let G be a reductive group over F, then for every sufficiently positive $T \in \mathfrak{a}_0$ and every $g \in G(\mathbb{A})^1$ with*

$$F^P(g;T)\,\tau_P^Q(H_P(g) - T_P) \neq 0$$

there exists $r > 0$ such that

$$e^{\|T\|} \ll \left(\min_{\varpi \in \hat\Delta_0^Q \setminus \hat\Delta_0^P} d_{P,\varpi}\right)^r \quad \text{and}$$

$$\|g\|_P \ll \left(\max_{\varpi \in \hat\Delta_0^Q \setminus \hat\Delta_0^P} d_{P,\varpi}\right)^r.$$

Proof. The proof is similar to [BPCZ22, Lemma 3.5.1.2]. For the convenience of readers, we reproduce it here.

We can assume $g \in \mathfrak{s}^P$. By [BPC, Lemma 2.3.3.1], for any $\varpi \in \hat\Delta_0^Q \setminus \hat\Delta_0^P$, we have $\langle \varpi, H_0(g)\rangle > \langle \varpi, T\rangle - \varepsilon\|T\|$. Therefore

$$d_{P,\varpi}(g) \asymp e^{\langle \varpi, H_0(g)\rangle} > e^{\varepsilon\|T\|}$$

for any $\varpi \in \hat\Delta_0^P \setminus \hat\Delta_0^Q$, the first inequality is thus proved.

By [LW13, Lemme 2.10.6], when $g \in G(\mathbb{A})^1$ we have

$$\|H_P(g) - T_P\| \ll \|H_P^Q(g) - T_P^Q\| \ll 1 + \max_{\varpi \in \hat\Delta_P^Q} \langle \varpi, H_0(g) - T\rangle, \tag{4.24}$$

where $H_P^Q(g)$ and T_P^Q stands for projection of $H_P(g)$ and T_P into $\mathfrak{a}_P^Q$ respectively. The condition $F^P(H_0(g) - T) \neq 0$ implies

$$\|H^P(g)\| \ll \|T\|, \tag{4.25}$$

where $H^P(g)$ stands for the projection of $H_0(g)$ to $\mathfrak{a}^P$. For $\varpi \in \hat\Delta_0^Q \setminus \hat\Delta_0^P$, let $\varpi = \varpi_P + \varpi^P$ be the decomposition of ϖ according to $\mathfrak{a}_0 = \mathfrak{a}_P \oplus \mathfrak{a}_0^{P,*}$, then ϖ_P runs through $\hat\Delta_P^Q$ as ϖ runs through $\hat\Delta_0^Q \setminus \hat\Delta_0^P$. Since for any $\alpha \in \Delta_0^P$, $\langle \varpi^P, \alpha^\vee\rangle = \langle \varpi, \alpha^\vee\rangle \leq 0$. Thus ϖ^P is a non-positive linear combination of Δ_0^P, hence

$$\langle \varpi, H_0(g) - T\rangle = \langle \varpi_P, H_0(g) - T\rangle + \langle \varpi^P, H_0(g) - T\rangle \ll \langle \varpi_P, H_0(g) - T\rangle. \tag{4.26}$$

Combining (4.24), (4.25) and (4.26), we obtain

$$\|H_0(g)\| \ll 1 + \|T\| + \max_{\varpi \in \hat\Delta_0^Q \setminus \hat\Delta_0^P} \langle \varpi, H_0(g)\rangle. \tag{4.27}$$

Combining with the first assertion, we have

$$\|H_0(g)\| \ll 1 + \max_{\varpi \in \hat\Delta_0^Q \setminus \hat\Delta_0^P} \langle \varpi, H_0(g)\rangle.$$

Finally note that $g \in \mathfrak{s}^P$, thus for some $r > 0$,

$$\|g\|_P \ll e^{\|H_0(g)\|} \ll \max_{\varpi \in \hat\Delta_0^Q \setminus \hat\Delta_0^P} e^{r\langle \varpi, H_0(g)\rangle} \ll \left(\max_{\varpi \in \hat\Delta_0^Q \setminus \hat\Delta_0^P} d_{P,\varpi}\right)^r.$$

This proves the lemma.

Proof of Proposition 4.27. Our proof follows the same line as [BPCZ22, Section 3.5]. Using Langlands partition formula (4.10) and

$$ {}^{R}_{P}\widehat{}_{R} = \sum_{Q \quad R} {}^{Q}_{P} $$

we have

$$
\begin{aligned}
\Lambda^{T} \,'\_(g) = \sum \sum_{R 2 F \; P \quad R} {}_{R} \\
\sum_{2 R(F) n G(F)} \; \sum_{2 P(F) n R(F)} \widehat{}_{R}(H_R(\; g) \quad T_R)_R \,'(\; g) F^{P}(\quad g; T) \; {}^{R}_{P}(H_P(\quad g) \quad T_P) \\
= \sum_{P \quad Q} \; \sum_{2 P(F) n G(F)} F^{P}(\; g; T) \; {}^{Q}_{P}(H_P(\; g) \quad T)_{P;Q} \,'(\; g)
\end{aligned}
$$

where

$$ {}_{P;Q} \,'(g) = \sum_{P \quad R \quad Q} {}_{R} \quad {}_{R} \,'(g) $$

Hence

$$ \Lambda^{T} \,'\_(g) \quad \Pi^{T} \,'\_(g) = \sum_{P (Q} \; \sum_{2 P(F) n G(F)} F^{P}(\; g; T) \; {}^{Q}_{P}(H_P(\; g) \quad T_P)_{P;Q} \,'(\; g): $$

Since E^{G}_{P} is a continuous map from $S^{0}(P(F)nG(F))$ to $S^{0}([G])$, we only need to show for every $c > 0; N > 0$, there exists a continuous seminorm on $T_F(G)$ such that

$$ j_{P;Q} \,'(g) j \qquad e^{\; ckTk} kgk_{P}^{\;\; N} k \,'\_k \tag{4.28} $$

holds every g with $F^{P}(g; T) \; {}^{Q}_{P}(H_P(g) \quad T_P) \neq 0$ and every $'\_ \; 2 \; T_F(G)$.

Fix $\quad 2 \; \Delta^{Q}_{0} \; n \; \Delta^{R}_{0}$. For a parabolic subgroup R with $P \quad R \quad Q$ and $\quad 2 \; \Delta^{R}_{0}$, define $R \quad$ such that $\Delta^{R}_{0} \quad = \Delta^{R}_{0} \; n \; f \quad g$. Then there exists $N_0 > 0$ such that for any $r > 0$, there exists a seminorm $k \quad k$ on $T_F(G)$ such that

$$ j_{P;Q} \,'(g) j \qquad \sum_{\substack{P \quad R \quad Q \\ 2 \Delta^{R}_{0}}} j_{R} \,'(g) \qquad {}_{R} \quad '(g) j \qquad kgk_{P}^{N_0} \sum_{\substack{P \quad R \quad Q \\ 2 \Delta^{R}_{0}}} d_{R \;\; ;} \; (g) \quad {}^{r} k \,' \_ k: $$

By first assertions of Lemma 4.29, $g \; 2 \; !^{Q}_{P}[> C] \qquad !^{R}_{P} \quad [> C^{0}]$, hence $d_{R \;\; ;} \; (g) \qquad d_{P;} \; (g)$. Hence

$$ j_{P;Q} \,'(g) j \qquad kgk_{P}^{N_0} \sum_{\substack{P \quad R \quad Q \\ 2 \Delta^{R}_{0}}} d_{P;} \; (g) \quad {}^{r} k \,' \_ k: $$

Finally (4.28) follows if we let $\quad$ vary and use the second assertion of Lemma 4.29, .

4.10. **Decomposition of Schwartz functions.** We begin by some general considerations. Let G be an algebraic group over F with a morphism $p : G \to \mathbf{A}^m_F$. Here $\mathbf{A}^m_F$ stands for the m-dimensional affine space over F, whose F-points will also be noted by F^m. Let $\Lambda \subset F^m_\infty$ be a full lattice (i.e. discrete finitely generated abelian group which generate F^m_∞ as an $\mathbb{R}$-vector space).

Let $U \subset F^m_\infty$ be a neighbourhood of 0, such that $U \cap \Lambda = \{0\}$, choose any $u \in C^\infty_c(F^m_\infty)$ such that $\operatorname{supp} u \subset U$ and $u(0) = 1$. For any $\lambda \in \Lambda$, define a function u_λ on $G(\mathbb{A})$ by $u_\lambda(g) = u(p(g_\infty) - \lambda)$, and $f_\lambda := f \cdot u_\lambda$. For $\lambda \in \Lambda$, and $g \in G(\mathbb{A})$ with $p(g) = \mu$, we have

$$f_\lambda(g) = \begin{cases} f(g) & \mu = \lambda \\ 0 & \mu \neq \lambda \end{cases} \tag{4.29}$$

Proposition 4.30. *For any $f \in S(G(\mathbb{A}))$, the sequence of functions $(f_\lambda)_{\lambda \in \Lambda}$ is absolutely summable in $S(G(\mathbb{A}))$.*

Proof. The space $S(G(\mathbb{A}))$ is a union of Fréchet spaces $S(G(\mathbb{A}); C; K)$ (cf. Subsection 4.3 with Z being trivial), and if $f \in S(G(\mathbb{A}); C; K)$, then $f_\lambda \in S(G(\mathbb{A}); C; K)$ for all λ. It suffices to prove that for all $N > 0$ and $X \in U(\mathfrak{g}_\infty)$

$$\sum_{\lambda \in \Lambda} \|f_\lambda\|_{X,N} < \infty.$$

(c.f. Remark 4.13).

By the Leibniz rule,

$$X f_\lambda = \sum_{Y,Y'} c_{Y,Y'} \, Y u_\lambda \cdot Y' f$$

for some universal constants $c_{Y,Y'}$, thus it suffices to show for any $Y, Y' \in U(\mathfrak{g}_\infty)$

$$\sum_{\lambda \in \Lambda} \sup_{g \in G(\mathbb{A})} \|g\|^N |Y u_\lambda(g)| \, |Y' f(g)| < \infty.$$

By the chain rule,

$$Y u_\lambda(g) = \sum_Z c_Z \cdot (Zu)(p(g_\infty) - \lambda) \cdot p_Z(g_\infty)$$

for some $Z \in U(\mathfrak{g})$, where c_Z are constants and p_Z are polynomials. Choose N_1 such that $|p_Z(g_\infty)| \leq \|g_\infty\|^{N_1} \leq \|g\|^{N_1}$ holds for all Z. Then it suffices to show for any $Y', Z \in U(\mathfrak{g}_\infty)$

$$\sum_{\lambda \in \Lambda} \sup_{g \in G(\mathbb{A})} \|g\|^{N+N_1} |Zu(p(g_\infty) - \lambda)| \, |Y' f(g)| < \infty.$$

Choose an norm $\|\cdot\|$ on F^n_∞. There exists C such that $u(x - \lambda) \neq 0$ implies $\|x\| \geq C\|\lambda\|$, by [Kot05, Proposition 18.1 (1)] there exists N_2 such that $Zu(p(g_\infty) - \lambda) \neq 0$ implies $\|g_\infty\|^{N_2} \geq \|\lambda\|$. Since f is Schwartz, for any $M > 0$

$$\sup_{g \in G(\mathbb{A})} |Y' f(g)| \, \|g\|^{N_1+N} \ll \|g\|^{-M} \leq \|g_\infty\|^{-M}.$$

Combining these, for any $N' > 0$

$$\sum_{\lambda\in\Lambda}\sup_{g\in G(\mathbb{A})}\|g\|^{N+N_1}|Zu(p(g_1)\ \ )|\,|Y'f(g)|\ll\sum_{\lambda\in\Lambda}\|\lambda\|^{-N'};$$

which is finite for N' large enough.

5. Jacobi groups

5.1. **Jacobi groups: general linear groups.** Put $L = E^n$ (column vectors) and $L^\vee = E_n$ (row vectors). The usual dot product gives a pairing between L and $L^\vee$. If W is a linear subspace of L we will write $W^\vee = \{{}^t x \in L^\vee \mid x \in W\}$ and $W^\perp = \{y \in L^\vee \mid xy = 0 \text{ for all } x \in W\}$. Denote by $(e_1, \dots, e_n)$ the standard basis of E^n, and $({}^te_1, \dots, {}^te_n)$ its dual basis. We always consider G_n as the subgroup of G_{n+1} acting trivially on e_{n+1} the last vector of the canonical basis of E^{n+1}.

Let $\tilde{S} = L^\vee \oplus L \oplus E$ be the Heisenberg group over E whose multiplication is given by

$$(u, v, t)\cdot(u', v', t') = \left(u + u', v + v', t + t' + \frac{uv' - u'v}{2}\right);$$

The center of $\tilde{S}$ consists of elements of the form $(0, 0, t)$, $t \in E$ and is isomorphic to $\mathbb{G}_{a,E}$. Put $S = \mathrm{Res}_{E/F}\tilde{S}$.

The group G_n acts on the left on S as group automorphism by $g\cdot(u, v, t) = (ug^{-1}, gv, t)$. We define the Jacobi group to be the semi-direct product

$$J_n = S \rtimes G_n$$

The center of J_n is $Z(J_n) = Z(S) \rtimes \{1\} \subset J_n$ which is isomorphic to $\mathrm{Res}_{E/F}\mathbb{G}_a$. We denote Z for both centers of S and J_n.

Remark 5.1. The group J_n is not exactly what is usually called a Jacobi group in the literature, but rather the restriction of scalars of them.

The group J_n satisfies the condition (SR), therefore we can speak of its standard D-parabolic subgroups as in Subsection 4.1. Here we recall that we have the upper triangular minimal parabolic subgroup B_n of G_n and standard D-parabolic subgroups of J_n are those that contains B_n. We denote by $\mathcal{F}$ the set of standard D-parabolic subgroups of J_n.

We now give an explicit description of $\mathcal{F}$. For $P \in \mathcal{F}$, we put $P_n = P \cap G_n$ and assume that P_n is the stabilizer of the flag

(5.1) $$0 = L_0 \subset L_1 \subset \cdots \subset L_r = L.$$

Then P is either of the form

(5.2) $$P = (L_k^\perp \oplus L_k \oplus E) \rtimes P_n$$

for some $0 \le k \le r$, in which case we call P of type I, or of the form

(5.3) $$P = (L_k^\perp \oplus L_{k+1} \oplus E) \rtimes P_n$$

for $0 \le k \le r-1$, in which case we call P of type II. If P is of type I, the D-Levi decomposition for P is

$$M_P = (0 \oplus 0 \oplus E) \rtimes M_{P_n}, \quad N_P = (L_k^\perp \otimes L_k \oplus 0) \rtimes N_{P_n}.$$

If P is of type II, the D-Levi decomposition for P is

$$M_P = (L_k^\perp / L_{k+1}^\perp \oplus L_{k+1}/L_k \oplus E) \rtimes M_{P_n}, \quad N_P = (L_{k+1}^\perp \otimes L_k \oplus 0) \rtimes N_{P_n}.$$

It is readily checked that P is of type II if and only if $P = P(\lambda)$ with some component of $\lambda : \mathbb{G}_m \to T$ being the trivial character, otherwise it is of type I.

We also define the Heisenberg part of these groups. For $X \in \{P, M_P, N_P\}$ put

$$X_S = X \cap S, \quad X_{L^-} = X \cap L^-, \quad X_L = X \cap L.$$

Let $\mathcal{F}_{\mathrm{RS}}$ be the set of Rankin-Selberg parabolic subgroups of $G_{n+1} \times G_n$ introduced in [BPCZ22, Section 3.1]. The set $\mathcal{F}_{\mathrm{RS}}$ consists of semistandard parabolic subgroups $P_{n+1} \times P_n$ of $G_{n+1} \times G_n$, such that $P_n = P_{n+1} \cap G_n$ and P_n is standard. Let $P \in \mathcal{F}$ be a standard parabolic subgroup of J_n. We construct a Rankin–Selberg parabolic subgroup $P_{n+1} \times P_n \in \mathcal{F}_{\mathrm{RS}}$ as follows. First let $P_n = P \cap G_n$. Assume that P_n stabilizes the flag (5.1). If P is of type I and is of the form (5.2), then let P_{n+1} be the parabolic subgroup of G_{n+1} stabilizing the flag

$$0 = L_0 \subset \cdots \subset L_k \subset L_k \oplus \mathrm{span}_E(e_{n+1}) \subset \cdots \subset L_r \oplus \mathrm{span}_E(e_{n+1}). \tag{5.4}$$

If P is of type II and is of the form (5.3), then let P_{n+1} be the parabolic subgroup of G_{n+1} stabilizing the flag

$$0 = L_0 \subset \cdots \subset L_k \subset L_{k+1} \oplus \mathrm{span}_E(e_{n+1}) \subset \cdots \subset L_r \oplus \mathrm{span}_E(e_{n+1}). \tag{5.5}$$

Lemma 5.2. *Let the notation be as above. The map $P \mapsto P_{n+1} \times P_n$ is a bijection from $\mathcal{F}$ to $\mathcal{F}_{\mathrm{RS}}$. Moreover, for any $P \in \mathcal{F}$, the map*

$$\Psi_{P_{n+1}} \to \Psi_P, \quad \alpha \mapsto \alpha|_{A_n}$$

is a bijection. Here we recall that Ψ_P (resp. $\Psi_{P_{n+1}}$) stands for the roots of A_n (resp. A_{n+1}) on $\mathfrak{n}_P$ (resp. $\mathfrak{n}_{P_{n+1}}$).

Proof. This is clear from the above construction.

Let $P = MN \in \mathcal{F}$. Let us now explain that the nontrivial additive character ψ of $Z(\mathbb{A})$, the subgroup $P(F)$ or $M(F)N(\mathbb{A})$ of $G(\mathbb{A})$ satisfies the condition (SL) in Subsection 4.3, and hence it makes sense to speak of the various spaces of functions on $P(F)\backslash G(\mathbb{A})$ or $[G]_P$ (with the character ψ) defined there. We will see in the argument that this crucially relies on the fact that ψ is nontrivial. Since $N(F)\backslash N(\mathbb{A})$ is compact, it is enough to explain that $M(F)N(\mathbb{A})$ satisfies the condition (SL). For simplicity of notation, we assume that P is of type I. The type II case can be treated in the same way. Assume that

$$L_k = \mathrm{span}_E(e_1, \ldots, e_a), \quad L_{\overline{k}} = \mathrm{span}_E({}^t e_1, \ldots, {}^t e_a).$$

We view them as subgroups of S. We write an element in $J_n(\mathbb{A})$ as

$$(u+v, u^- + v^-, t) g_f g_\infty$$

where $g_f \in G(\mathbb{A}_f)$, $g_\infty \in J_n(\mathbb{A}_f)$, $u \in L_k(\mathbb{A}_f)$, $v \in L_{\overline{k}}^{-,?}(\mathbb{A}_f)$, $u^- \in L_{\overline{k}}(\mathbb{A}_f)$, $v^- \in L_{\overline{k}}(\mathbb{A}_f)$. First we know that there is a compact subset C_1 of $G(\mathbb{A})$ such that

$$P_n(F)(C_1 \quad G(F_\infty)) = G(\mathbb{A}).$$

Thus we need to explain that for any open compact subgroup U of $J_n(\mathbb{A}_f)$, there are compact subsets $C_2 \quad L_{\overline{k}}(\mathbb{A}_f)$ and $C_3 \quad L_{\overline{k}}^{-,?}(\mathbb{A}_f)$ such that if $\in C^\infty(M(F)N(\mathbb{A})\backslash J_n(\mathbb{A}), \ _E)$ and

$$((u+v, u^- + v^-, t) g_f g_\infty) \neq 0,$$

then $u^- \in C_2$ and $v \in C_3$. This can be seen as follows. First we may assume that $g_f \in C_1$. Then consider

$$\bigcap_{g_f \in C_1} g_f U g_f^{-1}.$$

Since U is an open compact subgroup of $G(\mathbb{A}_f)$ and C_1 is compact, this is essentially a finite intersection and hence an open subgroup of $G(\mathbb{A}_f)$. It follows that

$$U' = \bigcap_{g_f \in C_1} g_f U g_f^{-1} \cap N_{L^-}(\mathbb{A})$$

is a nontrivial open compact subgroup of $N_{L^-}(\mathbb{A}_f)$. We pick $(x, y, 0)$ in U'. Then we have

$$\begin{aligned}((u+v, u^- + v^-, t) g_f g_\infty) &= ((x, y, 0)(u+v, u^- + v^-, t) g_f g_\infty) \\ &= (u^- x + yv) \ ((u+v, u^- + v^-, t)(x, y, 0) g_f g_\infty) \\ &= ((u+v, u^- + v^-, t) g_f g_\infty).\end{aligned}$$

Here the first equality is because is left $M(F)N(\mathbb{A})$-invariant, the second is because is -invariant by the central element, and the third is because $g_f^{-1}(x, y, 0) g_f \in U$ be our choices. It follows that if $((u+v, u^- + v^-, t) g_f g_\infty) \neq 0$ then $(u^- x + yv) = 1$. As $(x, y, 0)$ varies in the group U', we conclude that u^- and v should lie in an open compact subgroup of $L_{\overline{k}}(\mathbb{A}_f)$ and $L_{\overline{k}}^{-,?}(\mathbb{A}_f)$ which only depends on U'. Note that this is where the fact that is nontrivial is used. This proves that $M(F)N(\mathbb{A})$ satisfies the condition (SL).

For $P \in F$, the embedding $G_n \hookrightarrow G_{n+1}$ induces an embedding

$$[G_n]_{P_n} \hookrightarrow [G_{n+1}]_{P_{n+1}}.$$

We can then restrict any weight function on $[G_{n+1}]_{P_{n+1}}$ to obtain a weight function on $[G_n]_{P_n}$. The embedding also induces an embedding $A_n^\infty \to A_{n+1}^\infty$.

Lemma 5.3. *For $P, Q \in F$, we have $d_P^Q \quad d_{P_{n+1}}^{Q_{n+1}}|_{[G_n]_{P_n}}$. Here d_P^Q is the weight on $[G_n]_{P_n}$ defined in* (4.17).

Proof. It is enough to prove that

$$d_P^Q(a) \ll d_{P_{n+1}}^{Q_{n+1}}(a), \quad a \in A_n^{\infty}.$$

By the definitions of $d_{P_{n+1}}^{Q_{n+1}}$ and d_P^Q (see also Remark 5.4 after this proof), for all $a \in A_n^{\infty}$ we have

$$d_{P_{n+1}}^{Q_{n+1}}(a) \asymp \min_{\alpha \in \Psi_{P_{n+1}}^{Q_{n+1}}} \alpha(a), \quad d_P^Q(a) \asymp \min_{\alpha \in \Psi_P^Q} \alpha(a).$$

The lemma then follows from Lemma 5.2.

Remark 5.4. We temporarily denote by G a reductive group with a fixed minimal parabolic subgroup $P_0 = M_0 N_0$ and a maximal split torus $A_0 \subset M_0$. Let P, Q be two standard parabolic subgroups. In general, by definition we have

$$d_{P,\alpha}(x) = e^{\langle \alpha, H_0(x) \rangle}, \quad x \in \mathfrak{s}^P,$$

and

$$d_P^Q(x) = \min_{\alpha \in \Psi_P^Q} d_{P,\alpha}(x), \quad x \in [G]_P.$$

Put $W^P = \mathrm{Norm}_{M_P(F)}(A_0)/M_0(F)$ and let $\Lambda \subset \mathfrak{a}_0^*$ be a W^P-invariant subset. Then for all $a \in A_0^{\infty}$, we have

$$\min_{\alpha \in \Lambda} d_{P,\alpha}(a) \asymp \min_{\alpha \in \Lambda} e^{\langle \alpha, H_0(a) \rangle}. \tag{5.6}$$

So in particular, for all $a \in A_0^{\infty}$, we have

$$d_P^Q(a) \asymp \min_{\alpha \in \Psi_P^Q} e^{\langle \alpha, H_0(a) \rangle} \tag{5.7}$$

As a consequence, (5.7) holds for all semi-standard P, Q.

5.2. **Theta functions: general linear groups.** We now introduce theta functions on $J_n(\mathbb{A})$. Recall that we have fixed a character $\psi_E : E \backslash \mathbb{A}_E \to \mathbb{C}^\times$ whose restriction to $\mathbb{A}$ equals ψ. Define an action of $J_n(\mathbb{A})$ on $\mathcal{S}(\mathbb{A}_{E,n})$ by

$$\mathrm{R}_{\mu^{-1}}((u, v, t)g)\Phi(x) = \mu(\det g)^{-1} |\det g|_E^{1/2} \Phi((x+u)g)\psi_E\left(xv + \frac{uv}{2} + t\right) \tag{5.8}$$

This is an SLF-representation of $J_n(\mathbb{A})$.

For $P = MN$ a standard D-parabolic subgroup of J_n, we define the theta series ${}_P\Theta(\cdot, \Phi)$ on $[J_n]_P$ as

$${}_P\Theta(j, \Phi) = \int_{N_{L^-}(\mathbb{A})} \sum_{m \in M_{L^-}(F)} \mathrm{R}_{\mu^{-1}}(j)\Phi(m+n)dn, \quad j \in [J_n]_P. \tag{5.9}$$

When $P = J_n$, we omit the left subscript and write simply Θ. If $g \in G_n(\mathbb{A})$, we have

$$\Theta(g, \Phi) = \mu(\det g)^{-1} |\det g|^{\frac{1}{2}} \sum_{x \in E_n} \Phi(xg),$$

This is closely related to the mirabolic Eisenstein series, cf. [JS81, Section 4.1], which differs essentially by a term corresponding to $x = 0$ and an integral along the center.

Lemma 5.5. *There is an $N>0$ such that ${}_P\Theta(\cdot;\Phi)\in\mathcal{T}_N([J_n]_P;\psi_E)$ for $\Phi\in\mathcal{S}(\mathbb{A}_n)$.*

Proof. The fact that ${}_P\Theta(\cdot;\Phi)$ is invariant under $M_P(F)N_P(\mathbb{A})$ and transform by ψ_E under multiplication by Z is a direct calculation.

We next show that ${}_P\Theta$ is of uniform moderate growth. Recall that a height function $\|\cdot\|$ is fixed on $\mathbb{A}_{E,n}$ by (4.3). By [BP21a, Proposition A.1.1 (v)] we can pick a large N_1 such that

$$\int_{N_{L^-}(\mathbb{A})}\sum_{m\in M_{L^-}(F)}\|mn\|^{-N_1}\mathrm{d}n$$

is convergent. For any $\Phi\in\mathcal{S}(\mathbb{A}_{E,n})$ and $N_1>0$, we put

$$\|\Phi\|_{N_1}=\sup_{x\in\mathbb{A}_{E,n}}\|x\|^{N_1}|\Phi(x)|$$

Then we are reduced to showing that we can find an N_2 such that for all $X\in\mathcal{U}(\mathfrak{j}_\infty)$ (where $\mathfrak{j}=\mathrm{Lie}(J_n)$) and $\Phi\in\mathcal{S}(\mathbb{A}_{E,n})$ we have

$$\sup_{j\in J_n(\mathbb{A})}\|j\|^{-N_2}\|\mathrm{R}_{\psi^{-1}}(j)(\mathrm{R}(X)\Phi)\|_{N_1}<\infty.$$

This can be checked directly from the definition (5.8) the action $\mathrm{R}_{\psi^{-1}}$.

Lemma 5.6. *For $P\subset Q\in\mathcal{F}$, we have*

$$\int_{[N_{P_S}]}{}_Q\Theta(nj;\Phi)\mathrm{d}n={}_P\Theta(j;\Phi);\qquad\sum_{\gamma\in N_{P_S}(F)\backslash N_{Q_S}(F)}{}_P\Theta(\gamma j;\Phi)={}_Q\Theta(j;\Phi).$$

Proof. This is a direct calculation using (5.8).

5.3. **Jacobi groups: unitary groups.** Let (V,q_V) be a nondegenerate n-dimensional skew-Hermitian vector space over E and $\mathrm{U}(V)$ the corresponding unitary group.

Let $\mathrm{Res}\,V$ be the symplectic space over F whose underline vector space is V, and the symplectic form $\mathrm{Tr}_{E/F}\circ q_V$. Let $S(V)=\mathrm{Res}\,V\oplus F$ be the Heisenberg group, where the multiplication is given by

$$(v_1,t_1)\cdot(v_2,t_2)=\left(v_1+v_2,t_1+t_2+\frac{1}{2}\mathrm{Tr}_{E/F}\,q_V(v_1,v_2)\right).$$

The group $\mathrm{U}(V)$ acts on $S(V)$ by $x\cdot(v,t)=(xv,t)$ for $x\in\mathrm{U}(V)$ and $(v,t)\in S(V)$. Define the Jacobi group to be the semi-direct product

$$J(V)=S(V)\rtimes\mathrm{U}(V).$$

The center of $S(V)$ and $J(V)$ are both isomorphic to $\mathbb{G}_a$, we use Z to denote either of them.

Let m be the Witt index of V, and V_{an} be an anisotropic kernel of V. Choose a basis

$$e_1,\dots,e_m,e_{-1},\dots,e_{-m}$$

of the orthogonal complement of V_{an}, such that

$$q_V(e_i,e_j)=q_V(e_{-i},e_{-j})=0,\quad q_V(e_i,e_{-j})=\delta_{ij},\quad 1\le i,j\le m.$$

Let P_0 be the minimal parabolic subgroup of $\mathrm{U}(V)$ stabilizing the flag

$$0 \subset \mathrm{span}_E(e_1) \subset \mathrm{span}_E(e_1, e_2) \subset \cdots \subset \mathrm{span}_E(e_1, \cdots, e_m). \tag{5.10}$$

Let A_0 be the maximal split torus contained in P_0. As in the general linear group case, the Jacobi group $J(V)$ satisfies the condition (SR), hence we can speak of the standard D-parabolic subgroups of $J(V)$. Let $\mathcal{F}_V$ be the set of standard D-parabolic subgroups of $J(V)$ and $\mathcal{F}_V'$ be the set of standard parabolic subgroups of $\mathrm{U}(V)$. If $P \in \mathcal{F}_V$, then we put $P' = P \cap \mathrm{U}(V) \in \mathcal{F}_V'$.

Lemma 5.7. *The map*

$$\mathcal{F}_V \to \mathcal{F}_V', \quad P \mapsto P'$$

is a bijection.

Proof. By definition, $P' \in \mathcal{F}_V'$ is standard parabolic subgroup. Assume that it stabilizes an isotropic flag of the form

$$0 = X_0 \subset X_1 \subset \cdots \subset X_r \tag{5.11}$$

in V. Then there is a unique $P \in \mathcal{F}_V$ such that $P \mapsto P'$ given by $P = (X_r^\perp \times F) \rtimes P'$.

Assume that $P \in \mathcal{F}_V$ and $P' = P \cap \mathrm{U}(V)$ stabilizes the isotropic flag (5.11). Assume that $X_r = \mathrm{span}_E(e_1, \dots, e_{a_r})$. We put

$$X_{\bar{r}} = \mathrm{span}_E(e_{\bar{1}}, \dots, e_{\bar{a}_r}), \quad W_r = (X_r + X_{\bar{r}})^\perp.$$

Then the D-Levi decomposition for P is given by

$$M_P = (W_r \times F) \rtimes M_{P'}, \quad N_P = V_r \rtimes N_{P'}$$

In particular, M_P is a product of general linear groups and of the Jacobi group attached to the skew-hermitian space W_r.

We also define the Heisenberg part of these groups. For $X \in \{P, M_P, N_P\}$ put

$$X_S = X \cap S(V), \quad X_V = X \cap V.$$

Then we have

$$P_V = X_r + W_r, \quad M_{P_V} = W_r, \quad N_{P_V} = X_r.$$

Recall that we have defined weights d_P^Q and $d_{P'}^{Q'}$ at the end of Subsection 4.1.

Lemma 5.8. *For $P, Q \in \mathcal{F}_V$ with $P \subset Q$, then as weights on $[\mathrm{U}(V)]_{P'}$, we have*

$$\min\{d_{P'}^{Q'}, (d_{P'}^{Q'})^{\frac{1}{2}}\} \ll d_P^Q \ll d_{P'}^{Q'}.$$

Proof. By Lemma 4.24 we have $d_P^Q \ll d_{P'}^{Q'}$, which gives the second inequality.

Assume that P' stabilizes the flag (5.11), and that Q' stabilizes another flag

$$0 = X_0 \subset X_{k_1} \subset \cdots \subset X_{k_s},$$

with $1 < k_1 < \cdots < k_s \le r$, obtained by deleting some of the terms in the flag (5.11). If $X_{k_s} = X_r$, then one checks that $\mathfrak{n}_P^Q = \mathfrak{n}_{P'}^{Q'}$, and hence the lemma holds automatically. Assume that $X_{k_s} \neq X_r$, and write

$$V_{k_s} = \mathrm{span}_E(e_1, \dots, e_{a_s}), \quad V_r = \mathrm{span}_E(e_1, \dots, e_{a_r}).$$

Then $\mathfrak{n}_P^Q = \mathfrak{n}_{P'}^{Q'} \oplus \mathrm{span}_E(e_{a_s+1}, \dots, e_{a_r})$. For $i = a_s+1, \dots, a_r$, let $\alpha_i \in \Psi_P^Q$ be the root that appears in $\mathrm{span}_E(e_i) \otimes \mathrm{span}_E(e_{a_s+1}, \dots, e_{a_r})$. Then we observe that $2\alpha_i \in \Psi_{P'}^{Q'}$ for all i. It follows that we have

$$e^{\langle \alpha_i, H_0(x)\rangle} = \left(e^{\langle 2\alpha_i, H_0(x)\rangle}\right)^{\frac{1}{2}} \ge d_{P'}^{Q'}(x)^{\frac{1}{2}}, \quad x \in \mathfrak{s}^{P'}.$$

Therefore for $x \in \mathfrak{s}^{P'}$ we have

$$d_P^Q(x) \ge \min\{d_{P'}^{Q'}(x), \min_{a_s+1 \le i \le a_r} e^{\langle \alpha_i, H_0(x)\rangle}\} \ge \min\{d_{P'}^{Q'}(x), d_{P'}^{Q'}(x)^{\frac{1}{2}}\}.$$

This gives the first inequality and concludes the proof.

5.4. **Theta functions: unitary groups.** Let V be a nondegenerate n-dimensional skew-Hermitian space. Let $S(V)$ be the associated Heisenberg group. Fix a polarization $\mathrm{Res}\, V = L \oplus L^-$, i.e. L and L^- are maximal isotropic subspaces of $\mathrm{Res}\, V$ such that the pairing $\mathrm{Tr}_{E/F}\, q_V$ is nondegenerate when restricted to $L \otimes L^-$. We denote by $\omega = \omega_\psi$ the oscillator representation of $S(V)(\mathbb{A})$ on $\mathcal{S}(L^-(\mathbb{A}))$. For $\phi \in \mathcal{S}(L^-(\mathbb{A}))$, it is characterized by

$$(5.12) \quad (\omega(l + l', z))\phi(x) = \psi\left(z + (x, l) + \frac{1}{2}\mathrm{Tr}_{E/F}\, q_V(l', l)\right)\phi(x + l'), \quad l \in L(\mathbb{A}), \quad x, l' \in L^-(\mathbb{A}).$$

A different choice of the polarization gives another model, and the isomorphism between the two models are given by a partial Fourier transform, cf. [MVW87, Chapitre 2, I. 7].

Let $\mathrm{Mp}(\mathrm{Res}\, V)(\mathbb{A})$ be the metaplectic group attached to $\mathrm{Res}\, V$, which sits in a central extension

$$1 \to \mathbb{C}^1 \to \mathrm{Mp}(\mathrm{Res}\, V)(\mathbb{A}) \to \mathrm{Sp}(\mathrm{Res}\, V)(\mathbb{A}) \to 1,$$

where $\mathbb{C}^1$ stands for the complex numbers of norm 1. Note that even though we use this notation, $\mathrm{Mp}(\mathrm{Res}\, V)$ is not an algebraic group, and indeed does not make sense on its own. A theorem of Weil implies that the oscillating representation canonically extends to a representation of the group

$$S(V)(\mathbb{A}) \rtimes \mathrm{Mp}(\mathrm{Res}\, V)(\mathbb{A}).$$

By definition there is an embedding $\mathrm{U}(V) \to \mathrm{Sp}(\mathrm{Res}\, V)$. Recall that we have fixed a character $\chi : E^\times \to \mathbb{A}_E^\times$ extending the quadratic character η. Given such a χ, there is an explicit lift of this embedding $\iota_\chi : \mathrm{U}(V)(\mathbb{A}) \to \mathrm{Mp}(\mathrm{Res}\, V(\mathbb{A}))$, cf. [Kud94]. In this way we obtain a representation of $J(V)(\mathbb{A})$, realized on $\mathcal{S}(L^-(\mathbb{A}))$. This representation depends on two characters ψ and χ, and we denoted it by $\omega_{\psi,\chi}$, or simply ω when the characters are clear from the context.

We define the theta function

$$(5.13) \quad \theta_{\psi,\chi}(j, \phi) = \sum_{x \in L^-(F)} \omega_{\psi,\chi}(j)\phi(x), \quad j \in J(V)(\mathbb{A}), \quad \phi \in \mathcal{S}(L^-(\mathbb{A})).$$

As the different models are related by partial Fourier transforms, the Poisson summation formula ensures that the theta function is independent of the choice of polarization.

For our purposes, it is more convenient to choose a specific polarization $V = L \oplus L^-$ and interpret the model $\mathcal{S}(L^-(\mathbb{A}))$ as a mixed model, where the actions of parabolic subgroups are transparent. We follow the exposition in [GI16]. Though only local non-Archimedean cases were considered in [GI16], the formulae listed there are also valid in the Archimedean case, and taking product gives the formulae in the global situation. Recall that we have fixed a minimal parabolic subgroup P_0 of $\mathrm{U}(V)$ stabilizing the maximal isotropic flag (5.10). We fix a polarization $\mathrm{Res}\, V_{\mathrm{an}} = L_{\mathrm{an}} \oplus L_{\mathrm{an}}^-$. Put

$$L = L_{\mathrm{an}} \oplus \mathrm{span}_E(e_1, \dots, e_m), \quad L^- = L_{\mathrm{an}}^- \oplus \mathrm{span}_E(e_{\bar{1}}, \dots, e_{\bar{m}}).$$

Then $\mathrm{Res}\, V = L \oplus L^-$ is a polarization of $\mathrm{Res}\, V$.

Let $1 \leq k \leq m$ be a fixed integer. We put

$$X = \mathrm{span}_E(e_1, \dots, e_k), \quad X^- = \mathrm{span}_E(e_{\bar{1}}, \dots, e_{\bar{k}});$$
$$Y = \mathrm{span}_E(e_{k+1}, \dots, e_m), \quad Y^- = \mathrm{span}_E(e_{\overline{k+1}}, \dots, e_{\bar{m}}).$$

Set $V_0 = Y \oplus V_{\mathrm{an}} \oplus Y^-$. We have a decomposition $V = X \oplus V_0 \oplus X^-$. We view X, X^-, Y, Y^- as F-subspaces of $\mathrm{Res}\, V$. Put

$$L_0 = L_{\mathrm{an}} \oplus Y, \quad L_{\bar{0}} = L_{\mathrm{an}}^- \oplus Y^-.$$

Then $\mathrm{Res}\, V_0 = L_0 \oplus L_{\bar{0}}$ is a polarization of $\mathrm{Res}\, V_0$.

We denote by ω_0 the Weil representation of $J(V_0)$, realized on $\mathcal{S}_0 = \mathcal{S}(L_{\bar{0}}(\mathbb{A}))$. Via the canonical isomorphism $\mathcal{S}(L^-(\mathbb{A})) \simeq \mathcal{S}(X^-(\mathbb{A})) \hat{\otimes} \mathcal{S}_0$, we view elements in $\mathcal{S}(L^-(\mathbb{A}))$ as Schwartz functions on $X^-(\mathbb{A})$ valued in $\mathcal{S}_0$. We now describe the action. Let $P' = MN \subset \mathrm{U}(V)$ be the maximal parabolic subgroup stabilizing the isotropic subspace X in V. We write elements in P as $m_P(a) g_0 n_P(b) n_P(c)$ where $g_0 \in \mathrm{U}(V_0)$, $a \in \mathrm{GL}(X)$, $b \in \mathrm{Hom}(V_0, X)$, $c \in \mathrm{Herm}(X^-, X)$, and

$$m_P(a) = \begin{pmatrix} a & & \\ & 1_{V_0} & \\ & & (a^*)^{-1} \end{pmatrix}, \quad n_P(b) = \begin{pmatrix} 1_X & b & -\frac{1}{2}bb^* \\ & 1_{V_0} & -b^* \\ & & 1_{X^-} \end{pmatrix}, \quad n_P(c) = \begin{pmatrix} 1_X & 0 & c \\ & 1_{V_0} & 0 \\ & & 1_{X^-} \end{pmatrix},$$

and

$$\mathrm{Herm}(X^-, X) = \{c \in \mathrm{Hom}(X^-, X) \mid c^* = -c\}.$$

Here $a^* \in \mathrm{GL}(X^-)$, $b^* \in \mathrm{Hom}(X^-, V_0)$ and $c^* \in \mathrm{Hom}(X^-, X)$ are defined by

$$q_V(ax, x^-) = q_V(x, a^* x^-), \quad q_V(bv, x^-) = q_V(v, b^* x^-), \quad q_V(cx^-, y^-) = q_V(x^-, c^* y^-),$$

for $v \in V_0$, $x \in X$ and $x^-, y^- \in X^-$. We list the actions of various elements in $J(V)$ on $\mathcal{S}(L^-(\mathbb{A}))$. Take $\phi \in \mathcal{S}(L^-(\mathbb{A}))$ and $x \in X^-(\mathbb{A})$. The action of element in P are given as follows.

$$\omega(g_0)\phi(x) = \omega_0(g_0)(\phi(x)); \tag{5.14}$$

$$\omega(m_P(a))\phi(x) = \chi(\det a)|\det a|^{\frac{1}{2}}\phi(a\, x); \tag{5.15}$$

$$\omega(n_P(b))\phi(x) = \omega_0((b\, x; 0))(\phi(x)); \tag{5.16}$$

$$\omega(n_P(c))\phi(x) = \psi(\frac{1}{2}\mathrm{Tr}_{E/F}\, q_V(cx; x))(\phi(x)). \tag{5.17}$$

Define an isomorphism $I_X : X^- \to X$ by $I_X(e_{\bar{i}}) = e_i$, and put

$$w_P = \begin{pmatrix} & & I_X \\ & \mathbf{1}_{V_0} & \\ I_X^{-1} & & \end{pmatrix} \in \mathrm{U}(V).$$

Then

$$\omega(w_P)\phi(x) = \int_{X(\mathbb{A})} \phi(I_X^{-1}y)\,\psi(\mathrm{Tr}_{E/F}\, q_V(y; x))\mathrm{d}y. \tag{5.18}$$

Finally elements in $S(V)$ acts as follows. For $u \in X$, $u^- \in X^-$, $v_0 \in V_0$ and $\phi \in S(X^-(\mathbb{A}))$ we have

$$\omega((u + v_0 + u^-; 0))\phi(x) = \psi(\mathrm{Tr}_{E/F}\, q_V(x; u) + \frac{1}{2}\mathrm{Tr}_{E/F}\, q_V(u^-; u))\,\omega_0((v_0; 0))(\phi(x + u^-)); \tag{5.19}$$

We write $S_{\mathrm{an}} = S(L^-_{\mathrm{an}}(\mathbb{A}))$ for a model of the representation ω_{an} of $J(V_{\mathrm{an}})(\mathbb{A})$. In a similar fashion, via the canonical isomorphism $S(L_{\bar{0}}(\mathbb{A})) \simeq S(X_{\bar{0}}(\mathbb{A})) \widehat{\otimes} S_{\mathrm{an}}$, we may interpret elements in $S(L_{\bar{0}}(\mathbb{A}))$ as Schwartz functions on $X_0(\mathbb{A})$ valued in S_{an} and $S(L_{\bar{0}}(\mathbb{A}))$ as a mixed model.

We now define theta functions in general. Let $P \in \mathcal{F}_V$ be a standard parabolic subgroup of $J(V)$ and let $X \in \{P, M_P, N_P\}$. We set

$$X_L = X \cap L, \quad X_{L^-} = X \cap L^-.$$

We define

$$\theta_P(j; \phi) = \sum_{x \in M_{P_{L^-}}(F)} \omega(j)\phi(x), \quad j \in J(V)(\mathbb{A}).$$

Proposition 5.9. *For any $\phi \in S(L^-(\mathbb{A}))$, we have $\theta_P(\cdot; \phi) \in \mathcal{T}([J(V)]_P;\ )$. Moreover for any $j \in J(\mathbb{A})$, we have*

$$\int_{[N_{P_V}]} \theta(nj; \phi)\mathrm{d}n = \theta_P(j; \phi).$$

In particular, for any $\varphi \in \mathcal{T}([\mathrm{U}(V)])$, the constant term of the function $\varphi(\cdot)\theta(\cdot; \phi)$ on $[J(V)]$ along P equals $\varphi_{P'}(\cdot)\theta_P(\cdot; \phi)$.

Proof. The invariance of the theta function by $M(F)N(\mathbb{A})$, and the constant term calculation can be checked directly using mixed models. That it is of uniformly moderate growth can be proved in exactly the same way as Lemma 5.5.

For $P \in \mathcal{F}_V$, we also put

$$(5.20) \qquad \phi_P^{-}(j;\lambda) = \sum_{x \in M_{P_{L^-}}(F)} \omega^{-}(j)\phi(x).$$

Then $\phi_P^{-} \in \mathcal{T}([J]_P;\mu^{-1})$ and exhibits similar properties as in Proposition 5.9. We have

$$\phi_P^{-}(j;\lambda) = \overline{\phi_P(j;\overline{\lambda})}.$$

6. The coarse spectral expansion: general linear groups

6.1. **Notation.** We first list notation that will be used throughout this section.

Put $L = E^n$ and $L^- = E_n$.

Put $G = G_n \times G_n$, $G' = G'_n \times G'_n$, and $H = G_n$ which embeds in G diagonally. If $g \in G$ or G', we always write $g = (g_1, g_2)$ where $g_i \in G_n$ or G'_n. We view G as a subgroup of $G_{n+1} \times G_n$ via the embedding $G_n \to G_{n+1}$.

We fix good maximal compact subgroups (see Subsection 4.5) K_G, K_H and $K_{G'}$ of $G(\mathbb{A})$, $H(\mathbb{A})$ and $G'(\mathbb{A})$ respectively.

Define two characters of $G'(\mathbb{A})$ by

$$(6.1) \qquad \eta_{n+1}(g_1, g_2) = \eta(\det g_1 g_2)^{n+1}, \qquad \eta_{G'}(g_1, g_2) = \eta(\det g_1)^{n}\,\eta(\det g_2)^{n+1}.$$

For $k = n, n+1$, we have the subgroups B_k, T_k, A_k of G_k, and subgroups B'_k, T'_k of G'_k, cf. Subsection 3.3.

For $k = n, n+1$, put $\mathfrak{a}_k = \mathfrak{a}_{B_k}$ and $\mathfrak{a}'_k = \mathfrak{a}_{B'_k}$. A truncation parameter T is an element in $\mathfrak{a}'_{n+1}$.

Recall that we define the Jacobi group $J_n = S \rtimes G_n$ and we put $\widetilde{G} = J_n \times G_n$. There is a natural embedding $J_n \to \widetilde{G}$ which is identity on the first coordinate and the natural projection on the second coordinate. The image is denoted by $\widetilde{H}$.

Recall the convention that if Q is a D-parabolic subgroup, without saying the contrary, we write $Q = M_Q N_Q$ for its D-Levi decomposition.

Recall that $\mathcal{F}$ is the set of standard D-parabolic subgroup of J_n. Let

$$P \mapsto P_{n+1} \times P_n$$

be the bijection defined in Lemma 5.2, where P_k is a semistandard parabolic subgroup of G_k, $k = n, n+1$, and $P_n = P_{n+1} \cap G_n$ is standard.

If $P \in \mathcal{F}$, we put

$$\widetilde{P} = P \times P_n, \quad P_G = \widetilde{P} \cap G = P_n \times P_n, \quad P_H = \widetilde{P} \cap H.$$

and

$$P_{G'} = \widetilde{P} \cap G', \quad P'_n = P_n \cap G'_n, \quad P'_{n+1} = P_{n+1} \cap G'_{n+1}.$$

They are parabolic subgroups of $\widetilde{G}, G, H, G', G'_n, G'_{n+1}$ respectively. We also put $M_G = M_n \times M_n$ and $N_G = N_n \times N_n$ and hence $P_G = M_G N_G$ is a Levi decomposition.

Put $G_+ = G \times E_n$. This is viewed as a group over F, with the groups structure given by simply the product of the group G and the additive group E_n. Note that this is not a subgroup of $\widetilde{G}$, but merely a subvariety of it. For $P \in \mathcal{F}$, we define $M_{P,+} = M_G \times M_{L^-}$, $N_{P,+} = N_G \times N_{L^-}$, and $P_+ = P_G \times P_{L^-}$, where M_{L^-}, N_{L^-} and P_{L^-} are defined in Subsection 5.1. An element in P_+ is often written as (x,u) where $x \in P_G$ and $u \in P_{L^-}$, or as $m_+ n_+$ where $m_+ \in M_{P,+}$ and $n_+ \in N_{P,+}$. Note that the product $m_+ n_+$ is taken in G_+, not in $\widetilde{G}$.

There is a right action of $H \times G$ on G_+ given by

$$(x,u) \cdot (h,g) = (h^{-1} x g, uh),$$

and it restricts to an action of $M_{P_H} \times M_{P_G}$ on P_+ for any $P \in \mathcal{F}$.

6.2. **Technical preparations.** Let $P \in \mathcal{F}$ be a standard parabolic subgroup of J_n and w be a weight on $[G]_{P_G}$. We pull w back to a function on $[\widetilde{G}]_{\widetilde{P}}$, which we still denote by w. Let N be a nonnegative integer. Let $Z \simeq \mathrm{Res}_{E/F}\, \mathbb{G}_{a,E}$ be the central unipotent subgroup $Z \subset 1 \times \widetilde{G}$ and $\psi : [Z] \to \mathbb{C}^\times$ be a non trivial character. We then have various spaces of functions as defined in Subsection 4.3.

For $P, Q \in \mathcal{F}$ and $P \subset Q$, we define weights on $[G]_{P_G}$ by

$$\Delta_P(g) = \inf_{\gamma \in M_{P_n}(F) N_{P_n}(\mathbb{A})} \|g_1^{-1} \gamma g_2\|, \quad d_P^{Q,\Delta}(g) = \min\left\{ d_P^Q(g_1), d_{P_n}^{Q_n}(g_2) \right\}$$

Pulling back under the natural projection $[\widetilde{G}]_{\widetilde{P}} \to [G]_{P_G}$, we get two weights on $[\widetilde{G}]_{\widetilde{P}}$, which we still denote by Δ_P and $d_P^{Q,\Delta}$.

The "approximation by constant terms" for the group $\widetilde{G}$ takes the following form.

Proposition 6.1. *Let $N > 0, r \geq 0$ and for $P, Q \in \mathcal{F}$ with $P \subset Q$. There exists a continuous semi-norm $\|\cdot\|_{N,X,r}$ on $\mathcal{T}_N(\widetilde{Q}(F)\backslash\widetilde{G}(\mathbb{A}), \psi)$ such that*

$$|\mathrm{R}(X)\varphi(g) - \mathrm{R}(X)\varphi_{\widetilde{P}}(g)| \le \|g\|_{P_G}^N d_P^{Q,\Delta}(g)^{-r} \|\varphi\|_{N,X,r}$$

holds for all $\varphi \in \mathcal{T}_N([\widetilde{G}]_{\widetilde{Q}}, \psi)$ and $g \in G(\mathbb{A}) \subset \widetilde{G}(\mathbb{A})$.

Proof. For $g = (g_1, g_2) \in [G]_{P_G}$ we have

$$d_{\widetilde{P}}^{\widetilde{Q}}(g_1, g_2) \ge d_P^Q(g_1) d_{P_n}^{Q_n}(g_2) \ge d_P^{Q,\Delta}(g_1, g_2)^2.$$

Hence the result follows from Theorem 4.25.

We recall some function spaces introduced in [BPCZ22, Section 3.4.1]. Let $\mathcal{T}_F(G_n)$(resp. $\mathcal{T}_F(G'_n)$) be the space of tuples of functions

$$(\varphi_P)_{P \in \mathcal{F}} \in \prod_{P \in \mathcal{F}} \mathcal{T}([G_n]_{P_n}), \quad \text{resp. } (\varphi_P)_{P \in \mathcal{F}} \in \prod_{P \in \mathcal{F}} \mathcal{T}([G'_n]_{P'_n})$$

such that

$$\varphi_P - (\varphi_Q)_{P_n} \in S_{d_P^Q}([G_n]_{P_n}), \quad \text{resp. } \varphi_P - (\varphi_Q)_{P'_n} \in S_{d_P^Q}([G'_n]_{P'_n}),$$

for all $P \subsetneq Q \in \mathcal{F}$. These spaces agree with the spaces $\mathcal{T}_{\mathcal{F}_{\mathrm{RS}}}(G_n)$ (resp. $\mathcal{T}_{\mathcal{F}_{\mathrm{RS}}}(G'_n)$) defined in defined in [BPCZ22] by similar formulae, where the product ranges over $\mathcal{F}_{\mathrm{RS}}$ and the weights are $d^{Q_{n+1}}_{P_{n+1}}$ and $d^{Q'_{n+1}}_{P'_{n+1}}$ respectively. This is because there is a bijection between $\mathcal{F}$ and $\mathcal{F}_{\mathrm{RS}}$, and $d_P^Q \asymp d^{Q_{n+1}}_{P_{n+1}}|_{[G_n]_{P_n}}$ (cf. Lemma 5.3), and $d^{Q_{n+1}}_{P_{n+1}}|_{[G'_n]_{P'_n}} \asymp d^{Q'_{n+1}}_{P'_{n+1}}|_{[G'_n]_{P'_n}}$ (cf. [BPCZ22, Lemma 2.4.4.2]).

By the approximation by constant terms for the groups G and G' respectively, cf. Theorem 4.25, a family of functions $(\varphi_P)_{P\in\mathcal{F}}$ is in $\mathcal{T}_{\mathcal{F}}(G_n)$ (resp. $\mathcal{T}_{\mathcal{F}}(G'_n)$) if and only if there exists an integer $N_0 \geq 0$, such that for all $P \subsetneq Q \in \mathcal{F}$ and for any $g \in G_n(\mathbb{A})$ (resp. $G'_n(\mathbb{A})$), $X \in \mathcal{U}(\mathfrak{g}_{n,\infty})$ (resp. $\mathcal{U}(\mathfrak{g}'_{n,\infty})$), and $r > 0$ we have

$$|\mathrm{R}(X)\varphi_P(g) - \mathrm{R}(X)\varphi_Q(g)| \ll \|g\|_{P_n}^{N_0} d_P^Q(g)^{-r} \quad \text{resp. } \|g\|_{P'_n}^{N_0} d_P^Q(g)^{-r}. \tag{6.2}$$

As explained in [BPCZ22, Section 3.4.1], $\mathcal{T}_{\mathcal{F}}(G_n)$ and $\mathcal{T}_{\mathcal{F}}(G'_n)$ are LF spaces.

Let $\mathcal{T}_{\mathcal{F}}^{\Delta}(\widetilde{G})$ be the space of tuples of functions

$$(\varphi_P)_{P\in\mathcal{F}} \in \prod_{P\in\mathcal{F}} S_{\Delta_P}([\widetilde{G}]_{\widetilde{P},\Delta})$$

such that $\varphi_P - (\varphi_Q)_{\widetilde{P}} \in S_{d_P^{Q,\Delta}}([\widetilde{G}]_{\widetilde{P},\Delta})$ for all $P \subsetneq Q \in \mathcal{F}$. By Proposition 6.1, this condition is equivalent to the existence of N_0 such that for all $\widetilde{g} \in \widetilde{G}(\mathbb{A})$, all $X \in \mathcal{U}(\widetilde{\mathfrak{g}}_{\infty})$ and all $r \geq 0$ we have

$$\left|\mathrm{R}(X)\varphi_P(\widetilde{g}) - \mathrm{R}(X)\varphi_Q(\widetilde{g})\right| \ll \|\widetilde{g}\|_{\widetilde{P}}^{N_0} d_P^{Q,\Delta}(g)^{-r}. \tag{6.3}$$

In the same vein we define $\mathcal{T}_{\mathcal{F}}^{\Delta}(G)$ to be the space of tuples of functions

$$(\varphi_P)_{P\in\mathcal{F}} \in \prod_{P\in\mathcal{F}} S_{\Delta_P}([G]_{P_G})$$

such that $\varphi_P - (\varphi_Q)_{P_G} \in S_{d_P^{Q,\Delta}}([G]_{P_G})$ if $P \subsetneq Q \in \mathcal{F}$. By Theorem 4.25, this condition is equivalent to the existence of an N_0 such that for all $g \in G(\mathbb{A})$, all $X \in \mathcal{U}(\mathfrak{g}_{\infty})$ and all $r \geq 0$ we have

$$\left|\mathrm{R}(X)\varphi_P(g) - \mathrm{R}(X)\varphi_Q(g)\right| \ll \|g\|_{P_G}^{N_0} d_P^{Q,\Delta}(g)^{-r} \tag{6.4}$$

The space $\mathcal{T}_{\mathcal{F}}^{\Delta}(\widetilde{G})$ embeds in

$$\prod_{P\in\mathcal{F}} S_{\Delta_P}([\widetilde{G}]_{\widetilde{P},\Delta}) \times \prod_{P\subsetneq Q\in\mathcal{F}} S_{d_P^{Q,\Delta}}([\widetilde{G}]_{\widetilde{P},\Delta})$$

as a closed subspace, and hence inherits an LF topology. Similarly $\mathcal{T}_{\mathcal{F}}^{\Delta}(G)$ is an LF space.

Lemma 6.2. *If $(\varphi_P)_{P\in\mathcal{F}} \in \mathcal{T}_{\mathcal{F}}^{\Delta}(\widetilde{G})$, then*

$$\left(\varphi_P|_{[G]_{P_G}}\right)_{P\in\mathcal{F}} \in \mathcal{T}_{\mathcal{F}}^{\Delta}(G).$$

Proof. This follows directly from the characterizations (6.3) and (6.4).

Let w be a weight on $[G]_{P_G}$ and $\mu \in T_w^0([G]_{P_G})$ be a Radon measure. We define a Radon measure $\mu\,{}_P\Theta(\cdot;\Phi_0)$ on $[\widetilde{G}]_{\widetilde{P}}$ as follows. For $f \in C_c([\widetilde{G}]_{\widetilde{P}})$, we put

(6.5) $$\langle \mu\,{}_P\Theta(\cdot;\Phi_0), f\rangle = \int_{[G]_{P_G}}\int_{[S]_{P_S}} f(sg)\,{}_P\Theta(sg_1;\Phi_0)\,\mathrm{d}s\,\mu(g);$$

where $[S]_{P_S} = N_{P_S}(\mathbb{A})M_{P_S}(F)\backslash S(\mathbb{A})$, and where we recall that $g = (g_1, g_2)$, $g_1, g_2 \in G_n(\mathbb{A})$.

Lemma 6.3. *There is an $N_0 > 0$ such that for all $\Phi_0 \in \mathcal{S}(\mathbb{A}_{E,n})$ we have*

$$\mu\,{}_P\Theta(\cdot;\Phi_0) \in T^0_{w,N_0}([\widetilde{G}]_{\widetilde{P}};\ ).$$

Moreover the map

$$T^0_w([G]_{P_G}) \to T^0_{w,N_0}([\widetilde{G}]_{\widetilde{P}};\ ), \qquad \mu \mapsto \mu\,{}_P\Theta(\cdot;\Phi_0)$$

is continuous.

Proof. We may pick an N_0 such that for all Φ_0 we have

$$\sup_{g_1\in[G_n]_{P_n}} \|g_1\|_{P_n}^{N_0}\int_{[S]_{P_S}} |{}_P\Theta(sg_1;\Phi_0)|\,\mathrm{d}s < \infty.$$

It follows that

$$\int_{[\widetilde{G}]_{\widetilde{P}}} w(g)^{-1}\|g\|_{P_G}^{N_0}|\mu|(g)|{}_P\Theta(sg_1;\Phi_0)|\,\mathrm{d}s$$
$$\le \sup_{g_1\in[G]_{P_G}} \|g_1\|_{P_{G_n}}^{N_0}\int_{[S]_{P_S}} |{}_P\Theta(sg_1;\Phi_0)|\,\mathrm{d}s \int_{[G]_{P_G}} w(g)^{-1}|\mu|(g) < \infty.$$

The assertion on continuity follows from the same estimate.

Lemma 6.4. *If $P \subset Q \in \mathcal{F}$ and $\mu \in T^0([G]_{P_G})$, then*

$$(\mu\,{}_Q\Theta(\cdot;\Phi_0))_{\widetilde{P}} = \mu_{P_G}\,{}_P\Theta(\cdot;\Phi_0).$$

Proof. This follows from a direct computation using Lemma 5.6.

Lemma 6.5. *If $\varphi \in T^0([H])$, $\Phi_0 \in \mathcal{S}(\mathbb{A}_{E,n})$, and $\widetilde{f} \in \mathcal{S}(\widetilde{G}(\mathbb{A});\ ^{-1})$, then the family*

$$P \mapsto \mathrm{R}(\widetilde{f})(\varphi_{P_H}\,{}_P\Theta(\cdot;\Phi_0))$$

belongs to $T_F^{\Delta}(\widetilde{G})$.

Proof. We first note that the support of $\varphi_{P_H}\,{}_P\Theta(\cdot;\Phi_0)$ is contained in $[\widetilde{H}]_{P_{\widetilde{H}}}$, and Δ_P is bounded on it by definition. Thus by Lemma 4.14 we have

$$\mathrm{R}(\widetilde{f})(\varphi_{P_H}\,{}_P\Theta(\cdot;\Phi_0)) \in \mathcal{S}_{\Delta_P}([\widetilde{G}]_{\widetilde{P}};\ ).$$

We need to show that

$$\mathrm{R}(\widetilde{f})\left(\varphi_{P_H}\,{}_P\Theta(\cdot;\Phi_0) - (\varphi_{Q_H}\,{}_Q\Theta(\cdot;\Phi_0))_{\widetilde{P}}\right) \in \mathcal{S}_{d_P^{Q}}([\widetilde{G}]_{\widetilde{P}};\ ).$$

Indeed we have a slightly stronger result. Define on $[G]_{P_G}$ a weight

$$d_P^{Q,\Delta_n}(g) = \min\{d_{P_n}^{Q_n}(g_1), d_{P_n}^{Q_n}(g_2)\},$$

which pulls back to a weight on $[\tilde{G}]_{\tilde{P}}$. We have $d_P^{Q,\Delta} \ll d_P^{Q,\Delta_n}$ by [BPCZ22, Lemma 2.4.4.2]. We will show that

$$R(\tilde{P})\varphi_{P_H}\cdot{}_P\Theta(\cdot,\Phi_0) - (\varphi_{Q_H}\cdot{}_Q\Theta(\cdot,\Phi_0))_{\tilde{P}} \in \mathcal{S}_{d_P^{Q,\Delta_n}}([\tilde{G}]_{\tilde{P}},\sigma).$$

By Lemma 4.14, it suffices to show there is a positive real number C such that $\varphi_{P_H}\cdot{}_P\Theta(\cdot,\Phi_0)$ and $(\varphi_{Q_H}\cdot{}_Q\Theta(\cdot,\Phi_0))_{\tilde{P}}$ coincide on the set $\{\tilde{g} \in [\tilde{G}]_{\tilde{P}} \mid d_P^{Q,\Delta_n}(\tilde{g}) > C\}$.

By Lemma 6.4 we have

$$(\varphi_{Q_H}\cdot{}_Q\Theta(\cdot,\Phi_0))_{\tilde{P}} = (\varphi_{Q_H})_{P_G}\cdot{}_P\Theta(\cdot,\Phi_0).$$

We are thus reduced to show that there is a constant C such that φ_{P_H} and $(\varphi_{Q_H})_{P_G}$ coincide on the set $\{g \in [G]_{P_G} \mid d_P^{Q,\Delta_n}(g) > C\}$.

This is proved in the same way as [BPCZ22, Proposition 3.4.2.1(1)]. Using the adjunction relation (4.8), we are reduced to show that if $\phi \in \mathcal{S}^0([G]_{P_G})$ is supported in $\{g \in [G]_{P_G} \mid d_P^{Q,\Delta_n}(g) > C\}$, then we have

$$E_{P_H}^{Q_H}(\phi|_H) = E_{P_G}^{Q_G}\phi|_{[H]_{Q_H}}.$$

This in turn is equivalent to the fact that there is a $C > 0$ such that for $g \in P_G(F)N_{Q_G}(\mathbb{A})\backslash G(\mathbb{A})$ with $d_P^{Q,\Delta_n}(g) > C$, $\pi_{Q_G}^{P_G}(g) \in [H]_{Q_H}$ implies $g \in P_H(F)Q_H(\mathbb{A})\backslash H(\mathbb{A})$ (recall $\pi_{Q_G}^{P_G}$ is the map defined in (4.11)). By definition, $d_P^{Q,\Delta_n}(g) > C$ means $d_{P_n}^{Q_n}(g_i) > C$, $i = 1,2$, and $\pi_{Q_G}^{P_G}(g) \in [H]_{Q_H}$ means $\pi_{Q_n}^{P_n}(g_1) = \pi_{Q_n}^{P_n}(g_2)$. By Lemma 4.17(4) (applied to G_n), if C is sufficiently large, this implies $g_1 = g_2$, or equivalently $g \in P_H(F)Q_H(\mathbb{A})\backslash H(\mathbb{A})$.

Let $(\varphi_P)_{P \in \mathcal{F}} \in \mathcal{T}_{\mathcal{F}}^{\Delta}(G)$ and let $\varphi' \in \mathcal{T}^0([G_n'])$. For $g_1 \in G_n(\mathbb{A})$, we define a pairing

$$\langle\varphi_P, \varphi'_{P_n'}\rangle(g_1) = \int_{[G_n']_{P_n'}} \varphi_P(g_1, g_2')\varphi'_{P_n'}(g_2')dg_2' \tag{6.6}$$

We note that by definition for any weight w on $[G_n]_{P_n}$, there exists $N_0 > 0$ such that

$$w(g_2) \ll w(g_1)\Delta_P(g)^{N_0}. \tag{6.7}$$

By taking $w = \|\cdot\|_{P_n}$ in (6.7), we see that $\Delta_P(g)^{-N_0} \ll \|g_1\|_{P_n}\|g_2\|_{P_n}^{-1}$. Therefore, there exists $N_1 > 0$ such that for any $N > 0$

$$\varphi_P(g_1, g_2) \ll \|g_1\|_{P_n}^{N_1+N}\|g_2\|_{P_n}^{-N}.$$

Since $\|g_2\|_{P_n} \ll \|g_2'\|_{P_n'}$, we see that the integral in (6.6) is convergent.

Lemma 6.6. *Let the notation be as above. Then the family*

$$P \mapsto \langle\varphi_P, \varphi'_{P_n'}\rangle$$

belongs to $\mathcal{T}_{\mathcal{F}}([G_n])$.

Proof. Using Dixmier–Malliavin theorem for the smooth Fréchet representation $T_F^{\Delta}(G)^J$ for sufficiently small open compact subgroup $J \subset G_n(\mathbb{A}_f)$, we see that any $(\varphi_P) \in T_F^{\Delta}(G)$ is of the form $(\mathrm{R}(f)\varphi_{1,P})$ for $f \in C_c^{\infty}(G_n(\mathbb{A}))$ and $\varphi_{1,P} \in T_F^{\Delta}(G)$. Thus we are reduced to prove the same statement, but for

$$\langle \varphi_{1,P}, \mathrm{R}(f^{-})\varphi'_{P'_n} \rangle$$

where $f \in C_c^{\infty}(G_n(\mathbb{A}))$. By [BPCZ22, Proposition 3.4.2.1 (2)], the family of functions

$$P \in F \mapsto \mathrm{R}(f^{-})\varphi'_{P'_n}$$

satisfies

$$\mathrm{R}(f^{-})\varphi'_{P'_n} \in T([G_n]_{P_n}) \tag{6.8}$$

with

$$\mathrm{R}(f^{-})\varphi'_{P'_n} - (\mathrm{R}(f^{-})\varphi'_{Q'_n})_{P_n} \in S_{d_{P_n}^{Q_n}}([G_n]_{P_n}), \quad \text{for } P \subset Q \in F. \tag{6.9}$$

Note that there is a slight inconsistency of notation in [BPCZ22] here. In the notation of [BPCZ22], we indeed apply [BPCZ22, Proposition 3.4.2.1 (2)] to the case $\mathbf{G} = G_{n+1}$ and replace $n+1$ by n at all places.

For each $P \in F$, we define a function Ψ_P on $[G]_{P_G}$ by

$$\Psi_P(g_1, g_2) = \varphi_P(g_1, g_2)\mathrm{R}(f^{-})\varphi'_{P'_n}(g_2)$$

By the characterization (6.4) and (6.8), (6.9), the family of functions $\Psi = (\Psi_P)_{P \in F}$ belongs to $T_F^{\Delta}(G)$. Thus we are reduced to show that the family of functions

$$P \mapsto \left(g_1 \mapsto \int_{[G_n]_{P_n}} \Psi_P(g_1, g_2)\mathrm{d}g_2 \right)$$

belongs to $T_F(G_n)$.

Again by the characterization (6.4), we will need to show that there is an integer $N_0 > 0$ such that for all $(\Psi_P) \in T_F^{\Delta}(G)$, $P \subset Q$, all $X \in U(\mathfrak{g}_\infty)$, and $r \geq 0$ we have

$$\left| \int_{[G_n]_{P_n}} \mathrm{R}_1(X)\Psi_P(g_1, g_2)\mathrm{d}g_2 - \int_{[G_n]_{Q_n}} \mathrm{R}_1(X)\Psi_Q(g_1, g_2)\mathrm{d}g_2 \right| \ll_{r,X} \|g_1\|_{P_n}^{N_0} d_P^Q(g_1)^{-r}. \tag{6.10}$$

Here R_1 stands for the action of the universal enveloping algebra action on the first variable g_1. Up to replacing Ψ by $\mathrm{R}_1(X)\Psi$, we can assume $X = 1$.

The rest of the argument is similar to [BPCZ22, Proposition 3.4.3.1]. By (6.7) applied to $w = \|\cdot\|_{P_n}$ and $w = d_{P_n}^{Q_n}$ we see that there exists $N_1 > 0$ such that for any $N > 0$ and $r \geq 0$,

$$|\Psi_P(g)| \ll \|g_1\|_{P_n}^{N_1+N} \|g_2\|_{P_n}^{-N} d_{P_n}^{Q_n}(g_2)^r d_{P_n}^{Q_n}(g_1)^{-r} \tag{6.11}$$

and

$$|\Psi_Q(g)| \ll \|g_1\|_{Q_n}^{N_1+N} \|g_2\|_{Q_n}^{-N} d_{Q_n}^{P_n}(g_2)^r d_{Q_n}^{P_n}(g_1)^{-r} \ll \|g_1\|_{Q_n}^{N_1+N} \|g_2\|_{Q_n}^{-N} d_{Q_n}^{P_n}(g_2)^r d_{P_n}^{Q_n}(g_1)^{-r}. \tag{6.12}$$

Here in the second inequality of (6.12) we have made use of the fact that $d_{Q_n}^{P_n} \ll d_{P_n}^{Q_n}$, cf. [BPCZ22, (2.4.4.19)].

For $C > 0$, let $\omega = \omega_C = \{g \in P_n(F)N_{Q_n}(\mathbb{A})\backslash G_n(\mathbb{A}) \mid d_{P_n}^{Q_n}(g) > C\}$. Let ω_P and ω_Q be the image of ω under the projections $\pi_{P_n}^{Q_n}$ and $\pi_{Q_n}^{P_n}$ respectively. By Lemma 4.17(1)(2), $d_{Q_n}^{P_n}$ and $d_{P_n}^{Q_n}$ are bounded above on $[G_n]_{P_n}\backslash\omega_P$ and $[G_n]_{Q_n}\backslash\omega_Q$ respectively. Thus by (6.11), for any $r > 0$

$$\int_{[G_n]_{P_n}\backslash\omega_P} \Phi_P(g_1, g_2)\mathrm{d}g_2 \ll \|g_1\|_{P_n}^{N_1+N} d_{P_n}^{Q_n}(g_1)^{-r} \ll \|g_1\|_{P_n}^{N_1+N} d_P^Q(g_1)^{-r}$$

for N large enough. Here the second inequality follows from the fact that $d_P^Q \ll d_{P_{n+1}}^{Q_{n+1}} \ll d_{P_n}^{Q_n}$, cf. Lemma 5.3 and [BPCZ22, Lemma 2.4.4.2]. Similarly by (6.12), for any $r > 0$, we have

$$\int_{[G_n]_{Q_n}\backslash\omega_Q} \Phi_Q(g_1, g_2)\mathrm{d}g_2 \ll \|g_1\|_{P_n}^{N_1+N} d_P^Q(g_1)^{-r}.$$

It remains to estimate

$$\int_{\omega_P} \Phi_P(g_1, g_2)\mathrm{d}g_2 - \int_{\omega_Q} \Phi_Q(g_1, g_2)\mathrm{d}g_2.$$

We choose C large enough so that $\omega \to \omega_Q$ is injective. This is possible by Lemma 4.17(4). Thus the above difference equals to

$$\int_{\omega} \Phi_P(g_1, g_2) - \Phi_Q(g_1, g_2)\mathrm{d}g_2. \tag{6.13}$$

We assume $g_2 \in \omega$ for the rest of the proof. Since $d_{P_n}^{Q_n} \ll d_{Q_n}^{P_n}$ on ω by Lemma 4.17(2), thus by (6.11) and (6.12), we have for any $r \geq 0$ and $N > 0$

$$|\Phi_P(g) - \Phi_Q(g)| \ll \|g_1\|_{P_n}^{N_1+N}\|g_2\|_{P_n}^{-N} d_{P_n}^{Q_n}(g_2)^r d_{P_n}^{Q_n}(g_1)^{-r} \ll \|g_1\|_{P_n}^{N_1+N}\|g_2\|_{P_n}^{-N} d_{P_n}^{Q_n}(g_2)^r d_P^Q(g_1)^{-r}.$$

In particular, taking $r = 0$ in the first inequality, we have

$$|\Phi_P(g) - \Phi_Q(g)| \ll \|g_1\|_{P_n}^{N_1+N}\|g_2\|_{P_n}^{-N}.$$

Thus we conclude that

$$|\Phi_P(g) - \Phi_Q(g)| \ll \|g_1\|_{P_n}^{N_1+N}\|g_2\|_{P_n}^{-N} \max\{1, d_P^Q(g_1)d_{P_n}^{Q_n}(g_2)^{-1}\}^{-r}.$$

Since the family $(\Phi_P) \in \mathcal{T}_F^{\Delta}(G)$, there is an $N_0 > 0$ such that for any $r \geq 0$ we have

$$|\Phi_P(g) - \Phi_Q(g)| \ll \|g\|_{P_G}^{N_0} d_P^{Q,\Delta}(g)^{-r}.$$

Note that for any real numbers $a, b > 0$, we have $\max\{1, b/a\}\min\{a, b\} = b$. Thus there is an N_2 such that for any $r \geq 0$ and $N > 0$, we have

$$\begin{aligned}|\Phi_P(g) - \Phi_Q(g)| &\ll \|g_1\|_{P_n}^{N_2+N}\|g_2\|_{P_n}^{-N}\left(\max\{1, d_P^Q(g_1)d_{P_n}^{Q_n}(g_2)^{-1}\}d_P^{Q,\Delta}(g)\right)^{-r}\\ &= \|g_1\|_{P_n}^{N_2+N}\|g_2\|_{P_n}^{-N} d_P^Q(g_1)^{-r}.\end{aligned}$$

Fix a large N. It follows that there is an N_3 such that

$$(6.13) \ll \|g_1\|_{P_n}^{N_3} d_P^Q(g_1)^{-r}$$

for any $r > 0$. This proves the estimate (6.10).

6.3. **A modified kernel.** For $f \in \mathcal{S}(G(\mathbb{A}))$, $\chi \in \mathfrak{X}(G)$, and $P \in \mathcal{F}$, we let K_{f,P_G} and $K_{f,P_G,\chi}$ be the kernel functions defined in Subsection 4.7. If $\Phi \in \mathcal{S}(\mathbb{A}_{E,n})$, we have the theta series ${}_P\Theta(\cdot,\Phi)$ on $[J]_P$ defined in (5.9). For $(h,g) \in [H]_{P_H} \times [G]_{P_G}$ we define kernel functions for the test function $f \otimes \Phi$ by

$$K_{f\otimes\Phi,P}(h,g) = K_{f,P_G}(h,g)\,{}_P\Theta(h,\Phi), \quad K_{f\otimes\Phi,P,\chi}(h,g) = K_{f,P_G,\chi}(h,g)\,{}_P\Theta(h,\Phi). \tag{6.14}$$

We will need to extend these definitions to all test functions $f_+ \in \mathcal{S}(G_+(\mathbb{A}))$, not necessarily pure tensors. The case of $K_{f_+,P}$ is straightforward. Put

$$K_{f_+,P}(h,g) = \mu(\det h)^{-1}|\det h|^{\frac{1}{2}} \sum_{m_+ \in M_{P,+}(F)} \int_{N_{P,+}(\mathbb{A})} f_+(m_+ n_+ \cdot (h,g))\,\mathrm{d}n_+.$$

Then it is clear that if $f_+ = f \otimes \Phi$ then $K_{f_+,P}(h,g) = K_{f,P_G}(h,g)\,{}_P\Theta(h,\Phi)$. Moreover, for fixed h,g the linear form on $\mathcal{S}(G_+(\mathbb{A}))$ given by $f_+ \mapsto K_{f_+,P}(h,g)$ is continuous. The case of $K_{f_+,P,\chi}$ needs some work. First we note that for all $N_1 > 0$ there is an N_2 and a semi-norm $\|\cdot\|_{N_1,N_2}$ on $\mathcal{S}(G(\mathbb{A}))$ such that for all $f \in \mathcal{S}(G(\mathbb{A}))$ and $h \in [H]_{P_H}$, $g \in [G]_{P_G}$ we have

$$\sum_{\chi \in \mathfrak{X}(G)} |K_{f,P,\chi}(h,g)| \ll \|g_1\|_{P_n}^{N_2}\|g_2\|_{P_n}^{-N_1}\|h\|_{P_H}^{-N_1}\|f\|_{N_1,N_2}, \tag{6.15}$$

and a similar estimate with g_1 and g_2 swapped. This is a consequence of Lemma 4.21 applied to the weight $w = \Delta_P^{2N_1}\|\cdot\|^{-N_1}$. As ${}_P\Theta \in \mathcal{T}([H]_{P_H})$, we arrive at the following estimates: for all $N_1 > 0$ there is an N_2 and a semi-norm $\|\cdot\|_{N_1,N_2}$ on $\mathcal{S}(G_+(\mathbb{A}))$ such that for all $f_+ \in \mathcal{S}(G(\mathbb{A})) \otimes \mathcal{S}(\mathbb{A}_{E,n})$ (algebraic tensor) we have

$$\sum_{\chi \in \mathfrak{X}(G)} |K_{f_+,P,\chi}(h,g)| \ll \|g_1\|_{P_n}^{N_2}\|g_2\|_{P_n}^{-N_1}\|h\|_{P_H}^{-N_1}\|f_+\|_{N_1,N_2}, \tag{6.16}$$

and

$$\sum_{\chi \in \mathfrak{X}(G)} |K_{f_+,P,\chi}(h,g)| \ll \|g_1\|_{P_n}^{-N_2}\|g_2\|_{P_n}^{-N_2}\|h\|_{P_H}^{N_1}\|f_+\|_{N_1,N_2}. \tag{6.17}$$

If $f_+ \in \mathcal{S}(G_+(\mathbb{A}))$ is approximated by a sequence of functions $f_{+,i} \in \mathcal{S}(G(\mathbb{A})) \otimes \mathcal{S}(\mathbb{A}_{E,n})$, by the estimate (6.16), the sequence $K_{f_{+,i},P,\chi}$ is convergent to a function on $[H]_{P_H} \times [G]_{P_G}$, and this convergence along with all the derivative is locally uniformly for $(h,g) \in [H]_{P_H} \times [G]_{P_G}$. We denote this function by $K_{f_+,P,\chi}$. It is clearly independent of the choice of the sequence approximating f_+. Because the convergence of all the derivative is locally uniform, $K_{f_+,P,\chi}$ is a smooth function. Moreover the estimates (6.16) and (6.17) continue to hold for $K_{f_+,P,\chi}$.

We now give another interpretation of $K_{f_+,P,\chi}$. Fix a Schwartz function $\Phi_0 \in \mathcal{S}(\mathbb{A}_{E,n})$ with $\|\Phi_0\|_{L^2} = 1$. For any $\varphi \in \mathcal{T}^0([H]_{P_H})$, we push forward φ to a measure on $[G]_{P_G}$ and we let

$\varphi\ \ _P\Theta(\ ;\Phi_0)$ be the measure on $[\widetilde{G}]_{\widetilde{P}}$ defined in (6.5). If $f_+ \in S(G_+(\mathbb{A}))$ we define a function $\widetilde{f}_+ \in S(\widetilde{G}(\mathbb{A}))$ by

$$\widetilde{f}_+(gs) = \langle f_+(g^{-1};\ ),\ \mathrm{R}_{\ ^{-1}}(s)\Phi_0(\ )\rangle_{L^2}, \quad g \in G(\mathbb{A}),\ s \in S(\mathbb{A}).$$

This notation means that we evaluate f_+ at g^{-1} to obtain a Schwartz function on $\mathbb{A}_{E,n}$ and then take the L^2-inner product with $\mathrm{R}_{\ ^{-1}}(s)\Phi_0$. If $f_+ = f \otimes \Phi$, then $\widetilde{f}_+(gs) = f^{-}(g)\langle\Phi, \mathrm{R}_{\ ^{-1}}(s)\Phi_0\rangle$, where we recall that $f^{-}(g) = f(g^{-1})$.

We first note that the composition of the right translation

$$\varphi \mapsto \mathrm{R}(\widetilde{f}_+)(\varphi\ \ _P\Theta(\ ;\Phi_0))$$

and restriction to $[G]_{P_G}$ induces a continuous linear map

$$\mathrm{R}^G(\widetilde{f}_+) : \mathcal{T}^0([H]_{P_H}) \to \mathcal{T}([G]_{P_G}), \tag{6.18}$$

Let $\chi \in \mathfrak{X}(G)$ and define $\mathrm{R}^G_\chi(\widetilde{f}_+)$ to be the composition of $\mathrm{R}^G(\widetilde{f}_+)$ followed by projection to the χ-component $\mathcal{T}_\chi([G]_{P_G})$.

Lemma 6.7. *The maps $\mathrm{R}^G(\widetilde{f}_+)$ and $\mathrm{R}^G_\chi(\widetilde{f}_+)$ are represented by the kernel functions $K_{f_+,P}$ and $K_{f_+,P,\chi}$ respectively, i.e. for all $\varphi \in \mathcal{T}^0([H]_{P_H})$ and $g \in [G]_{P_G}$ we have*

$$\mathrm{R}^G(\widetilde{f}_+)(\varphi)(g) = \int_{[H]_{P_H}} K_{f_+,P}(h,g)\varphi(h), \quad \mathrm{R}^G_\chi(\widetilde{f}_+)(\varphi)(g) = \int_{[H]_{P_H}} K_{f_+,P,\chi}(h,g)\varphi(h). \tag{6.19}$$

In particular the map $\mathrm{R}^G(\widetilde{f}_+)$ is independent of the choice of Φ_0.

Note that the integrations appear in the first variable of the kernel function because we used g^{-1} in the definition of $\widetilde{f}_+$.

Proof. Since both sides of (6.19) are continuous linear forms in f_+, we may assume that $f_+ = f \otimes \Phi$ where $f \in S(G(\mathbb{A}))$ and $\Phi \in S(\mathbb{A}_{E,n})$.

Let us prove (6.19) for $\mathrm{R}^G(\widetilde{f}_+)$. The equality for $\mathrm{R}^G_\chi(\widetilde{f}_+)$ follows from it. For $\varphi \in \mathcal{T}^0([H]_{P_H})$ and $g \in [G]_{P_G}$, we have

$$\mathrm{R}^G(\widetilde{f}_+)(\varphi)(g) = \int_{H(\mathbb{A})}\int_{Z(\mathbb{A})\backslash S(\mathbb{A})} \widetilde{f}_+(g^{-1}uh)\ _P\Theta(uh,\Phi_0)\mathrm{d}u\,\varphi(h).$$

Then by the definition of $\widetilde{f}_+$, this equals

$$\int_{H(\mathbb{A})}\int_{Z(\mathbb{A})\backslash S(\mathbb{A})} f(h^{-1}g)\langle\Phi, \mathrm{R}_{\ ^{-1}}(h^{-1}uh)\Phi_0\rangle\ _P\Theta(uh,\Phi_0)\mathrm{d}u\,\varphi(h),$$

which simplifies to

$$\int_{[H]_{P_H}} K_{f,P_G}(h,g)\ _P\Theta(h,\Phi)\,\varphi(h).$$

This proves the lemma.

If $P \subset Q$, $P, Q \in \mathcal{F}$, we define a sign

$$\epsilon_P^Q = (-1)^{\dim \mathfrak{a}_{P_{n+1}}^{Q_{n+1}}}.$$

We simply write $\epsilon_P = \epsilon_P^{J_n}$. Let $T \in \mathfrak{a}_{n+1}$ be a truncation parameter and $(h, g') \in [H] \times [G']$. We define a modified kernel

$$K^T_{f_+}(h, g') = \sum_{P \in \mathcal{F}} \epsilon_P \sum_{\substack{\delta \in P_H(F) \backslash H(F) \\ \gamma \in P'(F) \backslash G'(F)}} \hat{\tau}_{P_{n+1}}(H_{P_{n+1}}(\gamma_1 g'_1) - T_{P_{n+1}}) K_{f_+, P}(\delta h, \gamma g').$$

For each $\chi \in \mathfrak{X}(G)$, we also define

$$K^T_{f_+, \chi}(h, g') = \sum_{P \in \mathcal{F}} \epsilon_P \sum_{\substack{\delta \in P_H(F) \backslash H(F) \\ \gamma \in P'(F) \backslash G'(F)}} \hat{\tau}_{P_{n+1}}(H_{P_{n+1}}(\gamma_1 g'_1) - T_{P_{n+1}}) K_{f_+, P, \chi}(\delta h, \gamma g').$$

By [Art78, Lemma 5.1], for (h, g') fixed, in each sum the component γ_1 can be taken in a finite set depending on g'_1. The absolute convergence of the sums therefore follows from the estimate (6.15).

Let $F^{G'_{n+1}}(\cdot, T)$ be the characteristic function of Arthur (characteristic function of the truncated Siegel set) for G'_{n+1} defined in Subsection 4.5. For a fixed T, it is compactly supported modulo the center of $G'_{n+1}(\mathbb{A})$. In particular it is compactly supported when restricted to $[G'_n]$.

Theorem 6.8. *For every $N > 0$, there is a continuous semi-norm $\|\cdot\|_{\mathcal{S}, N}$ on $\mathcal{S}(G_+(\mathbb{A}))$ such that*

$$\sum_{\chi \in \mathfrak{X}(G)} \left| K^T_{f_+, \chi}(h, g') - F^{G'_{n+1}}(g'_1, T) K_{f_+, \chi}(h, g') \right| \ll e^{-N\|T\|} \|h\|_H^{-N} \|g'\|_{G'}^{-N} \|f_+\|_{\mathcal{S}, N}. \tag{6.20}$$

holds for all sufficiently positive T. In particular, for T sufficiently positive

$$\sum_{\chi \in \mathfrak{X}(G)} \int_{[H]} \int_{[G']} \left| K^T_{f_+, \chi}(h, g') \right| \mathrm{d}g' \mathrm{d}h$$

is convergent and defines a continuous seminorm on $\mathcal{S}(G_+(\mathbb{A}))$.

Proof. We first note that the second statement follows directly from (6.16) and the estimate (6.20) by the fact that $F^{G'_{n+1}}(\cdot, T)$ is compactly supported when restricted to $[G'_n]$. So we only need to prove the estimate (6.20).

We recall that $\mathcal{S}^0([G'_n])$ is the space of (measurable) rapid decreasing function on $[G'_n]$, and it is a Fréchet space with the seminorms

$$\|\varphi\|_{\infty, N} = \sup_{g \in [G'_n]} \|g\|^N_{G'_n} |\varphi(g)|.$$

We also recall that $\mathcal{T}_F(G'_n)$ is an LF space. Two relative truncation operators

$$\Lambda^{T, G'_n}, \Pi^{T, G'_n} : \mathcal{T}_F(G'_n) \to \mathcal{S}^0([G'_n]).$$

are defined in [BPCZ22, Section 3.5]. For $\varphi = (\varphi_P)_{P\in\mathcal{F}} \in \mathcal{T}_{\mathcal{F}}(G'_n)$, and $g' \in [G'_n]$, we have

$$\Lambda^{T,G'_n}\varphi(g') = \sum_{P\in\mathcal{F}} \epsilon_P \sum_{\gamma\in P'_n(F)\backslash G'_n(F)} \widehat{\tau}_{P_{n+1}}(H_{P_{n+1}}(\gamma g') - T_{P_{n+1}})\varphi_P(\gamma g),$$

$$\Pi^{T,G'_n}\varphi(g') = F^{G'_{n+1}}(g',T)(\jmath_n\varphi)(g').$$

By [BPCZ22, Theorem 3.5.1.1], for every $c > 0$ and $N > 0$, there exists a continuous seminorm $\|\cdot\|_{c,N}$ on $\mathcal{T}_{\mathcal{F}}(G'_n)$ such that for all sufficiently positive truncation parameter $T \in \mathfrak{a}_{n+1}$ we have

$$\|\Lambda^{T,G'_n}\varphi - \Pi^{T,G'_n}\varphi\|_{\infty,N} \le e^{-c\|T\|}\|\varphi\|_{c,N}, \tag{6.21}$$

where $\|\cdot\|_{\infty,N}$ is the norm on $\mathcal{S}^0([G'_n])$.

For $f_+ \in \mathcal{S}(G(\mathbb{A})\times\mathbb{A}_{E,n})$ we consider the composition of the following sequence of linear maps:

$$\mathcal{T}^0([H])\otimes\mathcal{T}^0([G'_n]) \longrightarrow \mathcal{T}^{\Delta}_F(G)\otimes\mathcal{T}^0([G'_n]) \xrightarrow{\langle\cdot,\cdot\rangle} \mathcal{T}_F(G_n) \xrightarrow{j_{G'_n}} \mathcal{T}_F(G'_n) \overset{\Lambda^T}{\underset{\Pi^T}{\rightrightarrows}} \mathcal{S}^0([G'_n])$$

where the first map is $R^G(\widetilde{f}_+)\otimes\mathbf{1}$ (defined using any fixed $\Phi_0 \in \mathcal{S}(\mathbb{A}_{E,n})$), the second is the pairing, the third is the restriction, and the last is the truncation Λ^{T,G'_n} (resp. Π^{T,G'_n}), cf. Lemmas 6.2, 6.5, and 6.6 for the description of these maps. Their composition will be denoted by $L^T_{f_+}$ (resp. $P^T_{f_+}$). The fact that these maps are continuous is an easy consequence of the closed graph theorem, see Remark 6.9 after the proof.

By the definition of $K_{f_+,P}$ we see that the function $K^T_{f_+}$ (resp. $F^{G'_{n+1}}(\cdot,T)K_{f_+}$) is the kernel function of the map $L^T_{f_+}$ (resp. $P^T_{f_+}$). More precisely for any $\varphi\otimes\varphi' \in \mathcal{T}^0([H])\otimes\mathcal{T}^0([G'_n])$, we have

$$L^T_{f_+}(\varphi\otimes\varphi')(g'_1) = \int_{[H]}\int_{[G'_n]} K^T_{f_+}(h,g'_1,g'_2)\varphi(h)\varphi'(g'_2),$$

$$P^T_{f_+}(\varphi\otimes\varphi')(g'_1) = \int_{[H]}\int_{[G'_n]} F^{G'_{n+1}}(g'_1,T)K_{f_+}(h,g'_1,g'_2)\varphi(h)\varphi'(g'_2).$$

For $\chi \in \mathfrak{X}(G)$, define

$$L^T_{f_+,\chi}, P^T_{f_+,\chi} : \mathcal{T}^0([H])\otimes\mathcal{T}^0([G'_n]) \to \mathcal{S}^0([G'_n])$$

the same way as $L^T_{f_+}$ and $P^T_{f_+}$ respectively, except that we replace the first map by $R^G(\widetilde{f}_+)_\chi\otimes\mathbf{1}$. Then $L^T_{f_+,\chi}, P^T_{f_+,\chi}$ are separably continuous, and the functions $K^T_{f_+,\chi}$ and $F^{G'_{n+1}}(\cdot,T)K_{f_+,\chi}$ are the kernel functions of the operators $L^T_{f_+,\chi}$ and $P^T_{f_+,\chi}$ respectively.

Fix $N > 0$. By (6.21), for all $\varphi\otimes\varphi' \in \mathcal{T}^0_N([H])\otimes\mathcal{T}^0_N([G'_n])$, we have

$$\sum_{\chi\in\mathfrak{X}(G)} \left\| L^T_{f_+,\chi}(\varphi\otimes\varphi') - P^T_{f_+,\chi}(\varphi\otimes\varphi')\right\|_{\infty,N} \ll e^{-N\|T\|}.$$

As in [BPCZ22, Section 3.6], the uniform boundedness principle (applied to φ and φ') implies that

$$\sum_{\chi\in\mathfrak{X}(G)} \left\| L^T_{f_+,\chi}(\varphi\otimes\varphi') - P^T_{f_+,\chi}(\varphi\otimes\varphi')\right\|_{\infty,N} \ll e^{-N\|T\|}\|\varphi\|_{1,-N}\|\varphi'\|_{1,-N}$$

holds for all $\varphi \otimes \varphi' \in \mathcal{T}_N^0([H]) \otimes \mathcal{T}_N^0([G'_n])$ and T sufficiently positive. We apply this estimate to $\varphi = \varphi_h$ and $\varphi' = \varphi_{g'_2}$ and conclude that

$$\sum_\chi \left| K^T_{f_+,\chi}(h,g') - F^{G'_{n+1}}(g'_1,T) K_{f_+,\chi}(h,g') \right| \ll e^{-N\|T\|} \|h\|_H^N \|g_2\|_{G'}^N.$$

By the uniform bounded principle again (applied to f_+), we see that there exists a seminorm $\|\cdot\|_{\mathcal{S},N}$ on $\mathcal{S}(G(\mathbb{A}))$ such that

$$\sum_\chi \left| K^T_{f_+,\chi}(h,g') - F^{G'_{n+1}}(g'_1,T) K_{f_+,\chi}(h,g') \right| \ll e^{-N\|T\|} \|h\|_H^N \|g'\|_{G'}^N \|f_+\|_{\mathcal{S},N}.$$

This proves the estimate (6.20).

Remark 6.9. We explain how the closed graph theorem is used in the proof to check continuity. We temporarily denote by X and Y two topological spaces, and by $F(X)$ and $F(Y)$ certain spaces of functions on X and Y respectively. Let $T : F(X) \to F(Y)$ be a linear map. Assume that

- T is continuous when $F(Y)$ is equipped with the pointwise convergence topology,
- the topology on $F(Y)$ is finer than the pointwise convergence topology,
- $F(X)$ and $F(Y)$ satisfy the closed graph theorem, i.e. T is continuous if and only if its graph is closed (which is the case if $F(X)$ and $F(Y)$ are LF spaces).

Then we claim that T is continuous. Indeed, if (f_a) is a net in $F(X)$ such $\lim_{a\in\mathcal{A}} f_a = f$ and $\lim_{a\in\mathcal{A}} T(f_a) = g$, then we have for all $y \in Y$ the equality $T(f)(y) = \lim_{a\in\mathcal{A}} T(f_a)(y) = g(y)$, which implies that $T(f) = g$ and therefore that T is continuous.

6.4. **The coarse spectral expansion.** Let $f_+ \in \mathcal{S}(G_+(\mathbb{A}))$ and $\chi \in \mathfrak{X}(G)$. We put

$$I^T(f_+) = \int_{[H]} \int_{[G']} K^T_{f_+}(h,g') \theta_{n+1}(g') \mathrm{d}g' \mathrm{d}h$$

and

$$I^T_\chi(f_+) = \int_{[H]} \int_{[G']} K^T_{f_+,\chi}(h,g') \theta_{n+1}(g') \mathrm{d}g' \mathrm{d}h.$$

By Theorem 6.8, these integrals are absolutely convergent.

Theorem 6.10. *For T sufficiently positive, the functions $I^T(f_+)$ and $I^T_\chi(f_+)$ are the restrictions of exponential-polynomial functions whose purely polynomial parts are constants. We denote them by $I(f_+)$ and $I_\chi(f_+)$ respectively. Then I and I_χ are continuous as distributions on $\mathcal{S}(G_+(\mathbb{A}))$ and for any $f_+ \in \mathcal{S}(G_+(\mathbb{A}))$ we have*

$$\sum_{\chi\in\mathfrak{X}(G)} I_\chi(f_+) = I(f_+),$$

where the sum is absolutely convergent.

Before we delve into the proof of this theorem, let us first explain a variant of the construction of the modified kernel for parabolic subgroups. For $f'_+ \in \mathcal{S}(M_{Q_+}(\mathbb{A}))$ and $P \subset Q \in \mathcal{F}$, and $T \in \mathfrak{a}_{n+1}$, we define a kernel function $K_{f'_+,P\cap M_Q}(m_H,m_G)$ on $[M_{Q_H}]_{P_H\cap M_{Q_H}} \times [M_{Q_G}]_{P_G\cap M_{Q_G}}$ to be

$$\begin{cases} \eta^{-1}(\det m_H) \displaystyle\sum_{m_+\in M_{P_+}(F)} \int_{(N_{P_+}\cap M_{Q_+})(\mathbb{A})} f'_+(m_+n_+ \cdot (m_H,m_G))\mathrm{d}n_+, & Q \text{ is of type I;} \\ |\det m_{H,k}|_E^{\frac{1}{2}}\, \eta^{-1}(\det m_H) \displaystyle\sum_{m_+\in M_{P_+}(F)} \int_{(N_{P_+}\cap M_{Q_+})(\mathbb{A})} f'_+(m_+n_+ \cdot (m_H,m_G))\mathrm{d}n_+, & Q \text{ is of type II.} \end{cases}$$

Here we assume that Q_n stabilizes the flag (5.1). The Levi subgroup of $Q_H \simeq Q_n$ is isomorphic to $\prod_{i=1}^r \mathrm{GL}(L_i/L_{i-1})$, and we write $m_H \in M_{Q_H}(\mathbb{A})$ as $(m_{H,1},\dots,m_{H,r})$ where $m_{H,i} \in \mathrm{GL}(L_i/L_{i-1})(\mathbb{A}_E)$, and $\det m_H$ means taking the determinant of m_H as an element in G_n.

Similar to what we have done in Subsection 6.3, $K_{f'_+,P\cap M_Q}(m_H,m_G)$ is the kernel function of a continuous map $\mathcal{T}'([M_{Q_H}]_{P_H\cap M_{Q_H}}) \to \mathcal{T}([M_{Q_G}]_{P_G\cap M_{Q_G}})$. For $\chi' \in \mathfrak{X}(M_{Q_G})$, composing this map with the projection to the χ'-component of $\mathcal{T}([M_{Q_G}]_{P_G\cap M_{Q_G}})$ is given by another kernel function, denoted by $K_{f'_+,P\cap M_Q,\chi'}(m_H,m_G)$. Recall that there is a natural finite-to-one map $\mathfrak{X}(M_{Q_G}) \to \mathfrak{X}(G)$ which we temporarily denote by ι_Q. Then for $\chi \in \mathfrak{X}(G)$, we define

$$K_{f'_+,P\cap M_Q,\chi}(m_H,m_G) = \sum_{\chi'\in \iota_Q^{-1}(\chi)} K_{f'_+,P\cap M_Q,\chi'}(m_H,m_G).$$

For $T \in \mathfrak{a}_{n+1}$ and $(m_H,m') \in [M_{Q_H}] \times [M_{Q_{G'}}]$ we put

$$\begin{aligned} &K^{M_Q,T}_{f'_+,\chi}(m_H,m') \\ &= \sum_{\substack{P\in\mathcal{F} \\ P\subset Q}} \epsilon_P^Q \sum_{\substack{\gamma\in(P_H(F)\cap M_{Q_H}(F))\backslash M_{Q_H}(F) \\ \delta\in(P_{G'}(F)\cap M_{Q_{G'}}(F))\backslash M_{Q_{G'}}(F)}} \widehat{\tau}^{Q_{n+1}}_{P_{n+1}}(H_{P_{n+1}}(\delta_1 m'_1) - T_{P_{n+1}}) K_{f'_+,P\cap M_Q,\chi}(\gamma m_H, \delta m'). \end{aligned}$$

The group $A^\infty_{M_Q}$ which embeds in $M_{Q_H} \times M_{Q_{G'}}$ diagonally. Note if $Q = J_n$, then $A^\infty_{M_Q}$ is trivial. By the same method of proof of Theorem 6.8, one can show that for any $f'_+ \in \mathcal{S}(M_{Q,+}(\mathbb{A}))$ and T sufficiently positive, the expression

$$\sum_{\chi\in\mathfrak{X}(G)} \int_{A^\infty_{M_Q}\backslash[M_{Q_H}]\times[M_{Q_{G'}}]} \left|K^{M_Q,T}_{f'_+,\chi}(m_H,m')\right| \mathrm{d}m_H\mathrm{d}m' \tag{6.22}$$

is finite and defines a continuous seminorm on $\mathcal{S}(M_{Q_+}(\mathbb{A}))$. Define a distribution on $\mathcal{S}(M_{Q_+}(\mathbb{A}))$ by

$$I^{M_Q,T}_\chi(f'_+) = \int_{A^\infty_{M_Q}\backslash[M_{Q_H}]\times[M_{Q_{G'}}]} K^{M_Q,T}_{f'_+,\chi}(m_H,m')\, \eta_{n+1}(m')\mathrm{d}m_H\mathrm{d}m'.$$

Here and below in the proof, $\eta_{n+1}(m')$ means that we evaluate η_{n+1} at m' when m' is viewed as an element in $G'(\mathbb{A})$.

Proof of Theorem 6.10. By [Art81, Section 2], there exist functions $\Gamma'_{P_{n+1}}$ on $\mathfrak{a}^{G_{n+1}}_{P_{n+1}} \times \mathfrak{a}^{G_{n+1}}_{P_{n+1}}$, for $P \in \mathcal{F}$, that are compactly supported in the first variable when the second variable stays in a compact set and such that

$$\widehat{\tau}_{P_{n+1}}(H - X) = \sum_{\substack{Q \in \mathcal{F} \\ Q \supset P}} \epsilon_Q \widehat{\tau}^{Q_{n+1}}_{P_{n+1}}(H)\Gamma'_{Q_{n+1}}(H, X), \quad H, X \in \mathfrak{a}^{G_{n+1}}_{P_{n+1}}. \tag{6.23}$$

Here following Arthur, by $\Gamma'_{Q_{n+1}}(H, X)$ we mean evaluating $\Gamma'_{Q_{n+1}}$ at the projection of H and X to $\mathfrak{a}^{G_{n+1}}_{Q_{n+1}}$.

We set $\underline{\rho}_Q = \rho_{Q_{n+1}} - \rho_{Q_n} \in \mathfrak{a}^{G_{n+1},*}_{Q_{n+1}}$. If Q_n stabilizes the flag (5.1), then for $m \in M_{Q_n}(\mathbb{A})$ of the form $m = (m_1, \dots, m_r)$ with $m_i \in \mathrm{GL}(L_i/L_{i-1})(\mathbb{A}_E)$, one checks directly that

$$e^{\langle \underline{\rho}_Q, H_{Q_{n+1}}(m)\rangle} = \begin{cases} \prod_{i \le k} |\det m_i|_E^{\frac{1}{2}} \prod_{i > k} |\det m_i|_E^{-\frac{1}{2}}, & Q \text{ is of type I;} \\ \prod_{i \le k} |\det m_i|_E^{\frac{1}{2}} \prod_{i > k+1} |\det m_i|_E^{-\frac{1}{2}}, & Q \text{ is of type II.} \end{cases} \tag{6.24}$$

Define a function p_Q on $\mathfrak{a}^{G_{n+1}}_{Q_{n+1}}$ by

$$p_Q(X) = \int_{\mathfrak{a}^{G_{n+1}}_{Q_{n+1}}} e^{\langle \underline{\rho}_Q, H\rangle} \Gamma'_{Q_{n+1}}(H, X) \mathrm{d}H. \tag{6.25}$$

By [Zyd20, Lemma 3.5], p_Q is an exponential polynomial on $\mathfrak{a}^{G_{n+1}}_{Q_{n+1}}$ whose exponents are contained in the set $\{\underline{\rho}_R \mid R \supset Q\}$ and whose pure polynomial term is the constant $\epsilon_Q \widehat{\theta}_Q(\underline{\rho}_Q)^{-1}$. Here $\widehat{\theta}_Q$ is the homogeneous polynomial function on $\mathfrak{a}^{G_{n+1}}_{P_{n+1}}$ defined in [Art81, Section 2]. We do not need its precise definition.

For $f_+ \in \mathcal{S}(G_+(\mathbb{A}))$ we define for $(h, g') \in [H]_{Q_H} \times [G']_{Q_{G'}}$ a modified kernel function

$$K^{Q,T}_{f_+,\chi}(h, g') = \sum_{\substack{P \in \mathcal{F} \\ P \subset Q}} \epsilon^Q_P \sum_{\substack{\delta \in P_H(F)\backslash Q_H(F) \\ \gamma \in P_{G'}(F)\backslash Q_{G'}(F)}} \widehat{\tau}^{Q_{n+1}}_{P_{n+1}}(H_{P_{n+1}}(\gamma_1 g'_1) - T_{P_{n+1}}) K_{f_+,P,\chi}(\delta h, \gamma g');$$

By (6.23), for $T, T' \in \mathfrak{a}_{n+1}$, we have an inversion formula

$$K^T_{f_+,\chi}(h, g') = \sum_{Q \in \mathcal{F}} \sum_{\substack{\delta \in Q_H(F)\backslash H(F) \\ \gamma \in Q_{G'}(F)\backslash G'(F)}} \Gamma'_{Q_{n+1}}(H_{Q_{n+1}}(\gamma_1 g'_1) - T'_{Q_{n+1}}, T_{Q_{n+1}} - T'_{Q_{n+1}}) K^{Q,T'}_{f_+,\chi}(\delta h, \gamma g').$$

We integrate both sides over $[H] \times [G']$. For T sufficiently positive, the integral of $K^T_{f_+,\chi}(h, g')$ is absolutely convergent by Theorem 6.8. Moreover, when T' is also sufficiently positive, the computation below will show that the integral of each terms in the sum over Q is also absolutely convergent. It follows that $I^T_{f_+,\chi}$ equals

$$\sum_{Q \in \mathcal{F}} \int_{[H]_{Q_H}} \int_{[G']_{Q_{G'}}} \Gamma'_{Q_{n+1}}(H_{Q_{n+1}}(g'_1) - T'_{Q_{n+1}}, T_{Q_{n+1}} - T'_{Q_{n+1}}) K^{Q,T'}_{f_+,\chi}(h, g') \eta_{n+1}(g') \mathrm{d}h \mathrm{d}g'. \tag{6.26}$$

We now relate $K^{Q,T}_{f_+,\chi}$ and $K^{M_Q,T}_{f'_+,\chi}$ via parabolic descent. Recall that we have fixed good maximal compact subgroups K_H and $K_{G'}$ of $H(\mathbb{A})$ and $G'(\mathbb{A})$ respectively. For $f_+ \in S(G_+(\mathbb{A}))$, we define its parabolic descent as a function on $M_{Q,+}(\mathbb{A})$ given by

$$f_{+,Q}(m,l) = e^{\langle \rho_{Q_G}, H_{Q_G}(m)\rangle} \int_{K_H\times K_{G'}}\int_{N_{Q_G}(\mathbb{A})}\int_{N_{Q_{L^-}}(\mathbb{A})} f_+(k_H^{-1}mnk', (l+u)k_H)\,\mu_{n+1}(k')\,\mu(\det k_H)^{-1}\,\mathrm{d}u\,\mathrm{d}n\,\mathrm{d}k_H\,\mathrm{d}k'. \tag{6.27}$$

Here $m \in M_G(\mathbb{A})$ and $l \in M_{L^-}(\mathbb{A})$, and we take the convention that $l=0$ if Q is of type I. Then $f_{+,Q} \in S(M_{Q,+}(\mathbb{A}))$. For $(m_H,m') \in [M_{Q_H}]\times[M_{Q_G}]$ we have

$$\begin{aligned}&\int_{K_H\times K_{G'}} K_{f_+,P,\chi}(m_Hk_H, m'k')\,\mu_{n+1}(k')\,\mathrm{d}k_H\,\mathrm{d}k'\\ =&e^{\langle \rho_{Q_G}, H_{Q_G}(m')+H_{Q_G}(m_H)\rangle}\, e^{\langle \underline{\rho}_Q, H_{Q_{n+1}}(m_H)\rangle} K_{f_{+,Q},P\cap M_Q,\chi}(m_H,m_G).\end{aligned}$$

Indeed, using (6.24), we directly check this identity without the χ. The argument in [Zyd20, Lemma 1.3] shows that this implies the identity with the χ. Therefore for $T \in \mathfrak{a}_{n+1}$ and $(m_H,m') \in [M_{Q_H}]\times[M_{Q_{G'}}]$, we have

$$\begin{aligned}&\int_{K_H\times K_{G'}} K^{Q,T}_{f_+,\chi}(m_Hk_H, m'k')\,\mu_{n+1}(k')\,\mathrm{d}k_H\,\mathrm{d}k'\\ =&e^{\langle \rho_{Q_G}, H_{Q_G}(m')+H_{Q_G}(m_H)\rangle}\, e^{\langle \underline{\rho}_Q, H_{Q_{n+1}}(m_H)\rangle} K^{M_Q,T}_{f_{+,Q},\chi}(m_H,m').\end{aligned} \tag{6.28}$$

Fix a $Q \in \mathcal{F}$. By the Iwasawa decomposition, the summand corresponding to Q in (6.26) equals

$$\begin{aligned}\int_{[M_{Q_H}]\times[M_{Q_{G'}}]}\int_{K_H\times K_{G'}} & e^{\langle -2\rho_{Q_H}, H_{Q_H}(m_H)\rangle} e^{\langle -2\rho_{Q_{G'}}, H_{Q_{G'}}(m')\rangle}\\ &\Gamma'_{Q_{n+1}}(H_{Q_{n+1}}(m'_1)-T'_{Q_{n+1}}, T_{Q_{n+1}}-T'_{Q_{n+1}})\\ &K^{Q,T'}_{f_+,\chi}(m_Hk_H, m'k')\,\mu_{n+1}(m'k')\,\mathrm{d}m'\,\mathrm{d}m_H\,\mathrm{d}k'\,\mathrm{d}k_H.\end{aligned}$$

By (6.28), it equals

$$\begin{aligned}\int_{[M_{Q_H}]\times[M_{Q_{G'}}]} & e^{\langle \underline{\rho}_Q, H_{Q_{n+1}}(m_H)\rangle}\Gamma'_{Q_{n+1}}(H_{Q_{n+1}}(m'_1)-T'_{Q_{n+1}}, T_{Q_{n+1}}-T'_{Q_{n+1}})\\ &K^{M_Q,T'}_{f_{+,Q},\chi}(m_H,m')\,\mu_{n+1}(m')\,\mathrm{d}m_H\,\mathrm{d}m'.\end{aligned}$$

We break the integral into an integral over the split center $A^\infty_{M_Q}$ and another over $A^\infty_{M_Q}\backslash[M_{Q_H}]\times[M_{Q_{G'}}]$ to write the above integral as

$$\begin{aligned}\int_{A^\infty_{M_Q}\backslash[M_{Q_H}]\times[M_{Q_{G'}}]} & e^{\langle \underline{\rho}_Q, H_{Q_{n+1}}(m_H)-H_{Q_{n+1}}(m'_1)\rangle} K^{M_Q,T'}_{f_{+,Q},\chi}(m_H,m')\,\mu_{n+1}(m')\\ &\left(\int_{A^\infty_{M_Q}} e^{\langle \underline{\rho}_Q, H_{Q_{n+1}}(am'_1)\rangle}\Gamma'_{Q_{n+1}}(H_{Q_{n+1}}(am'_1)-T'_{Q_{n+1}}, T_{Q_{n+1}}-T'_{Q_{n+1}})\,\mathrm{d}a\right)\mathrm{d}m_H\,\mathrm{d}m'\end{aligned} \tag{6.29}$$

Note that the composition of the embeddings $\mathfrak{a}_Q \subset \mathfrak{a}_{Q_n} \subset \mathfrak{a}_{Q_{n+1}}$ with the projection $\mathfrak{a}_{Q_{n+1}} \to \mathfrak{a}_{Q_{n+1}}^{G_{n+1}}$ yields an isomorphism $\mathfrak{a}_Q \simeq \mathfrak{a}_{Q_{n+1}}^{G_{n+1}}$ whose Jacobian we denote by c_Q. The inner integral in (6.29) thus equals

$$c_Q \int_{\mathfrak{a}_{Q_{n+1}}^{G_{n+1}}} e^{\langle \lambda_Q, X + H_{Q_{n+1}}(m_1') \rangle} \Gamma'_{Q_{n+1}}(X + H_{Q_{n+1}}(m_1') - T'_{Q_{n+1}}, T_{Q_{n+1}} - T'_{Q_{n+1}}) \mathrm{d}X.$$

Changing the variable $X \mapsto X - H_{Q_{n+1}}(m_1') + T'_{Q_{n+1}}$, we obtain that this equals

$$c_Q \int_{\mathfrak{a}_{Q_{n+1}}^{G_{n+1}}} e^{\langle \lambda_Q, X + T'_{Q_{n+1}} \rangle} \Gamma'_{Q_{n+1}}(X, T_{Q_{n+1}} - T'_{Q_{n+1}}) \mathrm{d}X.$$

By definition, cf. (6.25), this equals

$$c_Q e^{\langle \lambda_Q, T' \rangle} p_Q(T - T').$$

In particular it is independent of m'. We then have

$$(6.29) = c_Q e^{\langle \lambda_Q, T' \rangle} p_Q(T - T') \int_{A_{M_Q}^1 \backslash [M_{Q_H}] \times [M_{Q_{G'}}]} e^{\langle \lambda_Q, H_{Q_{n+1}}(m_H) - H_{Q_{n+1}}(m_1') \rangle} K^{M_Q, T'}_{f_{+,Q},}(m_H, m') \eta_{n+1}(m') \mathrm{d}m_H \mathrm{d}m'.$$

This integral is absolutely convergent by (6.22). If we define

$$\widetilde{f}_{+,Q}(m, l) = e^{\langle -\lambda_Q, H_{Q_{n+1}}(m) \rangle} f_{+,Q}(m, l), \quad (m, l) \in M_{Q,+}(\mathbb{A}),$$

then

$$K^{M_Q, T'}_{\widetilde{f}_{+,Q},}(m_H, m') = e^{\langle \lambda_Q, H_{Q_{n+1}}(m_H) - H_{Q_{n+1}}(m_1') \rangle} K^{M_Q, T'}_{f_{+,Q},}(m_H, m'). \tag{6.30}$$

Thus

$$(6.29) = c_Q e^{\langle \lambda_Q, T' \rangle} p_Q(T - T') I^{M_Q, T'}(\widetilde{f}_{+,Q}).$$

In conclusion, we have

$$I^T(f_+) = \sum_{Q \in \mathcal{F}} c_Q e^{\langle \lambda_Q, T' \rangle} p_Q(T - T') I^{M_Q, T'}(\widetilde{f}_{+,Q}).$$

That is, $I^T(f_+)$ is an exponential-polynomial in T and the pure polynomial term is the constant

$$I(f_+) = \sum_{Q \in \mathcal{F}} c_Q\, \epsilon_Q\, \widehat{\theta}_Q(\lambda_Q)^{-1} e^{\langle \lambda_Q, T' \rangle} I^{M_Q, T'}(\widetilde{f}_{+,Q}). \tag{6.31}$$

This proves the theorem.

Remark 6.11. Using the same method, one can show that for any $Q \in \mathcal{F}$, and any $f'_+ \in \mathcal{S}(M_{Q_+}(\mathbb{A}))$ and T sufficiently positive, as a function of T, $I^{M_Q, T}(f'_+)$ is an exponential polynomial whose exponents are contained in $\{\lambda_P - \lambda_Q \mid P \supseteq Q\}$ and only depend on the image $T_{Q_{n+1}}$ of projection of T to $\mathfrak{a}_{Q_{n+1}}$, but the pure polynomial term is not necessarily a constant.

Proposition 6.12. *The distribution I is left $H(\mathbb{A})$-invariant and right $(G'(\mathbb{A}); {}_{G'})$-equivariant. More precisely for all $f_+ \in S(G_+(\mathbb{A}))$, $h \in H(\mathbb{A})$ and $g' \in G'(\mathbb{A})$ we have*

$$I\,((h;g')\cdot f_+) = {}_{n+1}(g')\,I\,(f_+)$$

where $((h;g')\cdot f_+)(g;u) = |\det h|_E^{1/2}\,(\det h)^{-1} f_+(h^{-1}gg';uh)$.

Proof. Fix an $(h_0;g'_0) \in H(\mathbb{A}) \times G'(\mathbb{A})$. Let $T \in \mathfrak{a}_{n+1}$ be sufficiently positive. By (6.23) again we have

$$\hat\tau_{P_{n+1}}(H_{P_{n+1}}({}_1g'_1g'_{0;1}) - T_{P_{n+1}}) = \sum_{P \subset Q} {}_Q\hat\tau_{P_{n+1}}^{Q_{n+1}}(H_{P_{n+1}}({}_1g'_1) - T_{P_{n+1}})$$
$$\Gamma'_{Q_{n+1}}(H_{Q_{n+1}}({}_1g'_1) - T_{Q_{n+1}}; H_{Q_{n+1}}({}_1g'_1) - H_{Q_{n+1}}({}_1g'_1g'_{0;1})).$$

Therefore we see that $K^T_{f_+;}(hh_0;g'g'_0)$ equals

$$\sum_{P \in F}\sum_{P \subset Q}\sum_{\substack{\in P_H(F)\backslash H(F)\\ \in P_{G'}(F)\backslash G'(F)}} {}_Q\hat\tau_{P_{n+1}}^{Q_{n+1}}(H_{P_{n+1}}({}_1g'_1) - T_{P_{n+1}})K_{f_+;P;}(\,hh_0;\,g'g'_0)$$
$$\Gamma'_{Q_{n+1}}(H_{Q_{n+1}}({}_1g'_1) - T_{Q_{n+1}}; H_{Q_{n+1}}({}_1g'_1) - H_{Q_{n+1}}({}_1g'_1g'_{0;1})).$$

Note that

$$K_{(h_0;g'_0)\cdot f_+;P;}(h;g') = K_{f_+;P;}(hh_0;g'g'_0).$$

Thus by summing over $P \in F$ first in the above expression, we see that it simplifies to

$$\sum_{Q \in F}\sum_{\substack{\in Q_H(F)\backslash H(F)\\ \in Q_{G'}(F)\backslash G'(F)}} \Gamma'_{Q_{n+1}}(H_{Q_{n+1}}({}_1g'_1) - T_{Q_{n+1}}; H_{Q_{n+1}}({}_1g'_1) - H_{Q_{n+1}}({}_1g'_1g'_{0;1})) \tag{6.32}$$
$$K^{Q;T}_{(h_0;g'_0)\cdot f_+;}(\,h;\,g').$$

We integrate $K^T_{f_+;}(hh_0;g'g'_0)\,{}_{n+1}(g')$ over $[H] \times [G']$. On the one hand by definition it equals

$$\int_{[H]\times[G']} K^T_{f_+;}(hh_0;g'g'_0)\,{}_{n+1}(g')\,\mathrm{d}g'\,\mathrm{d}h = {}_{n+1}(g'_0)\,I^T(f_+).$$

On the other hand, by (6.32), it equals the sum over all $Q \in F$ of the terms

$$\int_{[H]_{Q_H}}\int_{[G']_{Q_{G'}}} \Gamma'_{Q_{n+1}}(H_{Q_{n+1}}(g'_1) - T_{Q_{n+1}}; H_{Q_{n+1}}(g'_1) - H_{Q_{n+1}}(g'_1g'_{0;1})) \tag{6.33}$$
$$K^{Q;T}_{(h_0;g'_0)\cdot f_+;}(h;g')\,{}_{n+1}(g')\,\mathrm{d}h\,\mathrm{d}g'.$$

We calculate this as in the proof of Theorem 6.10. By the Iwasawa decomposition it equals

$$\int_{[M_{Q_H}]\times[M_{Q_{G'}}]}\int_{K_H \times K_{G'}} e^{\langle -2\rho_{Q_H}, H_{Q_H}(m_H)\rangle} e^{\langle -2\rho_{Q_{G'}}, H_{Q_{G'}}(m')\rangle} \tag{6.34}$$
$$\Gamma'_{Q_{n+1}}(H_{Q_{n+1}}(m'_1) - T_{Q_{n+1}}; -H_{Q_{n+1}}(k'_1g'_{0;1}))$$
$$K^{Q;T}_{(h_0;g'_0)\cdot f_+;}(m_Hk_H;m'k')\,{}_{n+1}(m'k')\,\mathrm{d}k'\,\mathrm{d}k_H\,\mathrm{d}m'\,\mathrm{d}m_H.$$

For $(m;l) \in M_{Q,+}(\mathbb{A})$, we put

$$f_{+,Q,h_0,g_0'}(m;l) = e^{\langle \rho_{Q_G}, H_{Q_G}(m)\rangle} \int_{K_H} \int_{K'} \int_{N_{Q_G}(\mathbb{A})} \int_{N_{Q_{L^-}}(\mathbb{A})} ((h_0;g_0') \cdot f_+)(k_H^{-1} mnk';(l+u)k_H)$$
$$p_Q(-H_{Q_{n+1}}(k_1' g_{0,1}'))\, \eta_{n+1}(k')\, \mu^{-1}(k_H)\, \mathrm{d}n\, \mathrm{d}u\, \mathrm{d}k_H\, \mathrm{d}k'.$$

One check directly that for an $(m_H;m') \in [M_{Q_H}] \times [M_{Q_{G'}}]$, the expression

$$\int_{K_H \times K_{G'}} \int_{A_{M_Q}^{1}} e^{\langle -2\rho_{Q_H}, H_{Q_H}(am_H)\rangle} e^{\langle -2\rho_{Q_{G'}}, H_{Q_{G'}}(am')\rangle} \Gamma'_{Q_{n+1}}(H_{Q_{n+1}}(am_1') - T_{Q_{n+1}}, -H_{Q_{n+1}}(k_1' g_{0,1}'))$$
$$K_{(h_0;g_0')\cdot f_+,P}(am_H k_H; am'k')\, \eta_{n+1}(am'k')\, \mathrm{d}a\, \mathrm{d}k_H\, \mathrm{d}k'$$

equals to

$$e^{\langle \underline{\rho}_Q, H_{Q_{n+1}}(m_H) - H_{Q_{n+1}}(m_1')\rangle} c_Q e^{\langle \underline{\rho}_Q, T\rangle} K^{M_Q}_{f_{+,Q,h_0,g_0'},P\cap M_Q}(m_H;m').$$

Then for any $\chi \in \mathfrak{X}(G)$,

$$\int_{K_H \times K_{G'}} \int_{A_{M_Q}^{1}} e^{\langle -2\rho_{Q_H}, H_{Q_H}(am_H)\rangle} e^{\langle -2\rho_{Q_{G'}}, H_{Q_{G'}}(am')\rangle} \Gamma'_{Q_{n+1}}(H_{Q_{n+1}}(am_1') - T_{Q_{n+1}}, -H_{Q_{n+1}}(k_1' g_{0,1}'))$$
$$K_{(h_0;g_0')\cdot f_+,P,\chi}(am_H k_H; am'k')\, \eta_{n+1}(am'k')\, \mathrm{d}a\, \mathrm{d}k_H\, \mathrm{d}k'$$

equals to

$$e^{\langle \underline{\rho}_Q, H_{Q_{n+1}}(m_H) - H_{Q_{n+1}}(m_1')\rangle} c_Q e^{\langle \underline{\rho}_Q, T\rangle} K^{M_Q}_{f_{+,Q,h_0,g_0'},P\cap M_Q,\chi}(m_H;m').$$

Therefore (6.34) equals

$$c_Q e^{\langle \underline{\rho}_Q, T\rangle} I^{M_Q,T}(\widehat{f_{+,Q,h_0,g_0'}}),$$

where

$$\widehat{f_{+,Q,h_0,g_0'}}(m;l) = e^{\langle -\underline{\rho}_Q, H_{Q_{n+1}}(m)\rangle} f_{+,Q,h_0,g_0'}(m;l), \quad (m;l) \in M_{Q,+}(\mathbb{A})$$

In conclusion, we have

$$\eta_{n+1}(g_0') I^T(f_+) = \sum_{Q \in \mathcal{F}} c_Q e^{\langle \underline{\rho}_Q, T\rangle} I^{M_Q,T}(\widehat{f_{+,Q,h_0,g_0'}}).$$

Each $I^{M_Q,T}(f_{+,Q,h_0,g_0'})$ is a exponential polynomial, whose exponents are in the set $\{\underline{\rho}_P - \underline{\rho}_Q \mid P \subset Q\}$. Since $\underline{\rho}_P$ is not trivial unless $P = J_n$, the only term on the right hand side that has a (possibly) nonzero purely polynomial part correspond to $Q = J_n$. In this case $f_{+,Q,h_0,g_0'} = (h_0;g_0') \cdot f_+$ and the pure polynomial part of the right hand side equals $I((h_0;g_0') \cdot f_+)$.

6.5. **A second modified kernel.** For later use we will need another modified kernel. For $f_+ \in \mathcal{S}(G_+(\mathbb{A}))$, $T \in \mathfrak{a}_{n+1}$, and $(h;g') \in [H] \times [G']$, we define

$$\kappa^T_{f_+,\chi}(h;g') = \sum_{P \in \mathcal{F}} \epsilon_P \sum_{\substack{\gamma \in P_H(F)\backslash H(F) \\ \delta \in P'(F)\backslash G'(F)}} \widehat{\tau}_{P_{n+1}}(H_{P_{n+1}}(\delta h) - T_{P_{n+1}}) K_{f_+,P,\chi}(\gamma h; \delta g').$$

Proposition 6.13. *For every $N > 0$, there is a continuous seminorm $\|\cdot\|_{S,N}$ on $\mathcal{S}(G(\mathbb{A}) \times \mathbb{A}_{E,n})$ such that for all $h \in [H]$ and $g' \in [G']$ we have*

$$\sum_{\chi} \left| K^T_{f_+,\chi}(h,g') - F^{G_{n+1}}(h,T) K_{f_+,\chi}(h,g') \right| \ll e^{-N\|T\|} \|h\|_H^{-N} \|g'\|_{G'}^{-N} \|f_+\|_{S,N}. \tag{6.35}$$

In particular

$$\sum_{\chi \in \mathfrak{X}(G)} \int_{[H]} \int_{[G']} \left| K^T_{f_+,\chi}(h,g') \right| \mathrm{d}g' \mathrm{d}h < \infty$$

for T sufficiently large.

Proof. As in the case of Theorem 6.8, we only need to prove the estimate (6.35), and second assertion on the absolute convergence follows from it. The proof is very similar to (and in fact simpler than) that of Theorem 6.8, so we will only sketch the differences.

We first note that we only need to prove that there are continuous semi-norms $\|\cdot\|_{S,1}$ and $\|\cdot\|_{S,2}$ on $\mathcal{S}(G(\mathbb{A}))$ and $\mathcal{S}(\mathbb{A}_{E,n})$ respectively, such that

$$\sum_{\chi} \left| K^T_{f \otimes \Phi,\chi}(h,g') - F^{G_{n+1}}(h,T) K_{f \otimes \Phi,\chi}(h,g') \right| \ll e^{-N\|T\|} \|h\|_H^{-N} \|g'\|_{G'}^{-N} \|f\|_{S,1} \|\Phi\|_{S,2}.$$

Once we have this, the estimate (6.35) holds for $f_+ \in \mathcal{S}(G(\mathbb{A})) \otimes \mathcal{S}(\mathbb{A}_{E,n})$. For fixed h and g', the left hand side of (6.35) is continuous with respect to f_+ by definition. Thus the estimate (6.35) is obtained by taking limits.

We assume $f_+ = f \otimes \Phi$ from now on. Define $\mathcal{T}^0_F(G)$ to be the space of tuples $(\varphi_P)_{P \in F}$ such that

$$\varphi_P \in \mathcal{T}([G]_{P_G}), \quad \varphi_P - (\varphi_Q)_{P_G} \in \mathcal{S}_{d^{Q_G}_{P_G}}([G]_{P_G}).$$

Let us consider the series of linear maps

$$\mathcal{T}^0([G]) \xrightarrow{\mathrm{R}(f)} \mathcal{T}^0_F(G) \longrightarrow \mathcal{T}_F(\mathcal{G}) \xrightarrow{j_H} \mathcal{T}_F(H) \underset{\Pi^{T,H}}{\overset{\Lambda^{T,H}}{\rightrightarrows}} \mathcal{S}^0([H])$$

The first map is

$$\varphi \mapsto \mathrm{R}(f)\varphi_{P_{G'}}.$$

The second map is given by

$$\varphi_P \mapsto \left((j,g_2) \mapsto \varphi_P(j,g_2)\, {}_P\Theta(j,\Phi)\right),$$

where $(j,g_2) \in J_n \times G_n$ and g_1 is the image of j in G_n. The third one is the restriction to $[H]_{P_H}$. The last map is one of the two truncation operators

$$\Lambda^{T,H}, \Pi^{T,H} : \mathcal{T}_F(H) \to \mathcal{S}^0([H]).$$

defined in [BPCZ22, Section 3.5]. The fact that the family $P \mapsto \mathrm{R}(f)\varphi_{P_{G'}}$ belongs to $\mathcal{T}^0_F(G)$ follows from [BPCZ22, Proposition 3.4.2.1.2]. The fact that rest of the maps make sense follows directly from the definition. Continuity of these maps follow from the closed graph theorem, cf. Remark 6.9.

The compositions of these maps are given by integral kernels $^{T}_{f \ \Phi}(h; g^{0})$ and $F^{G_{n+1}}(h; T) \ _{f \ \Phi}(h; g^{0})$ respectively.

Let $\ 2 \mathrm{X}(G)$. We modify the maps slightly, by using the map R (f) in the first one, where R (f) is the composition of R(f) followed by projection to the -component. The kernel function associated to the resulting maps are $^{T}_{f \ \Phi;}(h; g^{0})$ and $F^{G_{n+1}}(h; T) \ _{f \ \Phi;}(h; g^{0})$ respectively.

The rest of the proof is exactly the same as that of Theorem 6.8.

Let $T \ 2 \ \mathfrak{a}_{n+1}$ be sufficiently large and put

$$i^{T}(f_{+}) = \int_{[H]} \int_{[G^{0}]} {}^{T}_{f_{+};}(h; g^{0}) \ _{n+1}(g^{0}) \mathrm{d}g^{0} \mathrm{d}h$$

Proposition 6.14. *As a function of T, the function $i^{T}(f_{+})$ is the restriction of an exponential polynomial function whose purely polynomial part is a constant and equals $I \ (f_{+})$.*

Proof. By (6.23), we have

$$\widehat{}_{P_{n+1}}(H_{P_{n+1}}(\ h) \quad T_{P_{n+1}}) = \sum_{Q \ P} {}_{Q}\widehat{}^{Q_{n+1}}_{P_{n+1}}(H_{P_{n+1}}(\ _{1}g^{0}_{1}) \quad T_{P_{n+1}})$$
$$\Gamma^{0}_{Q_{n+1}}(H_{Q_{n+1}}(\ _{1}g^{0}_{1}) \quad T_{Q_{n+1}}; H_{Q_{n+1}}(\ _{1}g^{0}_{1}) \quad H_{Q_{n+1}}(\ h)):$$

Plugging into the definition of $^{T}_{f_{+};}$, we obtain that

$$^{T}_{f_{+};}(h; g^{0}) = \sum_{Q 2 F} \sum_{\substack{2 Q_{H}(F) n G(F) \\ 2 Q_{G^{0}}(F) n G^{0}(F)}} \Gamma^{0}_{Q_{n+1}}(H_{Q_{n+1}}(\ _{1}g^{0}_{1}) \quad T_{Q_{n+1}}; H_{Q_{n+1}}(\ _{1}g^{0}_{1}) \quad H_{Q_{n+1}}(\ h)) K^{Q;T}_{f_{+};}(\ h; \ g^{0}):$$

Define $f_{+;Q}$ by (6.28) and $f_{+;Q}^{\mathfrak{g}}$ by (6.30) as in the proof of Theorem 6.10, then the same computation as in the proof of Theorem 6.10 gives (we follow the same notation there)

$$i^{T}(f_{+}) = \sum_{Q 2 F} c_{Q} e^{h \ _{Q}; T i} I^{M_{Q}; T}(f_{+;Q}^{\mathfrak{g}});$$

As in the proof of Theorem 6.10 and Proposition 6.12, cf. also Remark 6.11, we see that $i^{T}(f_{+})$ is an exponential polynomial function of T. The only term on the right hand side with a (possibly) nonzero purely polynomial part is the one that correspond to $Q = J_{n}$. This term equals $I^{T}(f_{+})$ and the purely polynomial part is the constant $I \ (f_{+})$.

7. The coarse geometric expansion: general linear groups

7.1. **Geometric modified kernels.** We keep the notation from the previous section. We will need in addition the following list of notation.

Recall that $L = E^{n}$ and $L^{-} = E_{n}$. Let L and $L^{-;}$ be the purely imaginary part, i.e. $L \ = E^{\ ;n}$ and $L^{-;} \ = E_{n}$. If A is a subset of $L^{-} \quad L$, we define A to be its intersection with $L^{-;} \quad L$.

We put $G^{+} = G \times L_{-} \times L$, the group structure being the product of G and $L_{-} \times L$. Note that this is not a subgroup of $\tilde{G}$, but merely a subvariety. The group $H \times G'$ acts from the right on G^{+} by

$$(g; w; v) \cdot (h; g') = (h^{-1} g g'; w g'_1; g'^{-1}_1 v). \tag{7.1}$$

Let $P = MN \in F$. Put $M^{+} = M_G \times M_{L_-} \times M_L$, $N^{+} = N_G \times N_{L_-} \times N_L$, and $P^{+} = M^{+}N^{+}$. We often write an element in P^{+} as $m^{+}n^{+}$, but one should note that the product is the one in G^{+}, not the one in $\tilde{G}$.

Let $q : G^{+} \to A = G^{+}//(H \times G')$ be the GIT quotient. We define a morphism $G^{+} \to \mathrm{Res}_{E/F}\, \mathbf{A}_{2n;E}$ by

$$((g_1; g_2); w; v) \mapsto (a_1; \ldots; a_n; b_1; \ldots; b_n); \tag{7.2}$$

where $\mathbf{A}_{2n;E}$ denotes the $2n$-dimensional affine space over E, and we put

$$s = (g_1^{-1} g_2)(g_1^{-1} g_2)^{c;-1}; \quad a_i = \mathrm{Trace} \wedge^i s; \quad b_i = w s^i v; \quad i = 1; 2; \ldots; n.$$

This map descends to a locally closed embedding $A \hookrightarrow \mathrm{Res}_{E/F}\, \mathbf{A}_{2n;E}$. We always consider A as a locally closed subscheme of $\mathrm{Res}_{E/F}\, \mathbf{A}_{2n;E}$.

If $\alpha \in A(F)$, we define G^{+}_{α} to be the inverse image of α (as a closed subscheme of G^{+}). For $P = MN \in F$, we put $M^{+}_{\alpha} = G^{+}_{\alpha} \cap M^{+}$.

Let us now introduce the geometric counterparts of the modified kernels. For $\alpha \in A(F)$, $f^{+} \in S(G^{+}(\mathbb{A}))$, and $P \in F$, we define kernel functions on $[H]_{P_H} \times [G']_{P_{G'}}$ by

$$k_{f^{+};P;\alpha}(h; g') = \sum_{m^{+} \in M^{+}_{\alpha}(F)} \int_{N^{+}(\mathbb{A})} f^{+}(m^{+}n^{+} \cdot (h; g'))\,\mathrm{d}n^{+};$$

and

$$k_{f^{+};P}(h; g') = \sum_{m^{+} \in M^{+}(F)} \int_{N^{+}(\mathbb{A})} f^{+}(m^{+}n^{+} \cdot (h; g'))\,\mathrm{d}n^{+}.$$

Lemma 7.1. *There is an $N > 0$ and a seminorm $\| \cdot \|_S$ on $S(G^{+}(\mathbb{A}))$ such that for all $(h; g') \in [H]_{P_H} \times [G']_{P_{G'}}$, we have*

$$\sum_{m^{+} \in M^{+}(F)} \int_{N^{+}(\mathbb{A})} |f^{+}(m^{+}n^{+} \cdot (h; g'))|\,\mathrm{d}n\mathrm{d}l \ll \|f^{+}\|_S \|h\|^N_{P_H} \|g'\|^N_{P_{G'}}.$$

In particular the defining expressions of $k_{f^{+};P;\alpha}$ and $k_{f^{+};P}$ are absolutely convergent and we have

$$\sum_{\alpha \in A(F)} k_{f^{+};P;\alpha}(h; g') = k_{f^{+};P}(h; g').$$

Proof. For any large enough d and N, we can find a continuous seminorm $\| \cdot \|_d$ on $S(G^{+}(\mathbb{A}))$ such that

$$|f^{+}(m^{+}n^{+} \cdot (h; g'))| \ll \|m^{+}n^{+}\|^{-d}_{P^{+}(\mathbb{A})} \|f^{+}\|_d \|h\|^N_{P_H} \|g'\|^N_{P_{G'}}$$

where $\| \cdot \|_{P^+(\mathbb{A})}$ stands for a height function on $P^+(\mathbb{A})$. The lemma then follows from [BP21a, Proposition A.1.1.(v-vi)].

We make the following key observation. We identify $\mathrm{Res}_{E/F}\, \mathbf{A}_{2n,E}$ with the $4n$-dimensional affine space $\mathbf{A}^{4n}$ over F, and denote the morphism $G^+ \to \mathbf{A}^{4n}$ again by q. We extend the definition of $k_{f^+,P,\xi}$ to all $\xi \in F^{4n}$ by setting $k_{f^+,P,\xi} = 0$ if $\xi \notin A(F)$.

Let $C \subset G^+(\mathbb{A}_{F,f})$ be an open compact subset such that $\operatorname{supp} f^+ \subset C \times G^+(F_\infty)$. Since $q(C) \subset \mathbb{A}_{F,f}^{4n}$ is compact, there exists $d \in F$ depending only on C such that $q(C) \cap F^{4n} \subset (d\mathcal{O}_F)^{4n}$. Then if $k_{f^+,P,\xi}$ is not identically zero for some $\xi \in F^{4n}$ and $P \in \mathcal{F}$, then $\xi \in (d\mathcal{O}_F)^{4n}$.

Put $\Lambda = (d\mathcal{O}_F)^{4n}$ which is a lattice in F_∞^{4n}. For each $\xi \in \Lambda$, take $u_\xi \in C_c^\infty(F_\infty^{4n})$ and define the functions $f^+_\xi = f^+ u_\xi$ as in Subsection 4.10.

Lemma 7.2. *We have*

$$k_{f^+_\xi,P}(h,g') = k_{f^+_\xi,P,\xi}(h,g') = k_{f^+,P,\xi}(h,g')$$

for all $(h,g') \in [H]_{P_H} \times [G']_{P_{G'}}$. In particular, $k_{f^+_\xi,P}(h,g')$ is independent of the choice of the bump function u_ξ.

Proof. Take $\eta \in A(F)$. For any $(h,g') \in [H]_{P_H} \times [G']_{P_{G'}}$, and any $m^+ \in M^+(F)$, $n^+ \in N^+(\mathbb{A})$, we have $m^+n^+ \cdot (h,g') \in q^{-1}(\eta)$. By the definition of f^+_ξ, we have $f^+_\xi(m^+n^+ \cdot (h,g')) = f^+(m^+n^+ \cdot (h,g'))$ if $\xi = \eta$ and 0 if $\xi \neq \eta$. The result follows.

The kernel $K_{f_+,P}$ introduced in Subsection 6.3 is closely related to $k_{f^+,P}$. First we recall that the action $\mathrm{R}_{\mu^{-1}}$ of $G_n(\mathbb{A})$ on $\mathcal{S}(\mathbb{A}_{E,n})$ given by

$$\mathrm{R}_{\mu^{-1}}(g)\Phi(x) = \mu(\det g)^{-1} |\det g|_E^{\frac{1}{2}} \Phi(xg), \quad g \in G_n(\mathbb{A}), \quad \Phi \in \mathcal{S}(\mathbb{A}_{E,n}).$$

We define a Fourier transform $\mathcal{S}(\mathbb{A}_{E,n}) \to \mathcal{S}(\mathbb{A}_{E,n} \times \mathbb{A}_E^{n,-})$ by

$$\Phi^\dagger(w,v) = \int_{\mathbb{A}_n} \Phi(x+w)\, \psi((-1)^n\, xv)\,\mathrm{d}x, \quad (w,v) \in \mathbb{A}_{E,n} \times \mathbb{A}_E^{n,-}. \tag{7.3}$$

Then there is a unique action $\mathrm{R}^\dagger_{\mu^{-1}}$ of $G_n(\mathbb{A})$ on $\mathcal{S}(\mathbb{A}_{E,n} \times \mathbb{A}_E^{n,-})$ such that the Fourier transform is equivariant. Direct computation gives that if $g' \in G'_n(\mathbb{A})$ and $\Phi \in \mathcal{S}(\mathbb{A}_{E,n})$ then

$$\mathrm{R}^\dagger_{\mu^{-1}}(g')\Phi^\dagger(w,v) = \mu(\det g')\Phi^\dagger(wg', g'^{-1}v).$$

This Fourier transform extends to a continuous linear isomorphism

$$\mathcal{S}(G_+(\mathbb{A})) \to \mathcal{S}(G^+(\mathbb{A})), \quad f_+ \mapsto f_+^\dagger,$$

given by

$$f_+^\dagger(g,w,v) = \mathrm{R}^\dagger_{\mu^{-1}}(g_1^{-1}) f_+(g,\cdot)^\dagger(w,v), \quad (g,w,v) \in G^+(\mathbb{A}). \tag{7.4}$$

Here the expression on the right hand side is interpreted as follows. We evaluate f_+ at g first to obtain a Schwartz function on $\mathbb{A}_{E,n}$, then take the Fourier transform $\dagger$, then make g_1^{-1} act on it via $\mathrm{R}^{\dagger}_{\ \ 1}$, and finally evaluate the result at (w, v). We denote by $\ _{\dagger}$ the inverse integral transform of $\ ^{\dagger}$. If $f_+ = f \otimes \Phi$ where $f \in S(G(\mathbb{A}))$ and $\Phi \in S(\mathbb{A}_{E,n})$ then we have a cleaner expression

$$f_+^{\dagger}(g, w, v) = f(g) \ \mathrm{R}_{\ \ 1}(g_1^{-1})\Phi^{\ \dagger}(w, v) = f(g)\mathrm{R}^{\dagger}_{\ \ 1}(g_1^{-1})\Phi^{\dagger}(w, v).$$

Lemma 7.3. *For all* $(h, g') \in [H]_{P_H} \times [G']_{P_{G'}}$ *and* $f_+ \in S(G_+(\mathbb{A}))$, *we have*

$$K_{f_+,P}(h, g') = k_{f_+^{\dagger},P}(h, g') \ (\det g_1').$$

Proof. By Lemma 7.1, for fixed h and g', both sides are continuous linear forms on f_+. Therefore we only need to prove the lemma when $f_+ = f \otimes \Phi$ where $f \in S(G(\mathbb{A}))$ and $\Phi \in S(\mathbb{A}_{E,n})$. To best illustrate the ideas, we prove the lemma in the case $P = G$. The general case follows by the same computation but with messier notation. By definition $K_{f \otimes \Phi}(h, g')$ equals

$$\sum_{\ \in G(F)} \sum_{x \in E_n} f(h^{-1} \ g')\mathrm{R}_{\ \ 1}(h)\Phi(x).$$

This equals

$$\sum_{\ \in G(F)} \sum_{x \in E_n} f(h^{-1} \ g')\mathrm{R}_{\ \ 1}(g_1^{-1} \ _1^{-1}h)\Phi(xg_1')|\det g_1'| \ (\det g_1').$$

The Poisson summation formula then gives

$$\sum_{\ \in G(F)} \sum_{(w,v) \in E_n \times E^{n,}} f(h^{-1} \ g')(\mathrm{R}_{\ \ 1}(g_1'^{-1} \ _1^{-1}h)\Phi)^{\dagger}(w g_1, g_1^{-1}v) \ (\det g_1'),$$

which equals $k_{f_+^{\dagger},P}(h, g') \ (\det g_1')$.

Lemma 7.4. *Let* $f^+ \in S(G^+(\mathbb{A}))$. *We have*

$$(g_1')((h, g') \ f^+)_{\dagger} = (h, g') \ (f^+)_{\dagger},$$

where on the left hand, the action is the one induced from the action (7.1), *and on the right hand side the action is the one defined in Proposition 6.12.*

Proof. This is a direct computation.

Let $T \in \mathfrak{a}_{n+1}$ be a truncation parameter, and let $\ \in A(F)$. We define

$$k^T_{f^+,\ }(h, g') = \sum_{P \in F} \ _P \sum_{\substack{\ \in P_H(F)\backslash H(F) \\ \ \in P_{G'}(F)\backslash G'(F)}} \widehat{\tau}_{P_{n+1}}(H_{P_{n+1}}(\ _1 g_1') - T_{P_{n+1}})K_{f^+,P,\ }(\ h, \ g'),$$

and

$$k^T_{f^+}(h, g') = \sum_{P \in F} \ _P \sum_{\substack{\ \in P_H(F)\backslash H(F) \\ \ \in P_{G'}(F)\backslash G'(F)}} \widehat{\tau}_{P_{n+1}}(H_{P_{n+1}}(\ _1 g_1') - T_{P_{n+1}})K_{f^+,P}(\ h, \ g').$$

Lemma 7.5. *For $f^+ \in S(G^+(\mathbb{A}))$, if T is sufficiently positive the integral*

$$\int_{[H]\times[G']} |k^T_{f^+}(h,g')| \,\mathrm{d}h\mathrm{d}g'$$

is convergent and defines a seminorm on $S(G^+(\mathbb{A}))$. Put

$$i^T(f^+) = \int_{[H]\times[G']} k^T_{f^+}(h,g')\,\eta_{G'}(g')\mathrm{d}h\mathrm{d}g'.$$

Then $i^T(f^+)$ is the restriction of a polynomial exponential whose purely polynomial part is a constant that equals $I((f^+)_y)$.

Proof. It follows from Lemma 7.3 that if $f^+ \in S(G_+(\mathbb{A}))$ then

$$k^T_{f^+}(h,g')\,\eta(\det g'_1) = K^T_{f^+_y}(h,g')$$

for all $(h,g') \in [H]\times[G']$. The lemma then follows directly from Theorem 6.8.

7.2. **The coarse geometric expansion.** We now develop the coarse geometric expansion.

Theorem 7.6. *We have the following assertions.*

(1) *For T sufficiently positive, the expression*

$$\sum_{\xi\in A(F)} \int_{[G']}\int_{[H]} |k^T_{f^+,\xi}(h,g')| \,\mathrm{d}h\mathrm{d}g'$$

is convergent and defines a continuous semi-norm on $S(G^+(\mathbb{A}))$.

(2) *For $\xi \in A(F)$, we define*

$$i^T_\xi(f^+) = \int_{[G']}\int_{[H]} k^T_{f^+,\xi}(h,g')\,\eta_{G'}(g')\mathrm{d}h\mathrm{d}g'.$$

Then as a function of T, when T is sufficiently positive, $i^T_\xi(f)$ is the restriction of a polynomial exponential function whose purely polynomial part is a constant. We denote this constant by $i_\xi(f^+)$.

(3) *The distribution $f^+ \mapsto i_\xi(f^+)$ satisfies the invariance property that*

$$i_\xi((h,g')\cdot f^+) = \eta_{G'}(g')\,i_\xi(f^+),$$

where the action $(h,g')\cdot f^+$ is induced from the action (7.1).

Proof. Recall that we introduced a lattice $\Lambda \subset F_1^N$, and defined functions u_ν and f^+_ν for $\nu \in \Lambda$ before Lemma 7.2. By Proposition 4.30, $(f^+_\nu)_{\nu\in\Lambda}$ is absolutely summable in $S(G^+(\mathbb{A}))$. Therefore by Lemma 7.5, we have

$$\sum_{\nu\in\Lambda}\int_{[H]}\int_{[G']} |K^T_{f^+_\nu}(h,g')| \,\mathrm{d}h\mathrm{d}g' < \infty.$$

By Lemma 7.2, we have

$$K^T_{f^+_\nu}(h,g') = K^T_{f^+,\nu}(h,g').$$

By the construction of the lattice Λ we conclude that

$$\sum_{\alpha\in A(F)}\int_{[H]}\int_{[G']} K^T_{f^+,\alpha}(h,g')\,\mathrm{d}h\mathrm{d}g' = \sum_{\xi\in\Lambda}\int_{[H]}\int_{[G']} K^T_{f}(h,g')\,\mathrm{d}h\mathrm{d}g'.$$

This proves the first assertion on absolute convergence. By the uniform boundedness principle, it defines a continuous semi-norm.

By Lemma 7.2 and Lemma 7.3 we have

$$i^T_\alpha(f^+) = i^T_\alpha(f^+) = I^T_\alpha((f^+)_\dagger).$$

This implies the second assertion. The third assertion follows from Lemma 7.4 and Proposition 6.12.

7.3. **Synthesis of the results: the coarse relative trace formula.** We now summarize what we have done. For $f_+\in S(G_+(\mathbb{A}))$ we put

$$I^T_\alpha(f_+) = i^T_\alpha(f_+^\dagger);\quad I_\alpha(f_+) = i_\alpha(f_+^\dagger).$$

Then Theorem 7.6 tells us that if T is sufficiently positive then $I^T_\alpha(f_+)$ is the restriction of a polynomial exponential and its purely polynomial part is a constant that equals $I_\alpha(f_+)$. By Lemma 7.4, the distribution I_α is left $H(\mathbb{A})$-invariant and right $(G'(\mathbb{A}),\chi)$-equivariant in the sense of Proposition 6.12.

We summarize the coarse relative trace formulae on the general linear groups as the following theorem.

Theorem 7.7. *Let $f_+\in S(G_+(\mathbb{A}))$ be a test function. Then we have*

$$\sum_{\chi\in\mathfrak{X}(G)} I_\chi(f_+) = \sum_{\alpha\in A(F)} I_\alpha(f_+).$$

The summations on both sides are absolutely convergent and each summand is left $H(\mathbb{A})$-invariant and right $(G'(\mathbb{A}),\chi)$-equivariant.

This is simply a combination of Theorem 6.10, Proposition 6.12, Lemma 7.5 and Theorem 7.6.

8. The coarse spectral expansion: unitary groups

8.1. **Setup.** The following notation will be used throughout this section.

- Let (V,q_V) be a nondegenerate skew-Hermitian space of dimension n. Put $\mathrm{U}_V = \mathrm{U}(V)\times\mathrm{U}(V)$. If $g\in\mathrm{U}_V$, without mentioning explicitly the contrary, we will denote by $g=(g_1,g_2)$ where $g_i\in\mathrm{U}(V)$. Let $\mathrm{U}'_V\subset\mathrm{U}_V$ denote the diagonal subgroup, which is isomorphic to $\mathrm{U}(V)$.
- Let $S(V)$ be the Heisenberg group attached to V, and $J(V)=S(V)\rtimes\mathrm{U}(V)$ be the Jacobi group (see Subsection 5.3). Put $\widetilde{\mathrm{U}}_V=\mathrm{U}(V)\times J(V)$. If $g\in\mathrm{U}(V)$, an element in $J(V)$ whose image in $\mathrm{U}(V)$ is g is usually denoted by $\widetilde{g}$. An element in $\widetilde{\mathrm{U}}_V$ is usually denoted by $\widetilde{x}$. This means $\widetilde{x}=(x_1,\widetilde{x}_2)$, $x=(x_1,x_2)\in\mathrm{U}_V$ and the image of $\widetilde{x}$ in U_V is x.

The group U'_V diagonally embeds in $\widetilde{\mathrm{U}}_V$. Its image is again denoted by U'_V. There is a natural map $J(V) \to \widetilde{\mathrm{U}}_V$ and we let $\widetilde{\mathrm{U}}'_V$ its image.

We keep the notation from Subsection 5.3. In particular, we fix a minimal parabolic subgroup P_0 of $\mathrm{U}(V)$ which fixes the maximal isotropic flag (5.10). Standard parabolic subgroups of $\mathrm{U}(V)$ are those containing P_0. Let $\mathcal{F}_V$ be the subset of standard D-parabolic subgroups of $J(V)$, and let $\mathcal{F}'_V$ be the set of standard parabolic subgroup of U'_V. We put $P' = P \cap \mathrm{U}(V)$ for $P \in \mathcal{F}_V$. Then by Lemma 5.7, the map $P \mapsto P'$ is a bijection from $\mathcal{F}_V$ to $\mathcal{F}_{V'}$.

Let $P \in \mathcal{F}_V$. We denote by P_1 and P_2 respectively the subgroup P' of the first and second factor of U_V. We put

$$\widetilde{P} = P_1 \times P, \quad P_{\mathrm{U}} = P_1 \times P_2, \tag{8.1}$$

which are D-parabolic subgroups of $\widetilde{\mathrm{U}}_V$ and U_V respectively. The notation P' usually specifically means the parabolic subgroup of U'_V.

Let $\operatorname{Res} V$ be the symplectic space defined in Subsection 5.3. We fix a polarization $\operatorname{Res} V = L \oplus L^\vee$ as in Subsection 5.4. We have the Weil representation $\omega = \omega_{\psi,\mu}$ of $J(V)(\mathbb{A})$ realized on $\mathcal{S}(L^\vee(\mathbb{A}))$. For $\phi \in \mathcal{S}(L^\vee(\mathbb{A}))$ and $P \in \mathcal{F}_V$ we have theta theta function $\theta_P(\cdot,\phi)$. We also have $\omega^\vee = \omega_{\psi^{-1},\mu^{-1}}$ and $\theta_P^\vee(\cdot,\phi)$ defined in terms of $\omega^\vee$.

Put $\mathrm{U}_{V,+} = \mathrm{U}_V \ltimes (L^\vee \oplus L^\vee)$. The group structure is given by the product group structure of $\mathrm{U}(V)$ and the additive group $L^\vee \oplus L^\vee$. Let $P \in \mathcal{F}_V$. We put

$$M_{P,+} = M_{P_{\mathrm{U}}} \times M_{L^\vee} \times M_{L^\vee}, \quad N_{P,+} = N_{P_{\mathrm{U}}} \times N_{L^\vee} \times N_{L^\vee} = N_{P_{\mathrm{U}}}, \quad P_+ = M_{P,+}N_{P,+},$$

where we recall that $M_{L^\vee}$ and $N_{L^\vee}$ are defined in Subsection 5.4. These are subgroups of $\mathrm{U}_{V,+}$. We often write an element in $M_{P,+}$ as (m, l_1, l_2) where $m \in M_{P_{\mathrm{U}}}$ and $l_1, l_2 \in L^\vee$.

Put $\mathfrak{a}_0 = \mathfrak{a}_{P_0}$. A truncation parameter is an element in $\mathfrak{a}_0$.

8.2. **Technical preparations.** Let $P \in \mathcal{F}_V$ and w be a weight on $[\widetilde{\mathrm{U}}_V]_{\widetilde{P}}$. We can define various function spaces as in Subsection 4.3. Of particular interest to us are $\mathcal{S}_w([\widetilde{\mathrm{U}}_V]_{\widetilde{P}})$ and $\mathcal{T}_w([\widetilde{\mathrm{U}}_V]_{\widetilde{P}})$.

The "approximation by the constant term" for the group $\widetilde{\mathrm{U}}_V$ takes the following form.

Proposition 8.1. *Let $N > 0$, $r \geq 0$, $X \in \mathcal{U}(\widetilde{\mathfrak{u}}_{V,\infty})$ and $P, Q \in \mathcal{F}_V$. Then there exists a continuous seminorm $\|\cdot\|_{N,X,r}$ on $\mathcal{T}_N([\widetilde{\mathrm{U}}_V]_{\widetilde{Q}})$, such that*

$$\left|\mathrm{R}(X)\varphi(g) - \mathrm{R}(X)\varphi_{\widetilde{P}}(g)\right| \leq \|g\|_P^N d_{P_{\mathrm{U}}}^{Q_{\mathrm{U}}}(g)^{-r}\|\varphi\|_{N,X,r}$$

holds for all $\varphi \in \mathcal{T}_N([\widetilde{\mathrm{U}}_V]_{\widetilde{Q}})$ and $g \in \mathrm{U}_V(\mathbb{A})$.

Proof. For $g \in \mathrm{U}_V(\mathbb{A})$, we have $d_{\widetilde{P}}^{\widetilde{Q}}(g) = d_{P_1}^{Q_1}(g_1)d_P^Q(g_2)$ and $d_{P_{\mathrm{U}}}^{Q_{\mathrm{U}}}(g) = d_{P_1}^{Q_1}(g_1)d_{P_2}^{Q_2}(g_2)$. Therefore by Lemma 5.8, we have $\min\{d_{P_{\mathrm{U}}}^{Q_{\mathrm{U}}}(g), d_{P_1}^{Q_1}(g_1)d_{P_2}^{Q_2}(g_2)^{\frac{1}{2}}\} \ll d_{\widetilde{P}}^{\widetilde{Q}}(g) \ll d_{P_{\mathrm{U}}}^{Q_{\mathrm{U}}}(g)$, and hence the proposition follows from Theorem 4.25.

Similarly to what we have done in the case of general linear groups, we introduce various auxiliary spaces of functions.

Recall that we have the space of functions $T_{F_V^0}(\mathrm{U}_V^0)$ introduced in Subsection 4.9. Because there is a bijection between F_V and F_V^0, we denote it by $T_{F_V}(\mathrm{U}_V^0)$ and use F_V as indices. By Theorem 4.25, the space $T_{F_V}(\mathrm{U}_V^0)$ consists of tuples

$$(\varphi_P)_{P\in F_V} \in \prod_{P\in F_V} T([\mathrm{U}_V^0]_{P^0}),$$

such that

$$\varphi_P - (\varphi_Q)_{P^0} \in S_{d_{P^0}^{Q^0}}([\mathrm{U}_V^0]_{P^0}) \tag{8.2}$$

for any $P \subset Q \in F_V$.

Lemma 8.2. *If $(\varphi_P) \in T_{F_V}(\mathrm{U}_V^0)$ and $\phi \in S(L^-(\mathrm{A}))$, then the family of products*

$$P \mapsto (\varphi_P \cdot \phi_P(\cdot\,;\,\cdot))$$

belongs to $T_{F_V}(\mathrm{U}_V^0)$.

Proof. By Proposition 5.9 and the above characterization (8.2), the family $P \mapsto \phi_P(\cdot\,;\,\cdot)$ belongs to $T_{F_V}(\mathrm{U}_V^0)$. We thus prove a stronger statement: if $(\varphi_P), (\varphi'_P) \in T_{F_V}(\mathrm{U}_V^0)$, then the family

$$P \mapsto (\varphi_P \varphi'_P)$$

belongs to $T_{F_V}(\mathrm{U}_V^0)$. Using the Leibniz rule, we are reduced to that there exists $N_0 > 0$ such that for any $X, Y \in U((\mathfrak{u}_V^0)_{\infty})$ and any $r > 0$, we have

$$\left|\mathrm{R}(X)\varphi_P(g)\mathrm{R}(Y)\varphi'_P(g) - \mathrm{R}(X)\varphi_Q(g)\mathrm{R}(Y)\varphi'_Q(g)\right| \ll_{X,Y,r} \|g\|_{P^0}^{N} d_{P^0}^{Q^0}(g)^{-r}$$

for any $g \in [\mathrm{U}_V^0]_{P^0}$. The left hand side is bounded by

$$\left|\mathrm{R}(X)\varphi_P(g)\right|\left|\mathrm{R}(Y)\varphi'_P(g) - \mathrm{R}(Y)\varphi'_Q(g)\right| + \left|\mathrm{R}(Y)\varphi'_Q(g)\right|\left|(\mathrm{R}(X)\varphi_P(g) - \mathrm{R}(X)\varphi_Q(g))\right|.$$

The desired inequality then follows. ∎

For $P \in F_V$, define a weight function Δ_P on $[\mathrm{U}_V]_{P_{\mathrm{U}}}$ by

$$\Delta_P(g) = \inf_{\gamma\in M_{P^0}(F)N_{P^0}(\mathrm{A})} \|g_1^{-1}\gamma g_2\|_{P^0}.$$

For $P, Q \in F_V$ with $P \subset Q$, define a weight $d_P^{Q,\Delta}$ on $[\mathrm{U}_V]_{P_{\mathrm{U}}}$ by

$$d_P^{Q,\Delta}(g) = \min\{d_{P_1}^{Q_1}(g_1), d_{P_2}^{Q_2}(g_2)\}.$$

Pulling back under the projection $[\widetilde{\mathrm{U}}_V]_{\widetilde{P}} \to [\mathrm{U}_V]_{P_{\mathrm{U}}}$, we get two weights on $[\widetilde{\mathrm{U}}_V]_{\widetilde{P}}$, which are still denoted by Δ_P and $d_P^{Q,\Delta}$.

We define the space $T_{F_V}^{\Delta}(\mathrm{U}_V)$ (resp. $T_{F_V}^{\Delta}(\widetilde{\mathrm{U}}_V,\,\cdot)$) to be the space of functions

$$(\varphi_P) \in \prod_{P\in F_V} S_{\Delta_P}([\mathrm{U}_V]_{P_{\mathrm{U}}}) \quad \text{resp.} \quad \prod_{P\in F_V} S_{\Delta_P}([\widetilde{\mathrm{U}}_V]_{\widetilde{P}},\,\cdot). \tag{8.3}$$

such that for any $P \quad Q \in \mathcal{F}_V$, we have

$$ {}_P\varphi \quad ({}_Q\varphi)_P \in S_{d_P^{Q;}}([\mathrm{U}_V]_{P_\mathrm{U}}) \quad \text{resp.} \quad {}_P\varphi \quad ({}_Q\varphi)_{\tilde{P}} \in S_{d_P^{Q;}}([\widetilde{\mathrm{U}}_V]_{\tilde{P}}; \ ). $$

Since $d_P^{Q;\Delta} \quad d_P^Q$, by Theorem 4.25 and Proposition 8.1 respectively, that a family $({}_P\varphi)$ belongs to $\mathcal{T}_{\mathcal{F}_V}^{\Delta}(\mathrm{U}_V)$ (resp. $\mathcal{T}_{\mathcal{F}_V}^{\Delta}(\widetilde{\mathrm{U}}_V; \ )$) is equivalent to ${}_P\varphi \in S_{\Delta_P}([\mathrm{U}_V]_{P_\mathrm{U}})$ (resp. $S_{\Delta_P}([\widetilde{\mathrm{U}}_V]_{\tilde{P}}; \ )$ for each P and that for all $P \quad Q \in \mathcal{F}_V$, there exists an $N > 0$ such that for all $X \in \mathcal{U}((\mathfrak{u}_V)_{\infty})$ and all $r > 0$, we have

$$ |\mathrm{R}(X)_P\varphi(g) \quad \mathrm{R}(X)_Q\varphi(g)| \quad {}_{r;X} \|g\|_P^N d_P^{Q;\Delta}(g) \ ^r. \tag{8.4} $$

holds for all $g \in [\mathrm{U}_V]_{P_\mathrm{U}}$ (resp. $[\widetilde{\mathrm{U}}_V]_{\tilde{P}}$).

Lemma 8.3. *We have the following assertions.*

1. *For $({}_P\varphi) \in \mathcal{T}_{\mathcal{F}_V}^{\Delta}(\widetilde{\mathrm{U}}_V; \ )$, then the family of restrictions $P \mapsto ({}_P\varphi|_{[\mathrm{U}_V]_{P_\mathrm{U}}})$ belong to $\mathcal{T}_{\mathcal{F}_V}^{\Delta}(\mathrm{U}_V)$.*
2. *For $({}_P\varphi) \in \mathcal{T}_{\mathcal{F}_V}^{\Delta}(\mathrm{U}_V)$, then the family of restrictions $P \to ({}_P\varphi|_{[\mathrm{U}_V^0]_{P^0}})$ belongs to $\mathcal{T}_{\mathcal{F}_V}(\mathrm{U}_V^0)$.*

Proof. The first follows from the characterizations (8.4) of elements in $\mathcal{T}_{\mathcal{F}_V}^{\Delta}(\mathrm{U}_V)$ and $\mathcal{T}_{\mathcal{F}_V}^{\Delta}(\widetilde{\mathrm{U}}_V; \ )$. The second follows from the additional fact that $d_P^{Q;\Delta}|_{[\mathrm{U}_V^0]_{P^0}} = d_{P^0}^{Q^0}$.

We fix a $\ _0 \in \mathcal{S}(L^-(\mathbb{A}))$ such that $\| \ _0\|_{L^2} = 1$. For $f \in \mathcal{S}(\mathrm{U}_V(\mathbb{A}))$ and $\ _1 \in \mathcal{S}(L^-(\mathbb{A}))$, we define a function $\tilde{f} \in \mathcal{S}(\widetilde{\mathrm{U}}_V(\mathbb{A}); \ )$ by

$$ \tilde{f}(gs) = f(g^{\ 1})\langle \omega(s)\ _0, \overline{\ _1}\rangle, \quad s \in S(V)(\mathbb{A}), \ g \in \mathrm{U}_V(\mathbb{A}). \tag{8.5} $$

For $\varphi \in \mathcal{T}^0([\mathrm{U}_V]_{P_V})$, as in the case of general linear groups, we define a measure $\varphi \quad \overline{{}_P(\ ; \ _0)} \in \mathcal{T}^0([\widetilde{\mathrm{U}}_V]_{\tilde{P}}; \ )$ by

$$ \langle \varphi \quad \overline{{}_P(\ ; \ _0)}, \ \rangle = \int_{[\mathrm{U}_V]_{P_\mathrm{U}}} \int_{[S(V)]_{P_S}} (sg)\overline{{}_P(sg; \ _0)}\,\mathrm{d}s\,\varphi(g), \tag{8.6} $$

where $\ \in C_c([\widetilde{\mathrm{U}}_V]_{\tilde{P}})$.

Lemma 8.4. *Let $\varphi \in \mathcal{T}^0([\mathrm{U}_V^0])$, then the family*

$$ P \in \mathcal{F}_V \mapsto \mathrm{R}(\tilde{f}) \quad \varphi_{P^0} \quad \overline{{}_P(\ ; \ _0)} $$

belongs to $\mathcal{T}_{\mathcal{F}_V}^{\Delta}(\widetilde{\mathrm{U}}_V; \ ^{1})$. Moreover if $y \in [\mathrm{U}_V]_{P_\mathrm{U}}$ we have

$$ \mathrm{R}(\tilde{f})(\varphi \quad \overline{{}_P(\ ; \ _0)})(y) = \int_{[\mathrm{U}_V^0]_{P^0}} K_{f;P}(x;y) \ ^-(x; \ _1)\varphi(x). $$

In particular the composition of this map followed by the restriction to $[\mathrm{U}_V]_{P_\mathrm{U}}$ is independent from the choice of $\ _0$.

Proof. This is proved in the same way as in the case of general linear groups, and in particular Lemma 6.3, Lemma 6.5 and Lemma 6.7.

8.3. **A modified kernel.** Let $\chi \in \mathfrak{X}(\mathrm{U}_V)$ be a cuspidal datum. For $f_+ = f \otimes \phi_1 \otimes \phi_2$ as above, we put

$$K_{f\otimes\phi_1\otimes\phi_2,P,\chi}(\tilde x,\tilde y) = K_{f,P_{\mathrm{U}},\chi}(x,y)\,\theta^{\vee}_{P}(\tilde x_2,\phi_1)\,\theta_{P}(\tilde y_2,\phi_2), \quad \tilde x,\tilde y \in [\widetilde{\mathrm{U}}_V]_{\tilde P}. \tag{8.7}$$

Where we denote x,y to be the image of $\tilde x,\tilde y$ in $[\mathrm{U}_V]_{P_{\mathrm{U}}}$ respectively and $\tilde x_2,\tilde y_2 \in [J(V)]$ are the second component of $\tilde x$ and $\tilde y$. Let us now explain that the function $K_{f\otimes\phi_1\otimes\phi_2,P,\chi}$ extends continuously to a smooth function $K_{f_+,P,\chi}$ for all $f_+ \in \mathcal{S}(\mathrm{U}_{V,+}(\mathbb{A}))$. By Lemma 4.21 and Proposition 5.9, there exists an $N_0 > 0$ such that for any $N > 0$ and $X \in \mathcal{U}(\tilde{\mathfrak{u}}_{V,\infty})$, there is a continuous semi-norm $\|\cdot\|_{\mathcal{S}}$ on $\mathcal{S}(\mathrm{U}_{V,+}(\mathbb{A}))$ such that

$$\sum_{\chi\in\mathfrak{X}(\mathrm{U})} \left|\mathrm{R}_{\tilde y}(X)K_{f_+,P,\chi}(\tilde x,\tilde y)\right| \ll \|f_+\|_{\mathcal{S}}\|\tilde x\|_{\tilde P}^{-N}\|\tilde y\|_{\tilde P}^{N+N_0}. \tag{8.8}$$

holds for all $f_+ \in \mathcal{S}(\mathrm{U}_V(\mathbb{A})) \otimes \mathcal{S}(L^{\vee}(\mathbb{A})) \otimes \mathcal{S}(L^{\vee}(\mathbb{A}))$ (algebraic tensor product). Now let $f_+ \in \mathcal{S}(\mathrm{U}_{V,+}(\mathbb{A}))$ and $f_{+,n} \in \mathcal{S}(\mathrm{U}_V(\mathbb{A})) \otimes \mathcal{S}(L^{\vee}(\mathbb{A})) \otimes \mathcal{S}(L^{\vee}(\mathbb{A}))$ a sequence of functions approaching f_+. Because of the estimate (8.8), the sequence $K_{f_{+,n},P,\chi}$ is convergent to a function on $[\widetilde{\mathrm{U}}_V]_{\tilde P} \times [\widetilde{\mathrm{U}}_V]_{\tilde P}$, and this convergence is locally uniform for $(\tilde x,\tilde y) \in [\widetilde{\mathrm{U}}_V]_{\tilde P} \times [\widetilde{\mathrm{U}}_V]_{\tilde P}$. We denote this function by $K_{f_+,P,\chi}$. It is clearly independent of the choice of the sequence approximating f_+. Because the convergence is locally uniform, $K_{f_+,P,\chi}$ is a smooth function. Moreover the estimate (8.8) continues to hold for $K_{f_+,P,\chi}$. By the symmetry of $\tilde x$ and $\tilde y$, the estimate (8.8) holds when $\tilde x$ and $\tilde y$ on the right hand side are switched (and with a possibly different $\|\cdot\|_{\mathcal{S}}$).

For $T \in \mathfrak{a}_0$, $\chi \in \mathfrak{X}(\mathrm{U}_V)$, and $x,y \in [\widetilde{\mathrm{U}}_V]$, we define modified kernels

$$K^T_{f_+,\chi}(x,y) = \sum_{P\in\mathcal{F}_V} \epsilon_P \sum_{\substack{\gamma\in P'(F)\backslash \mathrm{U}'_V(F)\\ \delta\in P'(F)\backslash \mathrm{U}'_V(F)}} \hat\tau_{P'}(H_{P'}(\delta y) - T_{P'})K_{f_+,P,\chi}(\gamma x,\delta y).$$

and

$$K^T_{f_+}(x,y) = \sum_{P\in\mathcal{F}_V} \epsilon_P \sum_{\substack{\gamma\in P'(F)\backslash \mathrm{U}'_V(F)\\ \delta\in P'(F)\backslash \mathrm{U}'_V(F)}} \hat\tau_{P'}(H_{P'}(\delta y) - T_{P'})K_{f_+,P}(\gamma x,\delta y).$$

Here $\epsilon_P = (-1)^{\dim \mathfrak{a}_{P'}}$ and $\hat\tau_{P'}$ is the characteristic function of a certain cone in $\mathfrak{a}_{P'}$ defined in Subsection 4.5. In these definitions, the convergence of the inner sum can be seen as follows. For fixed x and y, there are only finitely many $\delta \in P'(F)\backslash \mathrm{U}'_V(F)$ (depending on y) such that $\hat\tau_{P'}(H_{P'}(\delta y) - T_{P'}) \neq 0$, cf. [Art78, Lemma 5.1]. Hence the estimate (8.8) implies that the summations defining $K^T_{f_+}$ and $K^T_{f_+,\chi}$ are absolutely convergent.

Proposition 8.5. *For every $N > 0$, there exists a continuous semi-norm $\|\cdot\|_{\mathcal{S},N}$ on $\mathcal{S}(\mathrm{U}_{V,+}(\mathbb{A}))$ such that for every $f_+ \in \mathcal{S}(\mathrm{U}_{V,+}(\mathbb{A}))$ and T sufficiently positive, we have*

$$\sum_{\chi\in\mathfrak{X}(\mathrm{U}_V)} \left|K^T_{f_+,\chi}(x,y) - K_{f_+,\chi}(x,y)F^{\mathrm{U}'_V}(x,T)\right| \ll e^{-N\|T\|}\|x\|_{\mathrm{U}'_V}^{-N}\|y\|_{\mathrm{U}'_V}^{-N}\|f_+\|_{\mathcal{S},N}. \tag{8.9}$$

In particular for $f_+ \in \mathcal{S}(\mathrm{U}_{V,+}(\mathbb{A}))$ and T sufficiently positive, the expression

$$\sum_{\chi \in \mathfrak{X}(\mathrm{U}_V)} \int_{[\mathrm{U}_V^0]} \int_{[\mathrm{U}_V^0]} \left| K^T_{f_+,\chi}(x,y) \right| \mathrm{d}x \mathrm{d}y$$

is finite and defines a continuous semi-norm on $\mathcal{S}(\mathrm{U}_{V,+}(\mathbb{A}))$.

Proof. First, since the center of U_V^0 is anisotropic, for a fixed T the function $x \mapsto F^{\mathrm{U}_V^0}(x,T)$ is compactly supported. Therefore the second assertion on the absolute convergence follows from the estimate (8.9).

Next we note that each summand in (8.9) is continuous in f_+. Thus by continuity we only need to prove (8.9) when $f_+ = f \otimes \phi_1 \otimes \phi_2$ where $f \in \mathcal{S}(\mathrm{U}_V(\mathbb{A}))$, $\phi_1, \phi_2 \in \mathcal{S}(L^-(\mathbb{A}))$. We will assume that this is the case from now on.

We fix a $\phi_0 \in \mathcal{S}(L^-(\mathbb{A}))$ with $\|\phi_0\|_{L^2} = 1$. Consider the following sequence of map

$$\mathcal{T}^0([\mathrm{U}_V^0]) \longrightarrow \mathcal{T}^{\Delta}_{\mathcal{F}_V}(\widetilde{\mathrm{U}}_V, \phi_1) \xrightarrow{j_{\mathrm{U}_V}} \mathcal{T}^{\Delta}_{\mathcal{F}_V}(\mathrm{U}_V) \xrightarrow{j_{\mathrm{U}_V^0}} \mathcal{T}_{\mathcal{F}_V}(\mathrm{U}_V^0) \longrightarrow \mathcal{T}_{\mathcal{F}_V}(\mathrm{U}_V^0) \underset{\Pi^T}{\overset{\Lambda^T}{\rightrightarrows}} \mathcal{S}^0([\mathrm{U}_V^0]),$$

where the first map sends $\varphi \in \mathcal{T}^0([\mathrm{U}_V^0])$ to the family

$$P \mapsto \mathrm{R}(\widetilde{f}) \, \varphi_{P^0} \, \overline{P(\cdot, \phi_0)} .$$

We recall that $\widetilde{f}$ is defined in (8.5). The second and the third maps are to restrict the family $(\varphi_P)_{P \in \mathcal{F}_V}$ to $\left(\varphi_P|_{[\mathrm{U}_V]_{P_{\mathrm{U}}}}\right)_{P \in \mathcal{F}_V}$ and then further to $\left(\varphi_P|_{[\mathrm{U}_V^0]_{P^0}}\right)_{P \in \mathcal{F}_V}$. The fourth map sends a family $(\varphi_{P^0})_{P \in \mathcal{F}_V}$ to the family $(\varphi_P \, \xi_P(\cdot, \phi_2))_{P \in \mathcal{F}_V}$. By Lemma 8.4, Lemma 8.3 and Lemma 8.2, the targets of these maps are as described. Using the closed graph theorem, one checks that all these maps are continuous, cf. Remark 6.9. The last map is one of the truncation operators defined in Proposition 4.27. We denote by L_{f,ϕ_1,ϕ_2} (resp. P_{f,ϕ_1,ϕ_2}) the composite of the sequence of the maps above, where we use Λ^T (resp. Π^T) in the last step.

Similarly, for $\chi \in \mathfrak{X}(\mathrm{U}_V)$, we consider the same chain of continuous linear maps as above, but we project to the χ-component before restricting to U_V^0. The resulting maps are denoted by $L_{f,\phi_1,\phi_2,\chi}$ and $P_{f,\phi_1,\phi_2,\chi}$ respectively.

As in the proof of Theorem 6.8, the functions $K^T_{f \otimes \phi_1 \otimes \phi_2}(x,y)$ and $K_{f \otimes \phi_1 \otimes \phi_2}(x,y) F^{\mathrm{U}_V^0}(x,T)$ are the kernel functions of L_{f,ϕ_1,ϕ_2} (resp. P_{f,ϕ_1,ϕ_2}) respectively. The same is true with a $\chi \in \mathfrak{X}(\mathrm{U}_V)$ in the subscript. By Proposition 4.27, for any fixed N and for all $\varphi \in \mathcal{T}^0([\mathrm{U}_V^0])$, we have

$$\sum_{\chi \in \mathfrak{X}(U)} \| L_{f,\phi_1,\phi_2,\chi}(\varphi) - P_{f,\phi_1,\phi_2,\chi}(\varphi) \| \ll e^{-N\|T\|} \|\varphi\|_{1,-N}.$$

The rest of the argument is to show that the implicit constant in this estimate can be taken to be continuous seminorms of f, ϕ_1 and ϕ_2. This is a consequence of the uniform boundedness principle, and the argument is the same as the proof of Theorem 6.8.

8.4. **The coarse spectral expansion.** For $\chi \in \mathfrak{X}(\mathrm{U}_V)$, $f_+ \in \mathcal{S}(\mathrm{U}_{V,+}(\mathbb{A}))$ and $T \in \mathfrak{a}_0$ a truncation parameter, we put

$$J^T(f_+) = \int_{[\mathrm{U}'_V]}\int_{[\mathrm{U}'_V]} K^T_{f_+}(x,y)\mathrm{d}x\mathrm{d}y, \quad J^T_\chi(f_+) = \int_{[\mathrm{U}'_V]}\int_{[\mathrm{U}'_V]} K^T_{f_+,\chi}(x,y)\mathrm{d}x\mathrm{d}y.$$

By Proposition 8.5, these integrals are absolutely convergent when T is sufficiently positive.

We define an action L_+ of $\mathrm{U}'_V(\mathbb{A})$ on $\mathcal{S}(\mathrm{U}_{V,+}(\mathbb{A}))$ by

$$\mathrm{L}_+(h)f_+(m,l_1,l_2) = \left(\omega^\vee(h)f_+(h^{-1}m,\cdot,l_2)\right)(l_1), \quad m \in \mathrm{U}_V(\mathbb{A}),\ l_1,l_2 \in L^\vee(\mathbb{A}).$$

The right hand side means that we first evaluate f_+ at $h^{-1}m$ and l_2 to obtain a Schwartz function in the variable l_1. We apply the Weil representation $\omega^\vee(h)$ to this Schwartz function and finally evaluate at l_1. We similarly define an action R_+ of $\mathrm{U}'_V(\mathbb{A})$ on $\mathcal{S}(\mathrm{U}_{V,+}(\mathbb{A}))$ by

$$\mathrm{R}_+(h)f_+(m,l_1,l_2) = \left(\omega(h)f_+(mh,l_1,\cdot)\right)(l_2), \quad m \in \mathrm{U}_V(\mathbb{A}),\ l_1,l_2 \in L^\vee(\mathbb{A}).$$

The right hand side is interpreted similarly as in the case L_+.

Using the action L_+ and R_+, for $P \in \mathcal{F}_V$ and $f_+ \in \mathcal{S}(\mathrm{U}_{V,+}(\mathbb{A}))$, the kernel function $K_{f_+,P}$ can also be written as

$$K_{f_+,P}(x,y) = \sum_{m \in M_{P_+}(F)} \int_{N_{P_+}(\mathbb{A})} \mathrm{L}_+(x)\mathrm{R}_+(y)f_+(mn)\mathrm{d}n \tag{8.10}$$

where $x,y \in [\mathrm{U}'_V]$.

Theorem 8.6. *As a function of T, the functions $J^T(f_+)$ and $J^T_\chi(f_+)$ are the restrictions of exponential polynomials whose purely polynomial term are constants denoted by $J(f_+)$ and $J_\chi(f_+)$ respectively. The linear forms $f_+ \mapsto J(f_+)$ and $f_+ \mapsto J_\chi(f_+)$ are continuous and bi-$\mathrm{U}'_V(\mathbb{A})$-invariant, i.e.*

$$J_\chi(\mathrm{L}_+(h_1)\mathrm{R}_+(h_2)f_+) = J_\chi(f_+), \quad J(\mathrm{L}_+(h_1)\mathrm{R}_+(h_2)f_+) = J(f_+), \quad h_1,h_2 \in \mathrm{U}'_V(\mathbb{A}).$$

Finally we have

$$J(f_+) = \sum_{\chi \in \mathfrak{X}(\mathrm{U}_V)} J_\chi(f_+)$$

where the sum is absolutely convergent.

Before we delve into the proof of this theorem, let us first explain a variant of the construction of the modified kernel for parabolic subgroups. Let us take $Q = M_Q N_Q \in \mathcal{F}_V$. Then $Q_{\mathrm{U}} = Q_1 \times Q_2$ where $Q_1 = Q_2$ are parabolic subgroup of $\mathrm{U}(V)$ and $Q' = Q_1 = Q_2$ is a parabolic subgroup of U'_V. Assume that Q' is the stabilizer of the isotropic flag

$$0 = X_0 \subset X_1 \subset \cdots \subset X_r \tag{8.11}$$

in V. Recall that we constructed a decomposition $V = X \oplus V' \oplus X^\vee$ in Subsection 5.3, where $X = X_r$, $V' = M_{Q_V}$, and V' is perpendicular to $X \oplus X^\vee$. We have a polarizations $\mathrm{Res}\, V = L \oplus L^\vee$

and another polarization $\operatorname{Res} V' = L' \oplus L'^{\vee}$ where $L' = V' \cap L$ and $L'^{\vee} = V' \cap L^{\vee}$. In particular $L = L' \oplus X$ and $L^{\vee} = L'^{\vee} + X^{\vee}$.

The Levi subgroup $M_{Q'}$ is isomorphic to

$$\prod_{i=1}^{r} \mathrm{GL}(X_i/X_{i-1}) \times \mathrm{U}(V_0).$$

We write $m \in M_{Q'}$ as $(m_\flat, m^\sharp)$ where $m_\flat = (m_1, \dots, m_r)$, $m_i \in \mathrm{GL}(X_i/X_{i-1})$, and $m^\sharp \in \mathrm{U}(V_0)$. Recall that in Subsection 8.1, we have defined

$$M_{Q_2,+} = M_{Q_2} \times M_{Q_{L^\vee}} \times M_{Q_{L^\vee}} = \left(\prod_{i=1}^{r} \mathrm{GL}(X_i/X_{i-1}) \times \mathrm{U}(V')\right) \times L'^{\vee} \times L'^{\vee}.$$

We have the D-Levi component $M_{\widetilde{Q}} = M_{Q_1} \times M_Q$ of $\widetilde{Q}$, where

$$M_Q = \prod_{i=1}^{r} \mathrm{GL}(X_i/X_{i-1}) \times J(V')$$

is the D-Levi component of Q. If $\widetilde{m} \in M_Q$, we still denote by $\widetilde{m}^\sharp$ its component in $J(V_0)$.

Take a $P \in \mathcal{F}_V$ with $P \subset Q$. Then $P \cap M_{Q_{\mathrm{U}}}$ is a parabolic subgroup of $M_{Q_{\mathrm{U}}}$. Denote temporarily by $\iota_{Q_{\mathrm{U}}} : \mathfrak{X}(M_{Q_{\mathrm{U}}}) \to \mathfrak{X}(\mathrm{U}_V)$ the natural finite-to-one map. For $\chi \in \mathfrak{X}(\mathrm{U}_V)$ and $f \in \mathcal{S}(M_{Q_{\mathrm{U}}}(\mathbb{A}))$, we put

$$K_{f, P \cap M_{Q_{\mathrm{U}}}, \chi} = \sum_{\chi' \in \iota_{Q_{\mathrm{U}}}^{-1}(\chi)} K_{f, P \cap M_{Q_{\mathrm{U}}}, \chi'}$$

where $K_{f, P \cap M_{Q_{\mathrm{U}}}, \chi'}$ is the kernel function on $M_{Q_{\mathrm{U}}}$. Let $\phi_1, \phi_2 \in \mathcal{S}(L'^{\vee}(\mathbb{A}))$ and $f'_+ = f \otimes \phi_1 \otimes \overline{\phi_2} \in \mathcal{S}(M_{Q,+}(\mathbb{A}))$ be a test function. We define

$$K_{f'_+, P \cap M_Q, \chi}(x,y) = K_{f, P \cap M_{Q_{\mathrm{U}}}, \chi}(x,y)\, \theta^{\vee}_{P \cap J(V')}(x, \phi_1)\, \overline{\theta^{\vee}_{P \cap J(V')}(y, \phi_2)}\, \mu^{-1}(\det x_\flat)\, \mu(\det y_\flat),$$

where $(x,y) \in [M_{Q'}]_{P' \cap M_{Q'}} \times [M_{Q'}]_{P' \cap M_{Q'}}$, and the intersection $P \cap J(V')$ is taken inside $J(V)$. As in Subsection 8.3, we define $K_{f'_+, P \cap M_Q, \chi}$ for all $f'_+ \in \mathcal{S}(M_{Q_+}(\mathbb{A}))$ by continuity.

For $T \in \mathfrak{a}_0$, define

$$K^{M_Q,T}_{f'_+,\chi}(x,y) = \sum_{\substack{P \subset Q \\ P \in \mathcal{F}_V}} \epsilon_P^Q \sum_{\substack{\gamma \in (M_{Q'} \cap P')(F) \backslash M_{Q'}(F) \\ \delta \in (M_{Q'} \cap P')(F) \backslash M_{Q'}(F)}} \widehat{\tau}_{P'}^{Q'}(H_{P'}(\delta y) - T_{P'})\, K_{f'_+, P, \chi}(\gamma x, \delta y).$$

where $(x,y) \in [M_{Q'}] \times [M_{Q'}]$. Using similar methods as the proof of Proposition 8.5, we can show that when T is sufficiently positive, for any $f'_+ \in \mathcal{S}(M_{Q_+}(\mathbb{A}))$, the integral

$$\int_{A^{\infty}_{Q'} \backslash [M_{Q'}] \times [M_{Q'}]} \left| K^{M_Q,T}_{f'_+,\chi}(x,y) \right| \mathrm{d}x \mathrm{d}y$$

is convergent and defines a continuous seminorm on $\mathcal{S}(M_{Q_+}(\mathbb{A}))$. Here $A^{\infty}_{Q'}$ embeds in $M_{Q'} \times M_{Q'}$ diagonally. We thus define a distribution

$$J^{M_Q,T}_\chi(f'_+) = \int_{A^{\infty}_{Q'} \backslash [M_{Q'}] \times [M_{Q'}]} K^{M_Q,T}_{f'_+,\chi}(x,y) \mathrm{d}x \mathrm{d}y.$$

Proof of Theorem 8.6. We define ${}_{Q}$ be the unique element in $\mathfrak{a}_{Q'}$ such that for any $m \in M_{Q'}(\mathbb{A})$ we have

$$e^{\langle {}_{Q}, H_{Q'}(m)\rangle} = |\det m|.$$

One check directly that ${}_{Q}$ concides with the definition of [Zyd20, Lemma 4.3].

By [Art81, Section 2], there exist functions $\Gamma'_{Q'}$ on $\mathfrak{a}_{Q'}^{\mathrm{U}'_V} \ \mathfrak{a}_{Q'}^{\mathrm{U}'_V}$, for $Q \in \mathcal{F}_V$, that are compactly supported in the first variable when the second variable stays in a compact and such that

$$\hat{\tau}_{Q'}(H \quad X) = \sum_{P \quad Q \in \mathcal{F}_V} {}^{\mathrm{U}'_V}_{Q'} \hat{\tau}_{P'}^{Q'}(H)\Gamma'_{Q'}(H, X). \tag{8.12}$$

Define a function p_Q on $\mathfrak{a}_{Q'}$ by

$$p_Q(X) = \int_{\mathfrak{a}_{Q'}} e^{\langle {}_{Q}, H\rangle}\Gamma'_{Q'}(H, X)\mathrm{d}H. \tag{8.13}$$

By [Zyd20, Lemma 4.3] p_Q is an exponential polynomial on $\mathfrak{a}_{Q'}$ with exponents contained in the set $\{{}_{R} \mid R \quad Q\}$ and the pure polynomial term is the constant ${}_{Q}\hat{\theta}_{Q'}({}_{Q})^{\ 1}$ where $\hat{\theta}_{Q'}$ is a homogeneous polynomial on $\mathfrak{a}_{Q'}$ defined in [Art81, Section 2].

For $Q \in \mathcal{F}_V, f_+ \in \mathcal{S}(\mathrm{U}_{V,+}(\mathbb{A}))$ and $T \in \mathfrak{a}_0$, define

$$K_{f_+,}^{Q,T}(x, y) = \sum_{\substack{P \quad Q \\ P \in \mathcal{F}_V}} {}^{Q}_{P} \sum_{\substack{\in P'(F)\setminus Q'(F) \\ \in P'(F)\setminus Q'(F)}} \hat{\tau}_{P'}^{Q'}(H_{P'}(\ y) \quad T_{P'})K_{f_+,P,}(\ x,\ y),$$

where $(x, y) \in [\mathrm{U}'_V]_{P'} \quad [\mathrm{U}'_V]_{P'}$. Here ${}^{Q}_{P} = (\ 1)^{\dim \mathfrak{a}_{P'}^{Q'}}$ and $\hat{\tau}_{P'}^{Q'}$ is the characteristic function defined in Subsection 4.5. Using the inversion formula (8.12), for $T, T' \in \mathfrak{a}_0$ we have

$$K_{f_+,}^{T}(x, y) = \sum_{Q \in \mathcal{F}_V} \sum_{\substack{\in Q'(F)\setminus \mathrm{U}'(F) \\ \in Q'(F)\setminus \mathrm{U}'(F)}} \Gamma'_{Q'}(H_{Q'}(\ y) \quad T'_{Q'}, T_{Q'} \quad T'_{Q'})K_{f_+,}^{Q,T'}(\ x,\ y).$$

We now relate $K_{f_+,}^{Q,T'}$ to $K_{f_+,}^{M_Q,T'}$ via parabolic descent. For $f_+ \in \mathcal{S}(\mathrm{U}_{V,+}(\mathbb{A}))$, we define its parabolic descent as

$$f_{+,Q}(m, l_1, l_2) = e^{\langle \ 2 \ {}_{Q'}+\frac{1}{2}{}_{Q}, H_{Q'}(m_1)\rangle} \int_{K'}\int_{K'}\int_{N_{Q_{\mathrm{U}}}(\mathbb{A})} (\mathrm{L}_+(k_1)\mathrm{R}_+(k_2)f_+)(mn_Q, l_1, l_2)\mathrm{d}n_Q\mathrm{d}k_1\mathrm{d}k_2,$$

where $m \in M_{Q_{\mathrm{U}}}(\mathbb{A})$, $l_1, l_2 \in L'(\mathbb{A})$. The element m_1 stands for the first component of m and we regard it as an element of $M_{Q'}$ under the natural identification $M_{Q_1} = M_{Q'}$. We have $f_{+,Q} \in \mathcal{S}(M_{Q_+}(\mathbb{A}))$, and for $x, y \in [M_{Q'}]$ and $P \quad Q \in \mathcal{F}_V$, we have

$$\int_{K'}\int_{K'} K_{f_+,P,}(xk_1, yk_2)\mathrm{d}k_1\mathrm{d}k_2 = e^{\langle 2 \ {}_{Q'}, H_{Q'}(x)\rangle}e^{\langle 2 \ {}_{Q'}+{}_{Q}, H_{Q'}(y)\rangle}K_{f_{+,Q},P\cap M_Q,}(x, y).$$

Indeed using (8.10) and the mixed model described in Subsection 5.4, direct calculations give the identity without . The argument in [Zyd20, Lemma 1.3] shows that this implies the identity with the . It follows that

$$\int_{K'}\int_{K'} K_{f_+,}^{T}(xk_1, yk_2)\mathrm{d}k_1\mathrm{d}k_2 = e^{\langle 2 \ {}_{Q'}, H_{Q'}(x)\rangle}e^{\langle 2 \ {}_{Q'}+{}_{Q}, H_{Q'}(y)\rangle}K_{f_{+,Q},}^{M_Q,T}(x, y).$$

The rest of the calculation is the same as that in the proof of Theorem 6.10. We omit the details and only record the final outcome. We have

$$J^T(f_+) = \sum_{Q \in \mathcal{F}_V} e^{\langle -\underline{\lambda}_{Q'}, T' \rangle} p_Q(T_{Q'} - T^0_{Q'}) J^{M_Q, T'}(f_{+,Q}).$$

It is an exponential polynomial in T whose purely polynomial term is a constant that equals

$$\sum_{Q \in \mathcal{F}_V} \epsilon_Q \widehat{\theta}_{Q'}(\underline{\lambda}_Q)^{-1} e^{\langle -\underline{\lambda}_{Q'}, T' \rangle} p_Q(T_{Q'} - T^0_{Q'}) J^{M_Q, T'}(f_{+,Q}).$$

The invariance of the linear form J is shown as that of Proposition 6.12. This concludes the proof.

8.5. **An alternative truncation operator.** For later use we will need another truncation operator for Fourier–Jacobi periods. It is an analogue of the regularized periods of Ichino and Yamana [IY19].

Take $P \in \mathcal{F}_V$. Then $\mathrm{U}(V) \times P$ is a D-parabolic subgroup of $\widetilde{\mathrm{U}}_V$. For $\varphi \in \mathcal{T}([\widetilde{\mathrm{U}}_V], \chi)$, we can speak of the constant term $\varphi_{\mathrm{U}(V) \times P}$. This amounts to viewing the function φ as a function in two variables $(x_1, \widetilde{x}_2) \in [\mathrm{U}(V)] \times [J(V)]$, fixing x_1, and taking the constant term along P in the second variable $\widetilde{x}_2$.

For $\varphi \in \mathcal{T}([\widetilde{\mathrm{U}}_V], \chi)$ and $x \in [\mathrm{U}'_V]$, we define

$$\Lambda_u^T \varphi(x) = \sum_{P \in \mathcal{F}_V} \epsilon_P \sum_{\delta \in P'(F) \backslash \mathrm{U}'_V(F)} \widehat{\tau}_{P'}(H_{P'}(\delta x) - T_{P'}) \varphi_{\mathrm{U}(V) \times P}(\delta x). \tag{8.14}$$

Proposition 8.7. *We have the following assertions.*

(1) $\Lambda_u^T \varphi \in \mathcal{S}^0([\mathrm{U}'_V])$. *And the map* $\varphi \mapsto \Lambda_u^T \varphi$ *induces a continuous map* $\mathcal{T}([\widetilde{\mathrm{U}}_V], \chi) \to \mathcal{S}^0([\mathrm{U}'_V])$.

(2) *For every* $N > 0$, *there exists a continious seminorm* $\| \cdot \|_N$ *on* $\mathcal{T}([\widetilde{\mathrm{U}}_V], \chi)$ *such that for all* $x \in [\mathrm{U}'_V]$ *and* $\varphi \in \mathcal{T}([\widetilde{\mathrm{U}}_V], \chi)$, *we have*

$$\left| \Lambda_u^T \varphi(x) - F^{\mathrm{U}'_V}(x, T)\varphi(x) \right| \ll e^{-N\|T\|} \|x\|_{\mathrm{U}'_V}^{-N} \|\varphi\|_N.$$

Proof. First note $d^{\mathrm{U}(V) \times Q}_{\mathrm{U}(V) \times P}|_{\mathrm{U}'_V} = d^Q_P$. Thus by Theorem 4.25 and Lemma 5.8, there exists an $N_0 > 0$ such that for any $P \subset Q \in \mathcal{F}_V$ and $X \in \mathcal{U}((\mathfrak{u}'_V)_\infty)$, we have

$$|\mathrm{R}(X)\varphi_{\mathrm{U}(V) \times P}(x) - \mathrm{R}(X)\varphi_{\mathrm{U}(V) \times Q}(x)| \ll d^{Q'}_{P'}(x)^{-r} \|x\|^{N_0}_{P'}. \tag{8.15}$$

By (8.2), this is equivalent to the fact that the family $P \mapsto \varphi_{\mathrm{U}(V) \times P}|_{[\mathrm{U}'_V]_{P'}}$ belongs to $\mathcal{T}_{\mathcal{F}'_V}(\mathrm{U}'_V)$. The second assertion follows from this by Proposition 4.27. The first assertion follows from the second since the function $x \mapsto F^{\mathrm{U}'_V}(x, T)$ is compactly supported. ∎

Recall that for $f_+ \in \mathcal{S}(\mathrm{U}_{V,+}(\mathbb{A}))$ and $\chi \in \mathfrak{X}(\mathrm{U}_V)$, we have a kernel function $K_{f_+,\chi}(\widetilde{x}, \widetilde{y})$ for $\widetilde{x}, \widetilde{y} \in [\widetilde{\mathrm{U}}_V]$ (cf. (8.7) for the definition for pure tensors). Applying Λ_u^T to the second variable, we get a function on $[\widetilde{\mathrm{U}}_V] \times [\mathrm{U}'_V]$ denoted by $K_{f_+,\chi}\Lambda_u^T$.

Proposition 8.8. *We have the following assertions.*

(1) *For $f_+ \in \mathcal{S}(\mathrm{U}_{V,+}(\mathbb{A}))$ and T sufficiently positive, the expression*

$$\sum_{\chi\in\mathfrak{X}(\mathrm{U}_V)} \int_{[\mathrm{U}'_V]}\int_{[\mathrm{U}'_V]} \left|K_{f_+,\chi}\Lambda^T_u(x,y)\right| \mathrm{d}x\mathrm{d}y$$

is finite and defines a continuous seminorm on $\mathcal{S}(\mathrm{U}_{V,+}(\mathbb{A}))$.

(2) *For any $r > 0$, there exists a continuous seminorm $\|\cdot\|$ on $\mathcal{S}(\mathrm{U}_{V,+}(\mathbb{A}))$ such that*

$$\left|J^T(f_+) - \int_{[\mathrm{U}'_V]}\int_{[\mathrm{U}'_V]} K_{f_+,\chi}\Lambda^T_u(x,y)\mathrm{d}x\mathrm{d}y\right| \ll e^{-r\|T\|}\|f_+\|.$$

for all T sufficiently positive and $f_+ \in \mathcal{S}(\mathrm{U}_{V,+}(\mathbb{A}))$. In particular, the absolutely convergent integral

$$\int_{[\mathrm{U}'_V]}\int_{[\mathrm{U}'_V]} K_{f_+,\chi}\Lambda^T_u(x,y)\mathrm{d}x\mathrm{d}y$$

is asymptotic to an exponential-polynomial in T, whose purely polynomial term is a constant that equals $J_\chi(f_+)$.

Proof. By Proposition 8.7, for every $N > 0$ there exists a continuous semi-norm $\|\cdot\|_T$ on $\mathcal{T}_N([\widetilde{\mathrm{U}}'_V], \ldots)$ such for all $x, y \in [\mathrm{U}'_V]$, we have

$$\sum_{\chi\in\mathfrak{X}(\mathrm{U}_V)} \left|K_{f_+,\chi}\Lambda^T_u(x,y) - F^{\mathrm{U}'_V}(y,T)K_{f_+,\chi}(x,y)\right| \ll e^{-N\|T\|}\|y\|^N_{\mathrm{U}'_V} \sum_{\chi\in\mathfrak{X}(\mathrm{U}_V)} \|K_{f_+,\chi}(x,\cdot)\|_T,$$

where $K_{f_+,\chi}(x,\cdot)$ is regarded as an element in $\mathcal{T}_N([\widetilde{\mathrm{U}}_V], \ldots)$. By (8.8), there exists $N_0 > 0$ and a continuous semi-norm $\|\cdot\|_{\mathcal{S}}$ on $\mathcal{S}(\mathrm{U}_{V,+}(\mathbb{A}))$ such that

$$\sum_{\chi\in\mathfrak{X}(\mathrm{U}_V)} \|K_{f_+,\chi}(x,\cdot)\|_T \ll \|f_+\|_{\mathcal{S}}\|x\|^{N+N_0}_{\mathrm{U}'_V}.$$

Combining the above two equations we obtain

$$\sum_{\chi\in\mathfrak{X}(\mathrm{U}_V)} \left|K_{f_+,\chi}\Lambda^T_u(x,y) - F^{\mathrm{U}'_V}(y,T)K_{f_+,\chi}(x,y)\right| \ll e^{-N\|T\|}\|f_+\|_{\mathcal{S}}\|x\|^{N+N_0}_{\mathrm{U}'_V}\|y\|^N_{\mathrm{U}'_V}.$$

This proves the first assertion. The second assertion follows from the first and Proposition 8.5.

9. The coarse geometric expansion: unitary groups

9.1. Geometric modified kernels. We keep the notation from the previous section. We also need the following additional notation.

We put $\mathrm{U}^+_V = \mathrm{U}_V \times V$. The group structure is given by the product of those of U_V and of the additive group V. Note that this is not a subgroup of $\widetilde{\mathrm{U}}_V$ but merely a closed subvariety. The group $\mathrm{U}'_V \times \mathrm{U}'_V$ acts on U^+_V from the right by $(g,v)\cdot(x,y) = (x^{-1}gy, y^{-1}v)$.

Let $P \in \mathcal{F}_V$, we put

$$P^+ = P_{\mathrm{U}} \times P_V, \quad M^+_P = M_{P_{\mathrm{U}}} \times M_{P_V}, \quad N^+_P = N_{P_{\mathrm{U}}} \times N_{P_V}.$$

These are subgroups of U^+_V.

By [CZ21, Lemma 15.1.4.1] the categorical quotient $\mathrm{U}_V^+//(\mathrm{U}_V'\times\mathrm{U}_V')$ is canonically identified with $A=G^+//(H\times G')$. The canonical morphism $q_V:\mathrm{U}_V^+\to A$ is given by

$$((g_1,g_2),v)\mapsto(a_1,\dots,a_n;b_1,\dots,b_n)$$

where

$$a_i=\mathrm{Trace}\wedge^i(g_1^{-1}g_2),\quad b_i=2(-1)^{n-1}\tau^{-1}q_V(g_1^{-1}g_2v,v).$$

Here A is viewed as a locally closed subscheme of $\mathrm{Res}_{E/F}\,\mathbf{A}_{2n,E}$ as in Subsection 7.1.

For $\xi\in A(F)$, let $\mathrm{U}_{V,\xi}^+$ be the preimage of ξ in U_V^+ as a closed subscheme. We also put

$$M_{P,\xi}^+=\mathrm{U}_{V,\xi}^+\cap M_P^+.$$

For $f^+\in\mathcal{S}(\mathrm{U}_V^+(\mathbb{A}))$ and $P\in\mathcal{F}_V$, we define

$$k_{f^+,P}(x,y)=\sum_{m^+\in M_P^+(F)}\int_{N_P^+(\mathbb{A})}f^+(m^+n^+\cdot(x,y))\mathrm{d}n^+,\quad x,y\in[\mathrm{U}_V']_{P'}.$$

For $\xi\in A(F)$, we define similarly

$$k_{f^+,P,\xi}(x,y)=\sum_{m^+\in M_{P,\xi}^+(F)}\int_{N_P^+(\mathbb{A})}f^+(m^+n^+\cdot(x,y))\mathrm{d}n^+.$$

Lemma 9.1. *There are an integer N and a semi-norm $\|\cdot\|_{\mathcal{S}}$ on $\mathcal{S}(\mathrm{U}_V^+(\mathbb{A}))$ such that for all*

$$\sum_{m\in M_P^+(F)}\int_{N_P^+(\mathbb{A})}\left|f^+(mn\cdot(x,y))\right|\mathrm{d}n\ll\|f^+\|_{\mathcal{S}}\|x\|^N_{\mathrm{U}_V'}\|y\|^N_{\mathrm{U}_V'}.$$

In particular the defining expressions of $k_{f^+,P,\xi}$ and $k_{f^+,P}$ are absolutely convergent and we have

$$\sum_{\xi\in A(F)}k_{f^+,P,\xi}(x,y)=k_{f^+,P}(x,y).$$

Proof. The proof is the same as Lemma 7.1.

Similar to what we have done in Subsection 7.1, we identify $\mathrm{Res}_{E/F}\,\mathbf{A}_{2n,E}$ with the affine space $\mathbf{A}^{4n}$ over F, and denote again by q_V the morphism $\mathrm{U}_V^+\to\mathbf{A}^{4n}$. We extend the definition of $k_{f^+,P,\xi}$ to all $\xi\in F^{4n}$ by setting $k_{f^+,P,\xi}=0$ if $\xi\notin A(F)$. There is a $d\in F^\times$ such that if $k_{f^+,P,\xi}$ is not identically zero for some $\xi\in F^{4n}$ and $P\in\mathcal{F}_V$, then $\xi\in(d\mathcal{O}_F)^{4n}$. Put $\Lambda=(d\mathcal{O}_F)^{4n}\cap F^{4n}$. For each $\xi\in\Lambda$, take $u_\xi\in C_c^\infty(F_\infty^{4n})$ and define the function $f_\xi^+=f^+u_\xi$ as in Subsection 4.10 .

Lemma 9.2. *We have*

$$k_{f_\xi^+,P}(x,y)=k_{f_\xi^+,P,\xi}(x,y)=k_{f^+,P,\xi}(x,y)$$

for all $x,y\in[\mathrm{U}_V']_{P'}$.

The proof is the same as Lemma 7.2.

We now relate the kernel function $k_{f^+,P}$ and the kernel function $K_{f_+,P}$ defined in Subsection 8.3. We first define a partial Fourier transform, cf. [Li92, Section 2]. For $\phi_1, \phi_2 \in \mathcal{S}(L^-(\mathbb{A}))$ we define a partial Fourier transform

$$\mathcal{S}(L^-(\mathbb{A})) \otimes \mathcal{S}(L^-(\mathbb{A})) \to \mathcal{S}(V(\mathbb{A})), \quad \phi_1 \otimes \phi_2 \mapsto (\phi_1 \otimes \phi_2)^{\ddagger}, \tag{9.1}$$

by

$$(\phi_1 \otimes \phi_2)^{\ddagger}(v) = \int_{L^-(\mathbb{A})} \phi_1(x+l')\phi_2(x-l')\psi(2\,\mathrm{Tr}_{E/F}\, q_V(x,l))\mathrm{d}x, \tag{9.2}$$

where we write $v = l + l'$ where $l \in L(\mathbb{A})$ and $L^-(\mathbb{A})$. In particular we have

$$(\phi_1 \otimes \phi_2)^{\ddagger}(0) = \langle \phi_1, \overline{\phi_2} \rangle_{L^2}$$

where $\langle\ ,\ \rangle_{L^2}$ stands for the L^2-inner product on $L^-(\mathbb{A})$. We also have

$$(\omega^-(g)\phi_1 \otimes \omega(g)\phi_2)^{\ddagger}(v) = (\phi_1 \otimes \phi_2)^{\ddagger}(g^{-1}v)$$

for $g \in \mathrm{U}(V)(\mathbb{A})$ and $v \in V(\mathbb{A})$.

The partial Fourier transform (9.1) extends to a continuous isomorphism

$$\mathcal{S}(\mathrm{U}_{V,+}(\mathbb{A})) \to \mathcal{S}(\mathrm{U}_V^+(\mathbb{A})),$$

which we still denote by $\ddagger$. Let $f^+ \mapsto f^+_{\ddagger}$ be its inverse. Concretely for $f_+ \in \mathcal{S}(\mathrm{U}_{V,+}(\mathbb{A}))$, we have

$$f_+^{\ddagger}((g_1,g_2),v) = \int_{L^-} \omega^-_{(1)}(g_1) f_+((g_1,g_2), x+l', x-l')\psi(2\,\mathrm{Tr}_{E/F}\, q_V(x,l))\mathrm{d}x, \tag{9.3}$$

where we write $v = l + l'$ where $l \in L(\mathbb{A})$ and $L^-(\mathbb{A})$. The notation is interpreted as follows. We first evaluate f_+ at (g_1, g_2) to obtain a Schwartz function on $L^-(\mathbb{A}) \times L^-(\mathbb{A})$. Then the Weil representation $\omega^-(g_1)$ acts on this Schwartz function on the first variable (this is what the subscript (1) indicates). Finally we take the partial Fourier transform.

Lemma 9.3. *For all $x, y \in [\mathrm{U}_V']_{P'}$, we have*

$$K_{f_+,P}(x,y) = k_{f_+^{\dagger},P}(x,y).$$

The proof is the same as Lemma 7.3.

Recall that we have defined in Subsection 8.4 the actions L_+ and R_+ of $\mathrm{U}_V'(\mathbb{A})$ on $\mathcal{S}(\mathrm{U}_{V,+}(\mathbb{A}))$. We also have the left and right translation of $\mathrm{U}_V'(\mathbb{A})$ on $\mathcal{S}(\mathrm{U}_V^+(\mathbb{A}))$ which we denote by L^+ and R^+ respectively. More precisely we have

$$\mathrm{L}^+(h)f^+(g,v) = f^+(h^{-1}g, v), \quad \mathrm{R}^+(h)f^+(g,v) = f^+(gh, h^{-1}v),$$

for $h \in \mathrm{U}_V'(\mathbb{A})$.

Lemma 9.4. *Let $f^+ \in \mathcal{S}(G^+(\mathbb{A}))$ and $x, y \in \mathrm{U}_V'(\mathbb{A})$. We have*

$$(\mathrm{L}^+(x)\mathrm{R}^+(y)f^+)_{\dagger} = \mathrm{L}_+(x)\mathrm{R}_+(y)(f^+_{\dagger}).$$

Proof. This is a direct computation.

For $T \in \mathfrak{a}_0$ and $x, y \in [\mathrm{U}'_V]$, we define the modified kernel
$$k^T_{f^+}(x,y) = \sum_{P \in \mathcal{F}_V} \epsilon_P \sum_{\gamma \in P'(F)\backslash \mathrm{U}'_V(F)} \sum_{\delta \in P'(F)\backslash \mathrm{U}'_V(F)} \hat{\tau}_{P'}(H_{P'}(\delta y) - T_{P'}) k_{f^+,P}(\gamma x, \delta y);$$
and for $\alpha \in \mathcal{A}(F)$ we put
$$k^T_{f^+,\alpha}(x,y) = \sum_{P \in \mathcal{F}_V} \epsilon_P \sum_{\gamma \in P'(F)\backslash \mathrm{U}'_V(F)} \sum_{\delta \in P'(F)\backslash \mathrm{U}'_V(F)} \hat{\tau}_{P'}(H_{P'}(\delta y) - T_{P'}) k_{f^+,\alpha,P}(\gamma x, \delta y).$$

Lemma 9.5. *For T sufficiently positive, the integral*
$$\int_{[\mathrm{U}'_V]\times[\mathrm{U}'_V]} |k^T_{f^+}(x,y)| \, \mathrm{d}x\mathrm{d}y$$
is absolutely convergent and defines a continuous seminorm on $\mathcal{S}(\mathrm{U}^+_V(\mathbb{A}))$. Put
$$j^T(f^+) := \int_{[\mathrm{U}'_V]\times[\mathrm{U}'_V]} k^T_{f^+}(x,y)\mathrm{d}x\mathrm{d}y.$$
Then j^T is the restriction of an exponential-polynomial function of T whose purely polynomial term is a constant that equals $J((f^+)_z)$.

Proof. The absolute convergence follows from Lemma 9.3 and Proposition 8.5, the second follows from Lemma 9.3 and Theorem 8.6.

9.2. **The coarse geometric expansion.** The following theorem gives the geometric expansions of relative trace formulae on unitary group.

Theorem 9.6. *We have the following assertions.*

(1) *For T sufficiently positive, the expression*
$$\sum_{\alpha \in \mathcal{A}(F)} \int_{[\mathrm{U}'_V]\times[\mathrm{U}'_V]} |k^T_{f^+,\alpha}(x,y)| \, \mathrm{d}x\mathrm{d}y$$
is finite and defines a continuous seminorm on $\mathcal{S}(\mathrm{U}^+_V(\mathbb{A}))$.

(2) *For $\alpha \in \mathcal{A}(F)$ and T sufficiently positive, put*
$$j^T_\alpha(f^+) = \int_{[\mathrm{U}'_V]\times[\mathrm{U}'_V]} k^T_{f^+,\alpha}(x,y)\mathrm{d}x\mathrm{d}y.$$
Then j^T_α coincides with the restriction of an exponential-polynomial function of T whose purely polynomial term is a constant $j_\alpha(f^+)$.

(3) *The linear form $f^+ \mapsto j_\alpha(f^+)$ is continuous and satisfies the invariant property that*
$$j_\alpha(\mathrm{L}^+(x)\mathrm{R}^+(y)f^+) = j_\alpha(f^+)$$
for all $x, y \in \mathrm{U}'_V(\mathbb{A})$.

The proof is the same as Theorem 7.6 and make use of Proposition 4.30.

9.3. **Synthesis of the results: the coarse relative trace formula.** We now summarize what we have done. For $f_+ \in \mathcal{S}(\mathrm{U}_{V,+}(\mathbb{A}))$ we put

$$J^T(f_+) = j^T(f_+^{\ddagger}), \quad J(f_+) = j(f_+^{\ddagger}).$$

Then Theorem 9.6 tells us that if T is sufficiently positive then $J^T(f_+)$ is the restriction of a polynomial exponential and the purely polynomial part is a constant that equals $J(f_+)$. By Lemma 9.4, the distribution J is bi-$\mathrm{U}_V'(\mathbb{A})$-invariant, i.e. for $x, y \in \mathrm{U}_V'(\mathbb{A})$ we have

$$J(\mathrm{L}_+(x)\mathrm{R}_+(y)f_+) = J(f_+).$$

We summarize the coarse relative trace formula on the unitary groups as the following theorem.

Theorem 9.7. *Let $f_+ \in \mathcal{S}(\mathrm{U}_{V,+}(\mathbb{A}))$ be a test function. Then we have*

$$\sum_{\in \mathfrak{X}(\mathrm{U}_V)} J(f_+) = \sum_{\in \mathcal{A}(F)} J(f_+).$$

Each summand on both sides are bi-$\mathrm{U}_V'(\mathbb{A})$-invariant.

This is simply a combination of Theorem 8.6, Lemma 9.5 and Theorem 9.6.

Part 3. Spectral expansions of automorphic periods

10. Periods and relative characters on general linear groups

10.1. **Preliminaries.** In addition to the groups we encountered in Part 2, we will use in this part the following notations.

- Set $n \geq 1$. Define $G_n = \mathrm{Res}_{E/F}\, \mathrm{GL}_n$, $G = G_n \times G_n$, $H = G_n$ embedded diagonally in G, $G' = \mathrm{GL}_{n,F} \times \mathrm{GL}_{n,F}$ seen as a subgroup of G and $G_+ = G \times E_n$.
- Let $e_n = (0, \dots, 0, 1) \in E_n$ and let P_n be the mirabolic subgroup of G_n which consists of matrices whose last row is e_n.
- Define $B = B_n \times B_n$, $T = T_n \times T_n$, $N = N_n \times N_n$, and $P_{n,n} = P_n \times P_n$ seen as subgroups of G.
- If A is a subgroup of G, set $A_H := A \cap H$.

We defined various spaces of functions in Part 2. We will use only use the following spaces, whose definitions are given in Subsection 4.3.

- The Schwartz spaces $\mathcal{S}(G(\mathbb{A}))$, $\mathcal{S}([G])$, and $\mathcal{S}(\mathbb{A}_E)$.
- The space $\mathcal{T}([G]) = \bigcup_{N \geq 0} \mathcal{T}_N([G])$ of functions of uniformly moderate growth.
- For $\chi \in \mathfrak{X}(G)$ a cuspidal datum, the spaces $\mathcal{S}_\chi([G])$ and $\mathcal{T}_\chi([G])$ (see Subsection 4.7).

In addition to these spaces, at one point in the proof of Theorem 10.4, we are going to make use of the Harish-Chandra Schwartz space $\mathcal{C}([G])$ defined in [BPCZ22, Section 2.5.8]. The precise definition is not needed in our argument and we will only use the two following facts.

1. There is a continuous inclusion $\mathcal{C}([G]) \subset \mathcal{T}([G])$ ([BPCZ22, Equation (2.4.5.25)]).
2. For all $f \in \mathcal{C}([G])$, we have $\int_{[G]} |f(g)|^2 \mathrm{d}g < \infty$, and moreover the map $f \in \mathcal{C}([G \times G]) \mapsto \int_{[G]} f(g,g)\mathrm{d}g$ is continuous ([BPCZ22, Equation (2.4.5.24)]).

The inclusion $\mathcal{C}([G]) \subset \mathcal{T}([G])$ induces by restriction of (4.6) a pairing $\mathcal{C}([G]) \times \mathcal{S}([G]) \to \mathbb{C}$. For any subset $\mathfrak{X} \subset \mathfrak{X}(G)$, define $\mathcal{C}_{\mathfrak{X}}([G])$ to be the orthogonal of $\mathcal{S}_{\mathfrak{X}^c}([G])$ in $\mathcal{C}([G])$, as in Subsection 4.7.

Denote characters of $[N_n]$ by

$$\psi_{n,\pm}(u) = \psi^E\left(\pm(-1)^n \sum_{i=1}^{n-1} u_{i,i+1}\right), \quad u \in [N_n], \tag{10.1}$$

and define a character of $[N]$ by $\psi_N = \psi_{n,+} \boxtimes \psi_{n,-}$. This is a generic character trivial on $[N_H]$. The $(-1)^n$ factor is for compatibility reasons in the local comparison of the relative trace formula.

10.2. **Regularity conditions on cuspidal data for** GL_n**.** Let Π be an irreducible automorphic representation of $G_n(\mathbb{A})$. Set $\Pi^c = \Pi \circ c = \{\varphi \circ c \mid \varphi \in \Pi\}$ and $\Pi^* = (\Pi^c)^\vee$.

Let $\chi = (\chi_n, \chi'_n) \in \mathfrak{X}(G) = \mathfrak{X}(\mathrm{GL}_n) \times \mathfrak{X}(\mathrm{GL}_n)$ be a cuspidal datum. Take (M_n, π) and (M'_n, π') to be representatives of χ_n and χ'_n respectively. Write $M_n = \prod \mathrm{GL}_{n_i}$, $M'_n = \prod \mathrm{GL}_{n'_j}$ and $\pi = \boxtimes \pi_i$, $\pi' = \boxtimes \pi'_j$ accordingly. We shall say that

- χ_n (resp. χ'_n) is regular if all the σ_i are mutually non-isomorphic (resp. all the σ'_j are mutually non-isomorphic),
- χ is G-regular if χ_n and χ'_n are regular,
- χ is (H, μ^{-1})-regular if $\mu^{-1}\sigma_i$ is never isomorphic to the contragredient $(\sigma'_j)^\vee$ for any i and j,
- χ is (G, H, μ^{-1})-regular if it is G and (H, μ^{-1})-regular.

Let $\chi_n \in \mathfrak{X}(G_n)$ be a cuspidal datum. Assume that it is represented by (M_n, σ) which admits the following decomposition

$$M_n = \prod_{i\in I} G_{n_i}^{d_i} \times \prod_{j\in J} G_{n_j}^{d_j} \times \prod_{k\in K} G_{n_k}^{d_k}, \quad \sigma = \bigotimes_{i\in I} \sigma_i^{\boxtimes d_i} \boxtimes \bigotimes_{j\in J} \sigma_j^{\boxtimes d_j} \boxtimes \bigotimes_{k\in K} \sigma_k^{\boxtimes d_k}, \tag{10.2}$$

where the σ_l's are distinct irreducible cuspidal automorphic representations of the $G_{n_l}(\mathbb{A})$ such that we have the following conditions.

- For all $i \in I$, $\sigma_i \neq \sigma_i^*$.
- For all $j \in J$, $\sigma_j = \sigma_j^*$ and $L(s, \sigma_j, \mathrm{As}^{(-1)^{n+1}})$ has no pole at $s = 1$.
- For all $k \in K$, $\sigma_k = \sigma_k^*$ and $L(s, \sigma_k, \mathrm{As}^{(-1)^{n+1}})$ has a pole at $s = 1$.

We say that χ_n is Hermitian if for all $i \in I$ there exists $i^* \in I$ such that $\sigma_{i^*} = (\sigma_i)^*$, and if for all $j \in J$, d_j is even. Therefore χ_n is Hermitian and regular if in the above decomposition we have $J = \emptyset$, for all $l \in I \cup K$ the exponent d_l is 1, and if for all $i \in I$ the index i^* is uniquely defined and the map $i \mapsto i^*$ is an involution of I with no fixed point. In this case, let $I' \subset I$ be a subset such that $I = I' \sqcup (I')^*$ and define

$$L_{\chi_n} = \prod_{i\in I'} G_{2n_i} \times \prod_{k\in K} G_{n_k}. \tag{10.3}$$

This is a Levi subgroup of G_n containing M_n.

For $\chi \in \mathfrak{X}(G)$ with decomposition $\chi = (\chi_n, \chi'_n)$, we shall say that χ is Hermitian if χ_n and χ'_n are Hermitian. In this case, we take (M_n, σ) and (M'_n, σ') to be representatives of χ_n and χ'_n respectively, and define $L_\chi = L_{\chi_n} \times L_{\chi'_n}$ which is a Levi subgroup of G.

Remark 10.1. We have defined in Section 1.1 the notion of regular Hermitian Arthur parameter, which is an irreducible automorphic representation $\Pi = \Pi_n \boxtimes \Pi'_n$ of $G(\mathbb{A})$. If $\Pi = I_P^G \sigma$ is a (G, H, μ^{-1})-regular Hermitian Arthur parameter, then (M_P, σ) represents a (G, H, μ^{-1})-regular Hermitian cuspidal datum χ of G. The requirement that Π_n is discrete corresponds to $I = \emptyset$ in (10.2). Therefore, if Π is a semi-discrete Hermitian Arthur parameter, then Π and χ are automatically (G, H, μ^{-1})-regular, as otherwise some $L(s, \sigma_k, \mathrm{As}^{(-1)^l})$ would have a pole at $s = 1$ for $l = n, n + 1$ which is not possible.

10.3. **Rankin–Selberg periods.** For $f \in \mathcal{T}([G])$ set

$$W_f(g) := \int_{[N]} f(ng)\psi_N^{-1}(n)\,dn, \quad g \in G(\mathbb{A}). \tag{10.4}$$

Recall that we have defined in Subsection 5.2 a representation $R_{\mu^{-1}}$ of $G_n(\mathbb{A})$ realized on $\mathcal{S}(\mathbb{A}_{E,n})$ and given by

$$(R_{\mu^{-1}}(h)\Phi)(x) = |\det h|_E^{\frac{1}{2}} \mu^{-1}(\det h)\Phi(xh), \quad h \in G_n, \quad x \in E_n.$$

Then for $f \in \mathcal{T}([G])$ and $\Phi \in \mathcal{S}(\mathbb{A}_{E,n})$, we set

$$Z^{\mathrm{RS}}(f,\Phi,\mu,s) = \int_{N_H(\mathbb{A})\backslash H(\mathbb{A})} W_f(h)|\det h|_E^{s}(R_{\mu^{-1}}(h)\Phi)(e_n)\mathrm{d}h, \tag{10.5}$$

for every $s \in \mathbb{C}$ such that this expression converges absolutely. Beware that $\det h$ is the determinant of h seen as an element of $G_n(\mathbb{A})$ and not $G(\mathbb{A})$.

If c is a real number, set $\mathcal{H}_{>c} := \{s \in \mathbb{C} \mid \Re(s) > c\}$, and if $c < C$, set $\mathcal{H}_{]c,C[} := \{s \in \mathbb{C} \mid c < \Re(s) < C\}$.

Lemma 10.2. *Let $N \geq 0$. There exists $c_N > 0$ such that the following holds.*

- *For every $f \in \mathcal{T}_N([G])$, $\Phi \in \mathcal{S}(\mathbb{A}_{E,n})$ and $s \in \mathcal{H}_{>c_N}$, the defining integral of $Z^{\mathrm{RS}}(f,\Phi,\mu,s)$ converges absolutely.*
- *For every $s \in \mathcal{H}_{>c_N}$, the bilinear form $(f,\Phi) \in \mathcal{T}_N([G]) \times \mathcal{S}(\mathbb{A}_{E,n}) \mapsto Z^{\mathrm{RS}}(f,\Phi,\mu,s)$ is separately continuous.*
- *For every $f \in \mathcal{T}_N([G])$ and $\Phi \in \mathcal{S}(\mathbb{A}_{E,n})$, the function $s \in \mathcal{H}_{>c_N} \mapsto Z^{\mathrm{RS}}(f,\Phi,\mu,s)$ is holomorphic and bounded in vertical strips.*

Proof. The proof follows the same line as that of [BPCZ22, Lemma 7.2.0.1]. Denote by V_B the F-vector space of additive characters $N \to \mathbb{G}_a$, and fix a height function $\|\cdot\|_{V_B}$ on $V_B(\mathbb{A})$ as in (4.3). The adjoint action of T on N induces a dual action of T on V_B, which we denote by Ad^*. We also fix a height function $\|\cdot\|_{\mathbb{A}_E}$ on $\mathbb{A}_E$.

Define an additive character $l \in V_B$ by

$$l(u,u') = \mathrm{Tr}_{E/F}\left((-1)^n\left(\sum_{i=1}^{n-1} u_{i,i+1} - \sum_{i=1}^{n-1} u'_{i,i+1}\right)\right), \quad (u,u') \in N.$$

Then $\psi_N = \psi \circ l$. It is readily checked that there exists $N_0 > 0$ such that

$$\prod_{i=1}^{n-1} \|t_i t_{i+1}^{-1}\|_{\mathbb{A}_E} \ll \|\mathrm{Ad}^*(t^{-1})l\|_{V_B}^{N_0}, \quad t \in T_n(\mathbb{A}). \tag{10.6}$$

By [BPCZ22, Lemma 2.6.1.1.2], for any $N_1 > 0$, there is a continuous seminorm $\|\cdot\|_N$ on $\mathcal{T}_N([G])$ such that

$$|W_f(t)| \ll \|\mathrm{Ad}^*(t^{-1})l\|_{V_P}^{-N_1}\|t\|_T^N\|f\|_N,$$

for all $t \in T(\mathbb{A})$ and $f \in \mathcal{T}_N([G])$. As Φ is Schwartz, for all $N_2 > 0$ there is a seminorm $\|\cdot\|$ such that

$$|\Phi(e_n t)| \ll \|t_n\|_{\mathbb{A}_E}^{-N_2}\|\Phi\|,$$

for all $t \in T_n(\mathbb{A})$.

By the Iwasawa decomposition for H and the estimates above, we are reduced to showing the existence of $c_N > 0$ such that for every $C > c_N$ there exists $N' > 0$ such that

$$\int_{T_n(\mathbb{A})} \prod_{i=1}^{n} \|t_i\|_{\mathbb{A}_E}^{-N'} \|t\|_{T_n}^{N} \delta_{B_n}(t)^{-1} |\det t|_E^{s} \mathrm{d}t$$

converges uniformly on vertical strips for $s \in \mathcal{H}_{]c_n, C[}$. This is an elementary calculation.

For every $f \in \mathcal{S}([G])$ and $\Phi \in \mathcal{S}(\mathbb{A}_{E,n})$, we define

$$\Theta'(h,\Phi) = \sum_{x \in E_n \setminus \{0\}} (\mathrm{R}_{\mu^{-1}}(h)\Phi)(x), \quad h \in [G_n],$$

and

$$Z_n^{\mathrm{RS}}(f,\Phi,\mu,s) = \int_{[H]} f(h)\Theta'(h,\Phi)|\det h|^{s}\mathrm{d}h. \tag{10.7}$$

Since $f \in \mathcal{S}([G])$ and $|\Theta'(h,\Phi)| \ll \|h\|_H^D$ for some D (cf. Lemma 5.5), we conclude by [BP21a, Proposition A.1.1 (vi)] that the integral converges absolutely for all $s \in \mathbb{C}$ and yields an entire function bounded on vertical strips.

Remark 10.3. Our definition of $\Theta'(.,\Phi)$ mirrors the Epstein–Eisenstein series of [JS81, Section 4]. It differs from the theta function $\Theta(.,\Phi)$ defined in Subsection 5.2 by the term $\mu(\det g)^{-1}|\det g|^{\frac{1}{2}}\Phi(0)$. When $f \in \mathcal{S}_\pi([G])$ with π a (H,μ^{-1})-regular cuspidal data, we have for all $s \in \mathbb{C}$

$$\int_{[H]} f(h)\mu(\det h)^{-1}|\det h|^{s+\frac{1}{2}}\mathrm{d}h = 0$$

It follows that for such an f we have

$$Z_n^{\mathrm{RS}}(f,\Phi,\mu,s) = \int_{[H]} f(h)\Theta(h,\Phi)|\det h|^{s}dh.$$

Theorem 10.4. *Let $\pi \in \mathfrak{X}(G)$ be a (G,H,μ^{-1})-regular cuspidal datum.*

(1) *For every $f \in \mathcal{T}_\pi([G])$ and $\Phi \in \mathcal{S}(\mathbb{A}_{E,n})$, the function $s \mapsto Z^{\mathrm{RS}}(f,\Phi,\mu,s)$, a priori defined on some right half-plane, extends to an entire function on $\mathbb{C}$ of finite order in vertical strips.*

(2) *For all $s \in \mathbb{C}$ and $\Phi \in \mathcal{S}(\mathbb{A}_{E,n})$, the restriction to $\mathcal{S}_\pi([G])$ of*

$$f \in \mathcal{S}([G]) \mapsto Z_n^{\mathrm{RS}}(f,\Phi,\mu,s),$$

extends continuously to a linear form on $\mathcal{T}_\pi([G])$.

Moreover, if we keep the same notations for these continuations, then for all $f \in \mathcal{T}_\pi([G])$, $\Phi \in \mathcal{S}(\mathbb{A}_{E,n})$ and $s \in \mathbb{C}$ we have

$$Z^{\mathrm{RS}}(f,\Phi,\mu,s) = Z_n^{\mathrm{RS}}(f,\Phi,\mu,s). \tag{10.8}$$

Proof. We first claim that if $f \in \mathcal{S}_\pi([G])$, $\Phi \in \mathcal{S}(\mathbb{A}_{E,n})$ and $\Re s \gg 0$, then the identity (10.8) holds. Assuming this for the moment, we fix a Φ and consider Z_n^{RS} and Z^{RS} as (families of) linear forms in f.

We temporarily set $w = (w_n, w_n) \in G(F)$, where w_n the longest Weyl group element in $G_n(F)$. For $f \in \mathcal{T}([G])$, set $\tilde{f}(g) = f({}^t g^{-1}) = f(w\,{}^t g^{-1}) \in \mathcal{T}([G])$.

For $s \in \mathbb{C}$, we consider the linear form on $\mathcal{T}([G])$ given by

$$f \mapsto \tilde{Z}^{\mathrm{RS}}(f, \Phi, \eta, s) := Z^{\mathrm{RS}}(\tilde{f}, \hat{\Phi}, \eta^{-1}, s), \tag{10.9}$$

where we denote by $\hat{\Phi}$ the Fourier transform of Φ defined as

$$\hat{\Phi}(y) = \int_{\mathbb{A}_{E,n}} \Phi(x) \psi_E(x\,{}^t y) \mathrm{d}x.$$

Note that if $f \in \mathcal{T}([G])$ then so does $\tilde{f}$. By Lemma 10.2 we see that Z^{RS} and $\tilde{Z}^{\mathrm{RS}}$ enjoy the following properties.

(1) For all $N > 0$ there exists $C_N > 0$ such that for all $s \in \mathcal{H}_{>C_N}$, $Z^{\mathrm{RS}}(\cdot, \Phi, \eta, s)$ and $\tilde{Z}^{\mathrm{RS}}(\cdot, \Phi, \eta, s)$ are continuous linear functionals on $\mathcal{T}_{N,\eta}([G])$.
(2) For all $N > 0$ and $f \in \mathcal{T}_{N,\eta}([G])$, $s \mapsto Z^{\mathrm{RS}}(f, \Phi, \eta, s)$ and $s \mapsto \tilde{Z}^{\mathrm{RS}}(f, \Phi, \eta, s)$ are holomorphic functions on $\mathcal{H}_{>C_N}$, bounded on vertical strips.
(3) By the claim that (10.8) holds for $f \in \mathcal{S}([G])$ and $\Re s \gg 0$, we conclude that for $f \in \mathcal{S}([G])$ the functions $s \mapsto Z^{\mathrm{RS}}(f, \Phi, \eta, s)$ and $s \mapsto \tilde{Z}^{\mathrm{RS}}(f, \Phi, \eta, s)$ have analytic continuations to all $s \in \mathbb{C}$.
(4) By remark 10.3 we may replace Θ' by Θ and use the Poisson summation formula to obtain for any $f \in \mathcal{S}([G])$ the functional equation

$$Z_n^{\mathrm{RS}}(f, \Phi, \eta, s) = Z_n^{\mathrm{RS}}(\tilde{f}, \hat{\Phi}, \eta^{-1}, -s).$$

Therefore it follows from (10.8) that

$$Z^{\mathrm{RS}}(f, \Phi, \eta, s) = \tilde{Z}^{\mathrm{RS}}(f, \Phi, \eta, -s). \tag{10.10}$$

Let $N' > 0$ and choose $N > 0$ such that we have a continuous inclusion $\mathrm{L}^2_{N'}([G])^\infty \subset \mathcal{T}_N([G])$ (see Subsection 4.3.6). By Proposition 4.20, the space $\mathcal{S}([G])$ is dense in $\mathrm{L}^2_{N',\eta}([G])^\infty$. It follows that the two functionals $f \mapsto Z^{\mathrm{RS}}(f, \Phi, \eta, s)$ and $f \mapsto \tilde{Z}^{\mathrm{RS}}(f, \Phi, \eta, s)$ defined on the LF-space $\mathcal{S}([G])$ satisfy the hypotheses of [BPCZ22, Corollary A.0.11.2]. As $\mathcal{T}([G]) = \bigcup_{N'>0} \mathrm{L}^2_{N',\eta}([G])^\infty$, this corollary shows the analytic continuation of Z^{RS} to $\mathbb{C}$ for all $f \in \mathcal{T}([G])$, and the continuous extension of Z_n^{RS} to $\mathcal{T}([G])$. Since the equality (10.8) is true for $f \in \mathcal{S}([G])$ and $\Re s \gg 0$, it is true for $f \in \mathcal{T}([G])$ by density, and finally for all s by analytic continuation. This concludes the proof.

It remains to prove our claim that the identity (10.8) holds for all $f \in \mathcal{S}([G])$, $\Phi \in \mathcal{S}(\mathbb{A}_{E,n})$ and $\Re s \gg 0$. The proof is a variant of the unfolding calculation in the classical Rankin–Selberg convolution.

Let us introduce the following notation. Let $1 \leq r \leq n$ and denote by $N_{r,n}$ the unipotent radical of the standard parabolic subgroup of G_n with Levi component $(G_r) \times (G_1)^{n-r}$. Set $N_r^G := N_{r,n} \times N_{r,n}$ and $N_r^H := N_r^G \cap H$. Let P_r be the mirabolic subgroup of G_r whose last row is e_r.

For $f \in \mathcal{S}([G])$ we define

$$f_{N_r^G,\psi}(g) = \int_{[N_r^G]} f(ug)\psi_N(u)^{-1}du, \quad g \in G(\mathbb{A}).$$

For $s \in \mathbb{C}$ and $\Phi \in \mathcal{S}(\mathbb{A}_{E,n})$ we set

$$Z_r^{\mathrm{RS}}(f,\Phi,\eta,s) = \int_{P_r(F)N_r^H(\mathbb{A})\backslash H(\mathbb{A})} f_{N_r^G,\psi}(h)(\mathrm{R}_{\eta^{-1}}(h)\Phi)(e_n)|\det h|_E^{\mathbf{s}}\mathrm{d}h.$$

For $r = 1$ we get $Z_1^{\mathrm{RS}}(f,\Phi,\eta,s) = Z^{\mathrm{RS}}(f,\Phi,\eta,s)$. The same argument as in the proof of Lemma 10.2 shows that for every $1 \leq r \leq n$ there exists $c_r > 0$ such that the following assertions hold.

- For all $f \in \mathcal{S}([G])$ and $\Phi \in \mathcal{S}(\mathbb{A}_{E,n})$, the expression defining $Z_r^{\mathrm{RS}}(f,\Phi,\eta,s)$ converges absolutely for $s \in \mathcal{H}_{>c_r}$.
- For all $s \in \mathcal{H}_{>c_r}$, $(f,\Phi) \mapsto Z_r^{\mathrm{RS}}(f,\Phi,\eta,s)$ is separately continuous.
- For every $f \in \mathcal{S}([G])$ and $\Phi \in \mathcal{S}(\mathbb{A}_{E,n})$, the function $s \in \mathcal{H}_{>c_r} \mapsto Z_r^{\mathrm{RS}}(f,\Phi,\eta,s)$ is holomorphic and bounded in vertical strips.

The desired identity (10.8) then follows from the following claim: for all $1 \leq r \leq n-1$, for all $f \in \mathcal{S}([G])$ and $\Phi \in \mathcal{S}(\mathbb{A}_{E,n})$ there exists $c > 0$ such that for all $s \in \mathcal{H}_{>c}$ we have

$$Z_{r+1}^{\mathrm{RS}}(f,\Phi,\eta,s) = Z_r^{\mathrm{RS}}(f,\Phi,\eta,s). \tag{10.11}$$

We now prove identity (10.11). First for r, f and Φ, we pick $c > 0$ such that $Z_{r+1}^{\mathrm{RS}}(f,\Phi,\eta,s)$ and $Z_r^{\mathrm{RS}}(f,\Phi,\eta,s)$ are well-defined for $s \in \mathcal{H}_{>c}$. Let U_{r+1} be the unipotent radical of P_r. Then $N_{r,n} = U_{r+1}N_{r+1,n}$. Set $U_{r+1}^G = U_{r+1} \times U_{r+1} \cap G$ and $U_{r+1}^H = U_{r+1} \cap H$. We then compute

$$\begin{aligned} Z_{r+1}^{\mathrm{RS}}(f,\Phi,\eta,s) &= \int_{P_{r+1}(F)N_{r+1}^H(\mathbb{A})\backslash H(\mathbb{A})} f_{N_{r+1}^G,\psi}(h)(\mathrm{R}_{\eta^{-1}}(h)\Phi)(e_n)|\det h|_E^{\mathbf{s}}\mathrm{d}h \\ &= \int_{G_r(F)N_r^H(\mathbb{A})\backslash H(\mathbb{A})}\int_{[U_{r+1}^H]} f_{N_{r+1}^G,\psi}(uh)(\mathrm{R}_{\eta^{-1}}(uh)\Phi)(e_n)|\det uh|_E^{\mathbf{s}}\mathrm{d}u\mathrm{d}h \\ &= \int_{G_r(F)N_r^H(\mathbb{A})\backslash H(\mathbb{A})}(\mathrm{R}_{\eta^{-1}}(h)\Phi)(e_n)|\det h|_E^{\mathbf{s}}\int_{[U_{r+1}^H]} f_{N_{r+1}^G,\psi}(uh)\mathrm{d}u\mathrm{d}h. \end{aligned}$$

By Fourier inversion on the compact abelian group $U_{r+1}^G(F)U_{r+1}^H(\mathbb{A})\backslash U_{r+1}^G(\mathbb{A})$ we get

$$\begin{aligned} \int_{[U_{r+1}^H]} f_{N_{r+1}^G,\psi}(uh)\mathrm{d}u &= \sum_{\gamma\in P_r(F)\backslash G_r(F)}\int_{[U_{r+1}^G]} f_{N_{r+1}^G,\psi}(u\gamma h)\psi_N(u)^{-1}\mathrm{d}u + \int_{[U_{r+1}^G]} f_{N_{r+1}^G,\psi}(uh)\mathrm{d}u \\ &= \sum_{\gamma\in P_r(F)\backslash G_r(F)} f_{N_r^G,\psi}(\gamma h) + \int_{[U_{r+1}^G]} f_{N_{r+1}^G,\psi}(uh)\mathrm{d}u. \end{aligned}$$

Therefore we have

$$Z_{r+1}^{\mathrm{RS}}(f,\Phi,\eta,s) = Z_r^{\mathrm{RS}}(f,\Phi,\eta,s) + F_r(s),$$

where

$$F_r(s) = \int_{G_r(F)N_r^H(\mathbb{A})\backslash H(\mathbb{A})}(\mathrm{R}_{\eta^{-1}}(h)\Phi)(e_n)|\det h|_E^{\mathbf{s}}\int_{[U_{r+1}^G]} f_{N_{r+1}^G,\psi}(uh)\mathrm{d}u\mathrm{d}h.$$

As our functions are holomorphic for $\Re(s) \gg 0$, it remains to show that F_r vanishes on some right half-plane of $\mathbb{C}$. Let Q_r be the standard parabolic subgroup of G with Levi component $L_r = (G_r \times G_{n-r}) \times (G_r \times G_{n-r})$, and set $Q_r^H = Q_r \cap H$. Let K_n be a good maximal compact subgroup of $G_n(\mathbb{A})$ as in Subsection 4.5. Let χ^L be the inverse image of χ in $\mathfrak{X}(L_r)$. We write any element in $\mathfrak{X}(L_r)$ as (χ_1, χ_2) where $\chi_1 \in \mathfrak{X}(G_r^2)$ and $\chi_2 \in \mathfrak{X}(G_{n-r}^2)$. Then by [BPCZ22, Section 2.9.6.10] we have the decomposition

$$\mathcal{C}_{\chi^L}([L_r]) = \bigoplus_{(\chi_1, \chi_2) \in \chi^L} \mathcal{C}_{\chi_1}([G_r^2]) \widehat{\otimes} \mathcal{C}_{\chi_2}([G_{n-r}^2]). \tag{10.12}$$

Define for all $s \in \mathbb{C}$ and $k \in K_n$

$$f_{Q_r,k,s} = \delta_{Q_r}^{-\frac{1}{2}+\frac{2s+1}{4n-4r}} \mathrm{R}(k) f_{Q_r}\big|_{[L_r]},$$

where f_{Q_r} is the constant term of f with respect to Q_r. As χ is G-regular, [BPCZ22, Corollary 2.9.7.2] applies and for $\Re s > 0$ and $k \in K_n$ we have $(f_{Q_r,k,s})\big|_{[L_r]} \in \mathcal{C}_{\chi^L}([L_r])$. Finally, for $k \in K_n$, set $\Phi_{k,n-r} = (\mathrm{R}(k)\Phi)\big|_{\{0\} \times \mathbb{A}_{E,n-r}} \in \mathcal{S}(\mathbb{A}_{E,n-r})$. Then using the Iwasawa decomposition $H(\mathbb{A}) = Q_r^H(\mathbb{A}) K_n$ we get just as in the proof of [BPCZ22, Proposition 7.2.0.2]:

$$F_r(s) = \int_{K_n} \left(P_{G_r}^{\eta^{-1}} \widehat{\otimes} Z^{\mathrm{RS}}_{n-r}\left(\cdot, \Phi_{k,n-r}, \mu, s + 2r\frac{2s+1}{4n-4r}\right)\right)(f_{Q_r,k,s})\, dk,$$

where

- $P_{G_r}^{\eta^{-1}}$ stands for the period integral over the diagonal subgroup G_r of G_r^2, against the character η^{-1}, which defines a continuous linear form on $\mathcal{C}([G_r^2])$ ([BPCZ22, Equation 2.4.5.24]),
- Z^{RS}_{n-r} is Z^{RS} for G_{n-r}, with $\mu^r = \mu((-1)^r \cdot)$. By Lemma 10.2, for every $\Phi' \in \mathcal{S}(\mathbb{A}_{E,n-r})$ it is a continuous linear form on $\mathcal{T}([G_{n-r}^2])$ provided that s lies in some right half-plane of $\mathbb{C}$ (that can be chosen independently of Φ'),
- the completed tensor product is taken relatively to the decomposition (10.12), recalling the continuous inclusion $\mathcal{C}([G_{n-r}^2]) \subset \mathcal{T}([G_{n-r}^2])$.

Since χ is (H, η^{-1})-regular, for every preimage $(\chi_1, \chi_2) \in \mathfrak{X}(G_r^2) \times \mathfrak{X}(G_{n-r}^2)$ with $\chi_1 = (\chi_1', \chi_1'')$ we have $\eta^{-1}\chi_1' \neq \chi_1''^{\vee}$. Thus, $P_{G_r}^{\eta^{-1}}$ vanishes identically on $\mathcal{C}_{\chi_1}([G_r^2])$, which implies that $F_r(s) = 0$ whenever $\Re(s) \gg 0$. This proves the identity (10.11) and hence finishes the proof of the theorem.

The linear form Z^{RS} given by Theorem 10.4 depends on a fixed Schwartz function Φ, but it is possible to obtain uniform bounds when varying Φ as asserted by the following lemma.

Lemma 10.5. *Let $N > 0$. There exist continuous semi-norms $\|.\|_N$ and $\|.\|$ on $\mathcal{T}_{N,\chi}([G])$ and $\mathcal{S}(\mathbb{A}_{E,n})$ respectively such that*

$$\left|Z^{\mathrm{RS}}(f, \Phi, \mu, 0)\right| \ll \|f\|_N \|\Phi\|, \quad f \in \mathcal{T}_{N,\chi}([G]), \ \Phi \in \mathcal{S}(\mathbb{A}_{E,n}).$$

Proof. Take c_N given by Lemma 10.2, and let $D > c_N$. Recall that $\tilde{Z}^{\mathrm{RS}}$ is defined in (10.9). By Theorem 10.4, the functions $s \mapsto e^{s^2} Z^{\mathrm{RS}}(f, \Phi, \mu, s)$ and $s \mapsto e^{s^2} \tilde{Z}^{\mathrm{RS}}(f, \Phi, \mu, s)$ are entire and of rapid decay in vertical strips, and moreover by the proof of Lemma 10.2 there exist continuous semi-norms $\|.\|_N$ and $\|.\|$ on $\mathcal{T}_{N,}([G])$ and $\mathcal{S}(\mathbb{A}_{E,n})$ respectively such that

$$(10.13)\qquad \begin{cases} |Z^{\mathrm{RS}}(f,\Phi,\ ,s)| \ll \|f\|_N\|\Phi\|, \\ |\tilde{Z}^{\mathrm{RS}}(f,\Phi,\ ,s)| \ll \|f\|_N\|\Phi\|, \end{cases} \quad s = D+it,\ t\in\mathbb{R},\ f\in\mathcal{T}_{N,}([G]),\ \Phi\in\mathcal{S}(\mathbb{A}_{E,n}).$$

For every f and Φ, consider the function

$$Z(f,\Phi,\ ,s) = \frac{e^{s^2}}{2}\left(\int_{-\infty}^{\infty} \frac{e^{(D+it)^2} Z^{\mathrm{RS}}(f,\Phi,\ ,D+it)}{D+it-s}\mathrm{d}t + \int_{-\infty}^{\infty} \frac{e^{(D+it)^2} \tilde{Z}^{\mathrm{RS}}(f,\Phi,\ ,D+it)}{D+it+s}\mathrm{d}t\right).$$

Z is well defined and holomorphic on $\mathcal{H}_{]-D,D[}$. Moreover, by (10.13) we have

$$(10.14)\qquad |Z(f,\Phi,\ ,0)| \ll \|f\|_N\|\Phi\|,\ f\in\mathcal{T}_{N,}([G]),\ \Phi\in\mathcal{S}(\mathbb{A}_{E,n}).$$

By the functional equation (10.10) and Cauchy's integration formula, we see that $Z(f,\Phi,\ ,s)$ and $Z^{\mathrm{RS}}(f,\Phi,\ ,s)$ coincide on $\mathcal{H}_{]-D,D[}$. Therefore, (10.14) concludes the proof of the lemma.

10.4. **Flicker–Rallis periods.** Let $\in \mathfrak{X}(G)$ be a $(G,H,\ ^{-1})$-regular cuspidal datum, represented by $(M_P,\ )$. Put $\Pi = I_{P(\mathbb{A})}^{G(\mathbb{A})}$ and for all $\in i\mathfrak{a}_P^{G,*}$ set $\Pi = I_{P(\mathbb{A})}^{G(\mathbb{A})}$. Assume that all the Π are generic and let $\mathcal{W}(\Pi,\ _N)$ be the corresponding Whittaker models for $_N$, cf. (10.1). For $\varphi\in\Pi$, $g\in G(\mathbb{A})$ and $\in\mathfrak{a}_{P,\mathbb{C}}^*$ the Eisenstein series $E(g,\varphi,\ )$ is absolutely convergent for $\Re$ in a suitable cone, and admits a meromorphic continuation to $\mathfrak{a}_{P,\mathbb{C}}^*$ which is regular on $i\mathfrak{a}_P^*$. By [Lap08, Theorem 2.2], for every $\in i\mathfrak{a}_P^*$ the map $\varphi\mapsto E(\ ,\varphi,\ )$ induces a continuous map $\Pi\to\mathcal{T}_N([G])$ for some $N>0$. If $\varphi\in\Pi$ and $\in i\mathfrak{a}_P^*$, set

$$W(g,\varphi,\ ) = \int_{[N]} E(ng,\varphi,\ )\ _N^{-1}(n)\mathrm{d}n,\quad g\in G(\mathbb{A}).$$

Then $W(\ ,\varphi,\ )\in\mathcal{W}(\Pi,\ _N)$.

If $f\in\mathcal{S}([G])$ and $\in i\mathfrak{a}_P^*$ we define as in (4.13)

$$(10.15)\qquad f_{\Pi} = \sum_{\varphi\in\mathcal{B}_{P,}} \langle f, E(\ ,\varphi,\ )\rangle E(\ ,\varphi,\ ).$$

Here $\mathcal{B}_{P,}$ is a K-basis of $\mathcal{A}_{P,\ ,\mathrm{cusp}}(G)$, cf. Subsection 4.6, and $\langle\ ,\ \rangle$ is the natural pairing between $\mathcal{S}([G])$ and $\mathcal{T}([G])$ defined in (4.6). By Proposition 4.18 (3), there exists an $N>0$ such that this series converges absolutely in $\mathcal{T}_N([G])$. We define a function

$$(10.16)\qquad W_{f,\Pi}(g) = W_f(g) = \int_{[N]} f_{\Pi}(ng)\ _N(n)\mathrm{d}n,\quad g\in G(\mathbb{A}),$$

which is the same expression as in (10.4) but with $_N$ for compatibility reasons. Note that $W_f\in\mathcal{W}(\Pi,\ _N^{-1})$.

If $f\in\mathcal{S}(G(\mathbb{A}))$, we denote by $I_P(\ ,f)$ the action of f on Π we get by transport from the action of $\mathcal{S}(G(\mathbb{A}))$ on Π via $\varphi\in\Pi\mapsto e^{\langle\ ,H_P(\ )\rangle}\varphi\in\Pi$.

If $f \in S([G])$, set

$$P_{G',\eta}(f) = \int_{[G']} f(g')\eta_{n+1}(g')\mathrm{d}g', \tag{10.17}$$

where $\eta_{n+1}(g_1, g_2) = \eta(\det g_1 g_2)^{n+1}$. The integral is absolutely convergent and defines a continuous functional on $S([G])$ by [BPCZ22, Theorem 6.2.6.1]. Let $P_{n,n} = P_n \times P_n$ be the product of mirabolic groups, $P'_{n,n} = P_{n,n} \cap G'$, and for S a sufficiently large finite set of places of F and $\lambda \in i\mathfrak{a}_P^*$, put

$$\lambda(W) = (\Delta_{G'}^{S,*})^{-1} L^{S,*}(1, \Pi, \mathrm{As}_G) \int_{N'(F_S)\backslash P'_{n,n}(F_S)} W(p_S)\eta_{n+1}(p_S)\mathrm{d}p_S, \quad W \in W(\Pi, \psi_N) \tag{10.18}$$

where we have set

$$L(s, \Pi, \mathrm{As}_G) = L(s, \Pi_1, \mathrm{As}^{(-1)^{n+1}})L(s, \Pi_2, \mathrm{As}^{(-1)^{n+1}})$$

with $\Pi = \Pi_1 \boxtimes \Pi_2$, and where $\Delta_{G'}^{S,*}$ is defined in Subsection 3.4. This expression is absolutely convergent and independent of S as soon as it is sufficiently large by [Fli88] and [JS81]. The same definition also applies to any $W \in W(\Pi, \psi_N^{-1})$

Recall that a Levi subgroup L_π attached to (M_P, π) was defined in (10.3). We equip $i\mathfrak{a}_{M_P}^{L_\pi,*}$ with the measure described in [BPC, Section 6.2.8], which is just a normalization of the usual Lebesgue measure on the induced $\mathbb{R}$-vector space. The following is [BPC, Corollary 6.2.7.1].

Theorem 10.6. *Let $\chi \in \mathfrak{X}(G)$ be a G-regular cuspidal datum represented by (M_P, π), and let $f \in S_\chi([G])$. If π is Hermitian, then the function $\lambda \in i\mathfrak{a}_{M_P}^{L_\pi,*} \mapsto \lambda(W_{f,\Pi_\lambda})$ is Schwartz, and the resulting map*

$$S_\chi([G]) \to S(i\mathfrak{a}_{M_P}^{L_\pi,*}), \quad f \mapsto (\lambda \mapsto \lambda(W_{f,\Pi_\lambda}))$$

is continuous. Moreover, we have

$$P_{G',\eta}(f) = \begin{cases} 2^{-\dim \mathfrak{a}_{L_\pi}} \displaystyle\int_{i\mathfrak{a}_{M_P}^{L_\pi,*}} \lambda(W_{f,\Pi_\lambda})\mathrm{d}\lambda, & \text{if } \pi \text{ is Hermitian}; \\ 0, & \text{otherwise.} \end{cases}$$

10.5. **The relative character I_Π.** Assume that $\chi \in \mathfrak{X}(G)$ is a (G, H, μ^{-1})-regular and Hermitian cuspidal datum represented by (M_P, π). For $f \in S([G])$ and $\Phi \in S(\mathbb{A}_{E,n})$ we put

$$P_H(f, \Phi) = \int_{[H]} f(h)\Theta'(g, \Phi)\mathrm{d}h. \tag{10.19}$$

It is absolutely convergent and for a fixed Φ, it defines a continuous linear form on $S([G])$. As χ is (G, H, μ^{-1})-regular, Theorem 10.4 says that the restriction of $P_H(\cdot, \Phi)$ to $S_\chi([G])$ extends continuously to $T_\chi([G])$ and we denote this extension by $P_H(\cdot, \Phi)$. By Theorem 10.4 we have

$$P_H(E(\cdot, \varphi, \lambda), \Phi) = Z^{\mathrm{RS}}(W(\varphi, \lambda), \mu, \Phi, 0). \tag{10.20}$$

Let $\|\cdot\|$ be a norm on the $\mathbb{R}$-vector space $i\mathfrak{a}_{M_P}^{L_\pi,*}$.

Lemma 10.7. *Let $f \in S(G(\mathbb{A}))$. There exists $N > 0$ such that for every $\lambda \in i\mathfrak{a}_{M_P}^{L,*}$, the function*

$$g \mapsto \sum_{\varphi \in B_{P,\pi}} E(g, I_P(\lambda, f)\varphi, \lambda)\overline{\beta(W(\varphi, \lambda))} \tag{10.21}$$

is defined by an absolutely convergent sum in $T_{N,\chi}([G])$. Moreover, for every continuous semi-norm $\| \cdot \|_N$ on $T_N([G])$ and every $d > 0$ there exists a continuous semi-norm $\| \cdot \|_d$ on $S(G(\mathbb{A}))$ such that

$$\left\| \sum_{\varphi \in B_{P,\pi}} E(\cdot, I_P(\lambda, f)\varphi, \lambda)\overline{\beta(W(\varphi, \lambda))} \right\|_N \leq \|f\|_d (1 + \|\lambda\|)^{-d}, \quad f \in S(G(\mathbb{A})), \quad \lambda \in i\mathfrak{a}_{M_P}^{L,*}. \tag{10.22}$$

Proof. Let us write $\Pi_\lambda^\vee$ for $(\Pi_\lambda)^\vee = (\Pi^\vee)_{-\lambda}$, and $\chi \in \mathfrak{X}(G)$ for the cuspidal datum represented by (M_P, π). It is (G, H, η^{-1})-regular and Hermitian.

By (10.15) and (10.16) we have

$$W_{K_{f,\chi}(g,\cdot),\Pi_\lambda^\vee} = \sum_{\varphi \in B_{P,\pi}} \langle K_{f,\chi}(g, \cdot), E(\cdot, \varphi, \lambda)\rangle \overline{W(\varphi, \lambda)}, \tag{10.23}$$

where the right hand is absolutely convergent in $W(\Pi_\lambda^\vee, \psi_N^{-1})$ by Proposition 4.18 (1). Note that

$$\langle K_{f,\chi}(g, \cdot), E(\cdot, \varphi, \lambda)\rangle = \langle K_f(g, \cdot), E(\cdot, \varphi, \lambda)\rangle = E(g, I_P(\lambda, f)\varphi, \lambda).$$

Therefore by continuity of β applied to (10.23) we see that for all $g \in [G]$

$$\beta\left(W_{K_{f,\chi}(g,\cdot),\Pi_\lambda^\vee}\right) = \sum_{\varphi \in B_{P,\pi}} E(g, I_P(\lambda, f)\varphi, \lambda)\overline{\beta(W(\varphi, \lambda))}. \tag{10.24}$$

It follows from Proposition 4.18 (1) and the fact that there exists N such that the map $\varphi \in \Pi \mapsto E(\varphi, \lambda) \in T_N([G])$ is continuous ([Lap08, Theorem 2.2]) that, as a function of g, the RHS sum of (10.24) is absolutely convergent in $T_{N,\chi}([G])$.

Let $d > 0$ and $X \in U(\mathfrak{g}_\infty)$. By Lemma 4.21, for every continuous semi-norm ν on $S([G])$, there is an $N_\nu > 0$ and a continuous semi-norm $\|.\|_\nu$ on $S(G(\mathbb{A}))$ such that

$$\nu(K_{f,\chi}(g, \cdot)) \ll \|g\|_G^{N_\nu}\|f\|_\nu, \quad g \in [G], \ f \in S(G(\mathbb{A})).$$

Note that $\mathrm{R}_g(X)\beta\left(W_{K_{f,\chi}(g,\cdot),\Pi_\lambda^\vee}\right) = \beta\left(W_{K_{\mathrm{L}(X)f,\chi}(g,\cdot),\Pi_\lambda^\vee}\right)$, where we use $\mathrm{R}_g(X)$ for $\mathrm{R}(X)$ applied to $g \mapsto \beta\left(W_{K_{f,\chi}(g,\cdot),\Pi_\lambda^\vee}\right)$. The first part of Theorem 10.6 then implies the existence of $N_d > 0$ and $\|.\|_d$ such that

$$\left|\beta\left(W_{K_{\mathrm{L}(X)f,\chi}(g,\cdot),\Pi_\lambda^\vee}\right)\right| \ll \|f\|_d\|g\|_G^{N_d}(1 + \|\lambda\|)^{-d}, \ g \in [G], \ f \in S(G(\mathbb{A})).$$

Combining this with (10.24) this proves the desired estimate (10.22).

It follows from Lemma 10.7, the continuity of $P_H(\cdot, \Phi)$ on $T([G])$ and (10.20) that we can define for every $\lambda \in i\mathfrak{a}_{M_P}^{L,*}$, $f \in S(G(\mathbb{A}))$ and $\Phi \in S(\mathbb{A}_{E,n})$ the relative character

$$I_{\Pi_\lambda}(f \otimes \Phi) = \sum_{\varphi \in B_{P,\pi}} Z^{\mathrm{RS}}(W(I_P(f, \lambda)\varphi, \lambda), \Phi, \mu, 0)\overline{\beta(W(\varphi, \lambda))}, \tag{10.25}$$

where this sum is absolutely convergent and does not depend on the choice of $B_{P,\pi}$.

Lemma 10.8. *For every* $\in i\mathfrak{a}_{M_P}^{L,*}$, $f \in S(G(\mathbb{A}))$ *and* $\Phi \in S(\mathbb{A}_{E,n})$ *we have*

$$I_{\Pi}(f \otimes \Phi) = P_H\left(g \mapsto W_{K_{f,}(g,),\Pi}, \Phi\right). \tag{10.26}$$

Moreover, the distribution I_{Π} *extends by continuity to* $S(G_+(\mathbb{A}))$ *and the linear functional* $f_+ \mapsto \int_{i\mathfrak{a}_{M_P}^{L,*}} I_{\Pi}(f_+) \mathrm{d}$ *is well defined and continuous.*

Proof. Equation (10.26) follows from applying $P_H(\,,\Phi)$ to (10.21), and the second part of the statement is a consequence of Lemma 10.5 and the estimate (10.22) of Lemma 10.7.

10.6. **Spectral expansion of I.** Let $f \in S(G(\mathbb{A}))$ and $\Phi \in S(\mathbb{A}_E)$. Let $\in \mathfrak{X}(G)$ be a cuspidal datum. Recall that we have defined in (6.14) the kernel $K_{f \otimes \Phi,}$. Let us put

$$K^H_{f \otimes \Phi,}(g) = \int_{[H]} K_{f \otimes \Phi,}(h, g)\mathrm{d}h, \quad K^{G'}_{f,}(g) = \int_{[G']} K_{f,}(g, g')\, _{n+1}(g')\mathrm{d}g', \quad g \in [G].$$

We now write a spectral expansion of I the relative character defined in Theorem 6.10.

Proposition 10.9. *Assume that is $(G, H, ^{1})$-regular. We have the following assertions.*

(1) *If is not Hermitian, then* $K^{G'}_{f,}(g) = 0$ *for every* $g \in [G]$*. Moreover* $I(f \otimes \Phi) = 0$*.*

(2) *Assume that is Hermitian. Then*

$$I(f \otimes \Phi) = \int_{[G']} K^H_{f \otimes \Phi,}(g')\, _{n+1}(g')\mathrm{d}g' = P_H\left(K^{G'}_{f,}, \Phi\right), \tag{10.27}$$

where the middle integral is absolutely convergent.

Proof. Let us assume first that is not Hermitian. By definition we have

$$K^{G'}_{f,}(g) = P_{G',}(K_{f,}(g,)).$$

Recall that $\in \mathfrak{X}(G)$ is the $(G, H, ^{1})$-regular cuspidal datum represented by $(M_P,)$. Since is not Hermitian, the linear form $P_{G',}$ vanishes identically on $S([G])$ by Theorem 10.6. It follows that $K^{G'}_{f,}(g) = 0$. By Proposition 6.13 we conclude that

$$i^T(f \otimes \Phi) \ll e^{-N\|T\|}$$

for some $N > 0$ and all T sufficiently positive. By Proposition 6.14, the function $T \mapsto i^T(f \otimes \Phi)$ is exponential polynomial with constant term $I(f \otimes \Phi)$. It follows that $I(f \otimes \Phi) = 0$.

We now assume that is Hermitian. By definition we have

$$K_{f \otimes \Phi,}(\,, g') = K_{f,}(\,, g')\Theta(.,\Phi).$$

By [BPCZ22, Lemma 2.10.1.3], the family of functions $g' \mapsto K_{f,}(\,, g')$ is absolutely integrable over $[G']$ in $T_N([G])$ for some N. By Theorem 10.4, the linear form $P_H(\,,\Phi)$ extends continuously from $S([G])$ to $T([G])$. Since

$$K^H_{f \otimes \Phi,}(g') = P_H\left(K_{f,}(\,, g'), \Phi\right) = P_H\left(K_{f,}(\,, g'), \Phi\right),$$

we conclude that the integral

$$\text{(10.28)} \qquad \int_{[G']} K^H_{f \quad \Phi;} (g') \quad _{n+1}(g')\mathrm{d}g'$$

is absolutely convergent, and equals

$$P_H \left(\int_{[G']} K_{f \quad \Phi;} \;(\;; g') \quad _{n+1}(g')\mathrm{d}g' ; \Phi \right) = P_H \left(K^{G'}_{f;} ; \Phi \right) .$$

The absolute convergence of (10.28), together with Theorem 6.8 and Theorem 6.10 implies that this integral equals $I \;(f \quad \Phi)$.

Theorem 10.10. *Assume that $\in \mathfrak{X}(G)$ is an Hermitian and $(G; H; \quad ^1)$-regular cuspidal datum, represented by $(M_P; \quad)$. Then for all $f_+ \in S(G_+(\mathbb{A}))$ we have*

$$\text{(10.29)} \qquad I \;(f_+) = 2^{\;\dim \mathfrak{a}_L} \int_{i\mathfrak{a}^L_{M_P},} I_\Pi \;(f_+)\mathrm{d} \; ;$$

where the integral on the right is absolutely convergent.

Proof. By Theorem 6.10 and Lemma 10.8, the two sides of (10.29) are continuous in f_+, and the RHS is absolutely convergent. We may therefore assume that $f_+ = f \quad \Phi$. Let us continue with the notations of Proposition 10.9. As $\;{}_{-}$ is Hermitian and $(G; H; \quad ^1)$-regular, Theorem 10.6 applies and yields for all $g \in [G]$

$$\text{(10.30)} \qquad K^{G'}_{f;} (g) = P_{G';} \,(K_{f;} \;(g; \;)) = 2^{\;\dim \mathfrak{a}_L} \int_{i\mathfrak{a}^L_{M_P},} W_{K_{f;} \;(g; \;);\Pi_-} \quad \mathrm{d} \; .$$

By Lemmas 10.7 and 10.8 we have

$$P_H \left(g \mapsto \int_{i\mathfrak{a}^L_{M_P},} W_{K_{f;} \;(g; \;);\Pi_-} \quad \mathrm{d} \; ; \Phi \right) = \int_{i\mathfrak{a}^L_{M_P},} I_\Pi \;(f_+)\mathrm{d} \; .$$

It now follows from Proposition 10.9 and (10.30) that

$$I \;(f \quad \Phi) = P_H \left(K^{G'}_{f;} ; \Phi \right) = 2^{\;\dim \mathfrak{a}_L} \int_{i\mathfrak{a}^L_{M_P},} I_\Pi \;(f_+)\mathrm{d} \; .$$

This proves the theorem.

11. Periods and relative characters on the unitary groups

11.1. **Notations.** We keep the notations from Section 8. We fix a nondegenerate skew c-Hermitian space V of dimension n, and a polarization $\operatorname{Res} V = L + L^-$. To shorten notation, we will drop the subscript V from all the groups, so that in particular $\mathrm{U} = \mathrm{U}_V = \mathrm{U}(V) \quad \mathrm{U}(V)$, $\mathrm{U}' = \mathrm{U}(V)$ seen as a subgroup of U by the diagonal embedding, $\mathrm{U}_+ = \mathrm{U} \quad L^- \quad L^-$. We will denote by $S(V)$ the Heisenberg group, $J(V) = S(V) \rtimes U(V)$ the Jacobi group (see Subsection 5.3) and write $\widetilde{\mathrm{U}} = \mathrm{U}(V) \quad J(V)$.

Recall that we have defined in Subsection 5.4 the Weil representation $\omega_{\;;\;;V}$ realized on $S(L^-(\mathbb{A}))$, and for any $\quad \in S(L^-(\mathbb{A}))$ the theta function $\;_{;\;;V}(\;;\quad) \in T([\mathrm{U}(V)])$. The characters and are fixed throughout this section, hence we simply write $(\,;\,)$ for $_{;\,;V}(\,;\,)$, and $^{-}(\,;\,)$ for $_{1;}\ _{1;V}(\,;\,)$.

11.2. **Regularity conditions on cuspidal data for unitary groups.** Let $=(\ _V;\ ^{0}_{V})\ 2\ \mathsf{X}(\mathrm{U})$ be a cuspidal datum represented by $(M_Q;\ )$ where M_Q is a Levi subgroup of a standard parabolic subgroup Q of U and is an irreducible cuspidal automorphic representation of $M_Q(\mathsf{A})$. We have the following decompositions.

$$M_Q = M_V \quad M^0_V,\ M_V = G_{n_1} \qquad G_{n_r} \quad \mathrm{U}(V_0),\ M^0_V = G_{n^0_1} \qquad G_{n^0_{r^0}} \quad \mathrm{U}(V^0_0).$$
$$= \ _V \quad ^0_V,\quad _V = \ _1 \qquad _r \quad _0,\quad ^0_V = \ ^0_1 \qquad ^0_{r^0} \quad ^0_0.$$

where

V_0 and V^0_0 are nondegenerate skew c-Hermitian vector spaces of dimension n_0 and n^0_0 respectively, $2(n_1 + \quad + n_r) + n_0 = 2(n^0_1 + \quad + n^0_{r^0}) + n^0_0 = n$,
$_i$ and ^0_i are cuspidal automorphic representations of the $G_{n_i}(\mathsf{A})$ and $G_{n^0_i}(\mathsf{A})$ respectively,
$_0$ and $^0_0$ are cuspidal automorphic representations of $\mathrm{U}(V_0)(\mathsf{A})$ and $\mathrm{U}(V^0_0)(\mathsf{A})$ respectively.

We shall say that

$_V$ is regular if the representations $_1;\,:::;\ _r;\ _1;\,:::;\ _r$ are pairwise distinct, and the same applies to ^0_V,
is U-regular if $_V$ and ^0_V are both regular,
is $(\mathrm{U}^0;\ ^1)$-regular if for any $i;j$, the representation $^1\ _i$ is neither isomorphic to $(\ ^0_j)^-$ nor $(\ ^0_j)^{\mathrm{c}}$,
is $(\mathrm{U};\mathrm{U}^0;\ ^1)$-regular if it is both U-regular and U^0-regular.

11.3. **Truncated Fourier–Jacobi periods.** Let $\ 2\ \mathsf{X}(\mathrm{U})$ be a cuspidal datum represented by $(M_Q;\ )$. For $'\ 2\ A_{Q;\ ;\mathrm{cusp}}(\mathrm{U})$ and $\ 2\ \mathfrak{a}_Q$ the Eisenstein series $E(\,;\,';\,)$ is meromorphic in and is holomorphic when $\ 2\ i\mathfrak{a}_Q$. Recall that we have defined in Subsection 8.5 the truncation operator Λ^T_u, where T is a parameter in $\mathfrak{a}_0$.

Proposition 11.1. *Let $T\ 2\ \mathfrak{a}_0$ be su ciently positive in the sense of Subsection 4.5. Then for any $'\ 2\ A_{Q;\ ;\mathrm{cusp}}(\mathrm{U})$, $\ 2\ i\mathfrak{a}_Q$ and $\ 2\ S(L^{-}(\mathsf{A}))$ we have the following assertions.*

(1) *The integral*

$$\int_{[\mathrm{U}^0]} \Lambda^T_u\ E(\,;\,';\,)\quad ^{-}(\,;\,)\ (x)\mathrm{d}x \tag{11.1}$$

is absolutely convergent.

(2) *If is $(\mathrm{U}^0;\ ^1)$-regular, the integral (11.1) is independent of T.*

Proof. The first assertion follows directly from Proposition 8.7. We now show the second assertion. We make use of the notations from Section 8. In particular, for any subgroup $H\quad \mathrm{U}$ we set $H^0 := H \setminus \mathrm{U}^0$.

Recall that $\mathcal{F}$ is the set of D-parabolic subgroups of $J(V)$ (see Subsection 5.3). For each $\phi \in \mathcal{T}([\widetilde{G}], \ )$, $R \in \mathcal{F}$ and $x \in R'(F)\backslash \mathrm{U}'(\mathbb{A})$, we define

$$\Lambda_u^{R,T}\phi(x) = \sum_{\substack{S \subset R \\ S \in \mathcal{F}}} \epsilon_S^R \sum_{\gamma \in S'(F)\backslash R'(F)} \widehat{\tau}_{S'}^{R'}(H_{S'}(\gamma x) - T_{S'})\phi_{\mathrm{U}(V)_S}(\gamma x),$$

where we recall that $\widehat{\tau}_{S'}^{R'}$ is defined in Subsection 4.5. This is the relative version of the truncation operator Λ_u^T defined in (8.14). One show similarly as in Proposition 8.7 that $\Lambda_u^{R,T}\phi$ is rapidly decreasing as a function on $R'(F)\backslash \mathrm{U}'(\mathbb{A})$. It follows that the integral

$$\int_{R'(F)\backslash \mathrm{U}'(\mathbb{A})} \Lambda_u^{R,T}\phi(x)\mathrm{d}x$$

converges absolutely. By (8.12) we have

$$\widehat{\tau}_{S'}(H_{S'}(x) - T - T') = \sum_{R \supset S} \epsilon_R \widehat{\tau}_{S'}^{R'}(H_{R'}(x) - T_{R'})\Gamma'_{R'}(H_{R'}(x) - T_{R'}, T'_{R'}).$$

When both T and $T+T'$ are sufficiently positive we therefore have

$$\Lambda_u^{T+T'}\phi(x) = \sum_{R \in \mathcal{F}} \sum_{\gamma \in R'(F)\backslash \mathrm{U}'(F)} \Gamma'_{R'}(H_{R'}(\gamma x) - T_{R'}, T'_{R'})\Lambda_u^{R,T}\phi(\gamma x).$$

Note that as functions on $[\mathrm{U}']$ we have $(\Lambda_u^{R,T}\phi)_{R'} = \Lambda_u^{R,T}\phi_{\widetilde{R}}$, where $\widetilde{R} = R' \times R$ is a D-parabolic subgroup of $\widetilde{G}$ (see (8.1)). It follows that

$$\begin{aligned}\int_{[\mathrm{U}']} \Lambda_u^{T+T'}\phi(x)\mathrm{d}x &= \sum_{R \in \mathcal{F}} \int_{R'(F)\backslash \mathrm{U}'(\mathbb{A})} \Gamma'_{R'}(H_{R'}(x) - T_{R'}, T'_{R'})\Lambda_u^{R,T}\phi(x)\mathrm{d}x \\ &= \sum_{R \in \mathcal{F}} \int_{[\mathrm{U}']_{R'}} \Gamma'_{R'}(H_{R'}(x) - T_{R'}, T'_{R'})(\Lambda_u^{R,T}\phi_{\widetilde{R}})(x)\mathrm{d}x \\ &= \sum_{R \in \mathcal{F}} \int_{[M_{R'}]} \int_{K'} e^{\langle -2\rho_{R'}, H_{R'}(m)\rangle}\Gamma'_{R'}(H_{R'}(m) - T_{R'}, T'_{R'})\Lambda_u^{R,T}\phi_{\widetilde{R}}(mk)\mathrm{d}m\mathrm{d}k.\end{aligned}$$

We now apply the previous calculation to $\phi = E(\ ,\varphi,\lambda)\overline{\theta(\ ,\ )}$ where $\varphi \in \mathcal{A}_{Q,\pi}(\mathrm{U})$, $\lambda \in i\mathfrak{a}_Q^*$ and $\ \in \mathcal{S}(L^\vee(\mathbb{A}))$. Set $R_{\mathrm{U}} = R' \cap R''$. It is a parabolic subgroup of U. By Proposition 5.9, we have

$$E(\ ,\varphi,\lambda)\overline{\theta(\ ,\ )}_{\widetilde{R}}(x) = E_{R_{\mathrm{U}}}(x,\varphi,\lambda)\overline{\theta_R^\vee(x,\ )},$$

where $\theta_R^\vee$ is defined in (5.20). We will therefore show for that $R \neq J(V)$ and $k \in K'$, the integral

$$\int_{[M_{R'}]} e^{\langle -2\rho_{R'}, H_{R'}(m)\rangle}\Gamma'_{R'}(H_{R'}(m) - T_{R'}, T'_{R'})\Lambda_u^{R,T}\left(E_{R_{\mathrm{U}}}(\ ,\varphi,\lambda)\overline{\theta_R^\vee(\ ,\ )}\right)(mk)\mathrm{d}m$$

vanishes, which concludes the proof of the proposition. By Lemma 5.7, we have $R \neq J(V)$ if and only if $R_{\mathrm{U}} \neq \mathrm{U}$. By the description of the mixed model (5.15), we see that for every $a \in A_{R'}^\infty$ we have

$$\Lambda_u^{R,T}\left(E_{R_{\mathrm{U}}}(\varphi,\lambda)\overline{\theta_R^\vee(\ )}\right)(amk) = e^{\langle \lambda + \rho_{R_{\mathrm{U}}}, H_{R_{\mathrm{U}}}(a)\rangle}\mu^{-1}(a)|a|^{\frac{1}{2}}\Lambda_u^{R,T}\left(E_{R_{\mathrm{U}}}(\varphi,\lambda)\overline{\theta_R^\vee(\ )}\right)(mk).$$

It is therefore enough to show that if $R_{\mathrm{U}} \neq \mathrm{U}$

$$\int_{[M_{R^0}]^1} \Lambda_u^{R;T} \quad E_{R_{\mathrm{U}}}(\; ; \varphi; \quad) \quad _{R} \; ^{-}(\; ; \quad) \quad (mk)\mathrm{d}m = 0.$$

The constant term of the cuspidal Eisenstein series is given by

$$E_{R_{\mathrm{U}}}(mk; \varphi; \quad) = \sum_{w \in W(Q;R_{\mathrm{U}})} E^{R_{\mathrm{U}}}(mk; M(w; \quad)\varphi; w \quad);$$

where $W(Q; R_{\mathrm{U}})$ is some subset of the Weyl group of U (see [MW95, II.1.7]). Note that the representation w is also $(\mathrm{U}^0; \quad ^1)$-regular, so that we may assume that $w = 1$ (which implies $Q \quad R_{\mathrm{U}}$). We are then left to show that the integral

$$\int_{[M_{R^0}]^1} \Lambda_u^{R;T} \quad E^{R_{\mathrm{U}}}(\; ; \varphi; \quad) \quad _{R} \; ^{-}(\; ; \quad) \quad (mk)\mathrm{d}m \tag{11.2}$$

vanishes when $R_{\mathrm{U}} \quad Q$ and $R_{\mathrm{U}} \neq \mathrm{U}$.

Form the decomposition $M_{R_{\mathrm{U}}} = M_{R;} \quad M_{R;}$, where $M_{R;}$ is isomorphic to a product of GL_{n_i} and $M_{R;}$ is isomorphic to a product of two identical unitary group. We write similarly $M_{R^0} = M_{R^0;} \quad M_{R^0;}$. Then $M_{R;} = M_{R^0;}^2$ and the embedding $M_{R^0;} \quad M_{R;}$ is the diagonal inclusion. Write $Q_1 \quad Q_2 = M_{R;} \cap Q$ and the restriction of the representation to $M_{Q_1}(\mathbb{A})$ and $M_{Q_2}(\mathbb{A})$ as $= {}_1 \quad {}_2$. We can assume that $\mathrm{R}(k)(\varphi)|_{[M_{R_{\mathrm{U}}}]} = (\varphi_1 \quad \varphi_2) \quad \varphi$, where $\varphi_1 \quad \varphi_2 \in A_{Q_1 \quad Q_2; \; 1 \quad 2}(M_{R;})$ and $\varphi \in A_{Q \cap M_{R;}}(M_{R;})$. Moreover, it follows from the description of the parabolic subgroups of $J(V)$ in Subsection 5.3 that the operator $\Lambda_u^{R;T}$ is a product of

- the usual Arthur operator Λ^T attached to the product of general linear groups $M_{R^0;}$, seen as an operator of the space $T([M_{R;}])$ acting on the second component (see [Art82]),
- the truncation operator Λ_u^T built in Section 8.5 attached to the unitary group $M_{R^0;}$, seen as an operator of the space $T([M_{R;}])$.

For $m \in [M_{R^0}]^1$ write $m = m \; m$ according to this decomposition. Using the description of mixed model (5.15), we see that $_{R} \; ^{-}(mk; \quad) = \quad ^1(m \quad) \quad _{R} \; ^{-}(m \; k; \quad)$. Thus the integral in (11.2) has

$$\int_{[M_{R^0;} \,]^1} E(m \; ; \varphi_{1;}; \quad)\Lambda^T E(m \; ; \varphi_{2;}; \quad) \quad ^1(m \quad)\mathrm{d}m \tag{11.3}$$

as a factor. The hypothesis $R_{\mathrm{U}} \neq \mathrm{U}$ implies that $M_{R^0;}$ is not trivial. By Langlands' formula for the inner product of truncated Eisenstein series applied to each general linear group ([Art82]), (11.3) is zero as is $(\mathrm{U}^0; \quad ^1)$-regular. This concludes the proof.

11.4. **The relative character $J_{Q;}$.** Assume now that $\in \mathfrak{X}(\mathrm{U})$ is a $(\mathrm{U}^0; \quad ^1)$-regular cuspidal datum represented by $(M_Q; \quad)$. For $\varphi \in A_{Q; \;;\mathrm{cusp}}(\mathrm{U})$, $\in S(L^{-}(\mathbb{A}))$ and T sufficiently positive, we put

$$P(\varphi; \; ; \quad) = \int_{[\mathrm{U}^0]} \Lambda_u^T \quad E(\; ; \varphi; \quad) \quad ^{-}(\; ; \quad) \quad (h)\mathrm{d}h;$$

which is the truncated Fourier–Jacobi period. By Proposition 11.1 this does not depend on T, justifying the notation. It is a meromorphic function in λ and is holomorphic when $\lambda \in i\mathfrak{a}_Q$. Moreover, it follows from the continuity of Λ_u^T shown in Proposition 8.7 that there are continuous semi-norms $\|\ \|$ and $\|\ \|_{L^-}$ on $\mathcal{A}_{Q,\pi,\mathrm{cusp}}(\mathrm{U})$ and $\mathcal{S}(L^-(\mathbb{A}))$ respectively such that

$$|P(\varphi, \phi, \lambda)| \leq \|\varphi\| \|\phi\|_{L^-}, \quad \varphi \in \mathcal{A}_{Q,\pi,\mathrm{cusp}}(\mathrm{U}), \quad \phi \in \mathcal{S}(L^-(\mathbb{A})). \tag{11.4}$$

Recall that we have fixed in Subsection 4.5 a maximal compact subgroup K of $\mathrm{U}(\mathbb{A})$. Let $\mathcal{B}_{Q,\pi}$ be a basis of $\mathcal{A}_{Q,\pi,\mathrm{cusp}}(\mathrm{U})$ as in Subsection 4.6. By Proposition 4.18 (1), for all $f \in \mathcal{S}(\mathrm{U}(\mathbb{A}))$ the series

$$\sum_{\varphi \in \mathcal{B}_{Q,\pi}} (I_Q(\lambda, f)\varphi) \otimes \overline{\varphi}$$

is absolutely convergent in $\mathcal{A}_{Q,\pi,\mathrm{cusp}}(\mathrm{U}) \widehat{\otimes} \overline{\mathcal{A}_{Q,\pi,\mathrm{cusp}}(\mathrm{U})}$. We can define for all $\lambda \in i\mathfrak{a}_Q$ and $\phi_1, \phi_2 \in \mathcal{S}(L^-(\mathbb{A}))$ the relative character

$$J_{Q,\pi}(\lambda, f \otimes \phi_1 \otimes \phi_2) = \sum_{\varphi \in \mathcal{B}_{Q,\pi}} P(I_Q(\lambda, f)\varphi, \phi_1, \lambda)\overline{P(\varphi, \overline{\phi_2}, \lambda)}. \tag{11.5}$$

This is independent of the choice of basis and for fixed f, ϕ_1 and ϕ_2 it is an holomorphic expression in $\lambda \in i\mathfrak{a}_Q$. It follows from (11.4) and Proposition 4.18 (1) that for fixed λ, the map $f \otimes \phi_1 \otimes \phi_2 \mapsto J_{Q,\pi}(\lambda, f \otimes \phi_1 \otimes \phi_2)$ extends by continuity to $\mathcal{S}(\mathrm{U}_+(\mathbb{A}))$. Moreover we have the following functional equation.

Lemma 11.2. *Let (M_{Q_1}, π_1) and (M_{Q_2}, π_2) be two pairs representing the same (U', μ^{-1})-regular cuspidal datum χ. Then for all $w \in W(Q_1, Q_2)$ (see Subsection 4.5) such that $\pi_2 = w\pi_1$, and for every $\lambda \in i\mathfrak{a}_{Q_1}$, $f \in \mathcal{S}(\mathrm{U}(\mathbb{A}))$, $\phi_1, \phi_2 \in \mathcal{S}(L^-(\mathbb{A}))$, we have*

$$J_{Q_1,\pi_1}(\lambda, f \otimes \phi_1 \otimes \phi_2) = J_{Q_2,\pi_2}(w\lambda, f \otimes \phi_1 \otimes \phi_2).$$

Proof. This follows from the functional equation of Eisenstein series and the fact that the intertwining operator sends $\mathcal{B}_{Q_1,\pi_1}$ to a K-basis $\mathcal{B}_{Q_2,\pi_2}$.

Proposition 11.3. *There exist continuous semi-norms $\|\ \|$ on $\mathcal{S}(\mathrm{U}(\mathbb{A}))$ and $\|\ \|_{L^-}$ on $\mathcal{S}(L^-(\mathbb{A}))$ such that*

$$\sum_{(M_Q,\pi)} \int_{i\mathfrak{a}_Q} |J_{Q,\pi}(\lambda, f \otimes \phi_1 \otimes \phi_2)| \, d\lambda \leq \|f\| \|\phi_1\|_{L^-} \|\phi_2\|_{L^-}, \quad f \in \mathcal{S}(\mathrm{U}(\mathbb{A})), \quad \phi_1, \phi_2 \in \mathcal{S}(L^-(\mathbb{A})), \tag{11.6}$$

where (M_Q, π) ranges over a set of representatives of $(\mathrm{U}, \mathrm{U}', \mu^{-1})$-regular cuspidal data of U.

Proof. Fix ϕ_1 and ϕ_2. We begin by showing that the functional $f \mapsto (((Q, \pi), \lambda) \mapsto J_{Q,\pi}(\lambda, f, \phi_1, \phi_2))$ valued in the space of L^1 functions with variables $((Q, \pi), \lambda)$ (given the product of the counting measure and the Lebesgue measure) is continuous. To prove this, it is enough to show that for every $f \in S(\mathrm{U}(\mathbb{A}))$ the sum

$$\sum_{(M_Q,\pi)} \int_{i\mathfrak{a}_Q^*} |J_{Q,\pi}(\lambda, f \ast \phi_1 \ast \phi_2)| \,\mathrm{d}\lambda \tag{11.7}$$

is finite. Indeed, if it is, then our functional is the pointwise limit of a sequence of continuous forms on $S(\mathrm{U}(\mathbb{A}))$, hence it is continuous by the closed graph theorem.

Let $f \in S(\mathrm{U}(\mathbb{A}))$. By Proposition 11.1, for T positive enough (11.7) is

$$\sum_{(M_Q,\pi)} \int_{i\mathfrak{a}_Q^*} \left| \sum_{\varphi \in \mathcal{B}_{Q,\pi}} \int_{[\mathrm{U}']} \int_{[\mathrm{U}']} \Lambda_u^T E(I_Q(\lambda, f)\varphi, \lambda)_{-\lambda}(\phi_1)(h_1)\overline{\Lambda_u^T E(\varphi, \lambda)_{-\lambda}(\overline{\phi_2})}(h_2)\,\mathrm{d}h_1\mathrm{d}h_2 \right| \mathrm{d}\lambda .$$

By Proposition 4.18 (1), we can switch the last sum and integral, so that this expression is bounded above by

$$\sum_{(M_Q,\pi)} \int_{i\mathfrak{a}_Q^*} \int_{[\mathrm{U}']} \int_{[\mathrm{U}']} \left| \sum_{\varphi \in \mathcal{B}_{Q,\pi}} \Lambda_u^T E(I_Q(\lambda, f)\varphi, \lambda)_{-\lambda}(\phi_1)(h_1)\overline{\Lambda_u^T E(\varphi, \lambda)_{-\lambda}(\overline{\phi_2})}(h_2) \right| \mathrm{d}h_1\mathrm{d}h_2\mathrm{d}\lambda . \tag{11.8}$$

By the Dixmier–Malliavin theorem, we may assume that $f = f_1 \ast f_2^{\ast}$ where $f_2^{\ast}(g) = \overline{f_2}(g^{-1})$. By change of basis, the inner term in (11.8) is

$$\sum_{\varphi \in \mathcal{B}_{Q,\pi}} \Lambda_u^T E(I_Q(\lambda, f_1)\varphi, \lambda)_{-\lambda}(\phi_1)(h_1)\overline{\Lambda_u^T E(I_Q(\lambda, f_2)\varphi, \lambda)_{-\lambda}(\overline{\phi_2})}(h_2).$$

It follows from the Cauchy–Schwarz inequality that, up to replacing $\overline{\phi_2}$ by ϕ_2, (11.8) is bounded above by the square root of the product over $i = 1, 2$ of

$$\sum_{(M_Q,\pi)} \int_{i\mathfrak{a}_Q^*} \int_{[\mathrm{U}']} \int_{[\mathrm{U}']} \sum_{\varphi \in \mathcal{B}_{Q,\pi}} \left| \Lambda_u^T E(I_Q(\lambda, f_i)\varphi, \lambda)_{-\lambda}(\phi_i)(h_i) \right|^2 \mathrm{d}h_1\mathrm{d}h_2\mathrm{d}\lambda . \tag{11.9}$$

Set $g_i = f_i \ast f_i^{\ast}$. By Proposition 4.18 (2), the series

$$\sum_{\varphi \in \mathcal{B}_{Q,\pi}} E(I_Q(\lambda, g_i)\varphi, \lambda)_{-\lambda}(\phi_i) \otimes \overline{E(\varphi, \lambda)_{-\lambda}(\phi_i)}$$

is absolutely convergent in $\mathcal{T}([\mathrm{U} \times \mathrm{U}])$. Using again a change of variable and Proposition 8.7 we have

$$\sum_{\varphi \in \mathcal{B}_{Q,\pi}} \left| \Lambda_u^T E(I_Q(\lambda, f_i)\varphi, \lambda)_{-\lambda}(\phi_i) \right|^2 = \sum_{\varphi \in \mathcal{B}_{Q,\pi}} \Lambda_u^T E(I_Q(\lambda, g_i)\varphi, \lambda)_{-\lambda}(\phi_i) \otimes \overline{\Lambda_u^T (E(\varphi, \lambda)_{-\lambda}(\phi_i))}$$

$$= \Lambda_u^T \left(\sum_{\varphi \in \mathcal{B}_{Q,\pi}} E(I_Q(\lambda, g_i)\varphi, \lambda)_{-\lambda}(\phi_i) \otimes \overline{E(\varphi, \lambda)_{-\lambda}(\phi_i)} \right) \Lambda_u^T , \tag{11.10}$$

where we use $\Lambda_u^T F \Lambda_u^T$ to denote truncation in the left and right variables of a function F. Moreover, we can switch the integrals by positivity and (11.9) becomes

(11.11)
$$\mathrm{vol}([\mathrm{U}^0]) \int_{[\mathrm{U}^0]} \sum_{(M_Q, \;)} \int_{i\mathfrak{a}_Q} \Lambda_u^T \left(\sum_{\varphi \in \mathcal{B}_{Q,\;}} E(I_Q(\;, g_i)\varphi, \;) \, \overline{\;}(\;_i) \, \overline{E(\varphi, \;)\, \overline{\;}(\;_i)} \right) \Lambda_u^T(h, h) \mathrm{d}\; \mathrm{d}h.$$

Set
$$K_{g_i,\mathrm{reg}} = \sum_{(\mathrm{U},\mathrm{U}^0,\;^1)\ \mathrm{reg}} K_{g_i,\;}$$

where the sum ranges over $(\mathrm{U},\mathrm{U}^0,\;^1)$-regular cuspidal data of U. By Lemma 4.22 $K_{g_i,\mathrm{reg}}$ is well-defined, belongs to $\mathcal{T}([\mathrm{U} \;\; \mathrm{U}])$, and we have the spectral expansion

$$K_{g_i,\mathrm{reg}}(x,y)\,\overline{\;}(x,\;_i)\,(y,\;_i) = \sum_{(M_Q,\;)} \int_{i\mathfrak{a}_Q} \sum_{\varphi \in \mathcal{B}_{Q,\;}} E(x, I_Q(\;, g_i)\varphi, \;)\,\overline{\;}(x,\;_i)\overline{E(y,\varphi,\;)\,\overline{\;}(y,\;_i)}\mathrm{d}\; .$$

That this decomposition coincides with the one given in Lemma 4.22 follows from the functional equation of Eisenstein series and the U-regularity of the considered cuspidal data (see Remark 4.23). Moreover, the double integral $\sum_{(M_Q,\;)} \int_{i\mathfrak{a}_Q}$ is absolutely convergent for fixed x and y by Lemma 4.22. Therefore, for fixed $h \in [\mathrm{U}^0]$ we have

$$\begin{aligned}
&\sum_{(M_Q,\;)} \int_{i\mathfrak{a}_Q} \Lambda_u^T \left(\sum_{\varphi \in \mathcal{B}_{Q,\;}} E(I_Q(\;, g_i)\varphi, \;)\,\overline{\;}(\;_i)\,\overline{E(\varphi,\;)\,\overline{\;}(\;_i)} \right) \Lambda_u^T(h,h)\mathrm{d}\; \\
&= \Lambda_u^T \left(\sum_{(M_Q,\;)} \int_{i\mathfrak{a}_Q} \sum_{\varphi \in \mathcal{B}_{Q,\;}} E(I_Q(\;, g_i)\varphi, \;)\,\overline{\;}(\;_i)\,\overline{E(\varphi,\;)\,\overline{\;}(\;_i)}\mathrm{d}\; \right) \Lambda_u^T(h,h) \\
&= \Lambda_u^T \left(K_{g_i,\mathrm{reg}}\, \overline{\;}_x(\;_i)\, \;_y(\;_i) \right) \Lambda_u^T(h,h),
\end{aligned}$$

where we write $\overline{\;}_x(\;_i)$ (resp. $\;_y(\;_i)$) for $\overline{\;}(\;_i)$ seen as a function of the first variable (resp. $(\;_i)$ as a function of the second variable). Indeed, it follows from the definition given in (8.14) and [Art78, Lemma 5.1] that for any $F \in \mathcal{T}([\mathrm{U} \;\; \mathrm{U}])$, $\Lambda_u^T F \Lambda_u^T F(h,h)$ is a finite sum of left-translates of constant terms of F (independent of F) evaluated at h. These constant terms are integrals over compact subsets and therefore commute with the absolutely convergent double integral $\sum_{(M_Q,\;)} \int_{i\mathfrak{a}_Q}$. This implies that

$$(11.11) \quad \mathrm{vol}([\mathrm{U}^0]) \int_{[\mathrm{U}^0]} \Lambda_u^T \left(K_{g_i,\mathrm{reg}}\, \overline{\;}_x(\;_i)\, \;_y(\;_i) \right) \Lambda_u^T(h,h)\mathrm{d}h,$$

and this expression is finite by Proposition 8.7. Therefore, (11.7) is also finite and this concludes the proof of the claim.

It remains to obtain uniform bounds on the $\;_i$. As the g_i depend only on f, we see that our bound is uniform in $\;_1$ and $\;_2$, in the sense that there exists a continuous semi-norm $\|.\|_{L^-}$ on

$\mathcal{S}(L_-(\mathbb{A}))$ such that for every f

$$\sup_{\substack{\phi\in\mathcal{S}(L_-(\mathbb{A}))_{(M_Q,\pi)}\\ \|\phi\|_{L_-}\le 1}} \sum_{\varphi} \int_{i\mathfrak{a}_Q^*} |J_{Q,\pi}(\lambda, f\otimes\phi_1\otimes\phi_2)|\,\mathrm{d}\lambda < \infty.$$

The bound (11.6) now follows from the uniform boundedness principle.

As a byproduct of the proof, we obtain the following lemma.

Lemma 11.4. *Let $\chi\in\mathfrak{X}(\mathrm{U})$ be a $(\mathrm{U},\mathrm{U}',\chi^{-1})$-regular cuspidal datum represented by (M_Q,π). Let $f_+\in\mathcal{S}(\mathrm{U}_+(\mathbb{A}))$. Then for T sufficiently positive we have*

$$\int_{i\mathfrak{a}_Q^*} J_{Q,\pi}(\lambda, f_+)\,\mathrm{d}\lambda = \int_{[\mathrm{U}']}\int_{[\mathrm{U}']} \Lambda_u^T K_{f_+,\chi}\Lambda_u^T(x,y)\,\mathrm{d}x\,\mathrm{d}y. \tag{11.12}$$

Proof. As both sides are continuous in f_+ by Proposition 11.3 and Proposition 8.7, we may assume that $f_+ = f\otimes\phi_1\otimes\phi_2$. By definition (see (8.7))

$$K_{f_+,\chi}(x,y) = K_{f,\chi}(x,y)\,\theta(x,\phi_1)\overline{\theta(y,\phi_2)}.$$

By Proposition 11.1, the finiteness of (11.8), the dominated convergence theorem and Proposition 8.7, we have

$$\begin{aligned}
\int_{i\mathfrak{a}_Q^*} J_{Q,\pi}(\lambda, f_+)\,\mathrm{d}\lambda &= \int_{i\mathfrak{a}_Q^*}\int_{[\mathrm{U}']}\int_{[\mathrm{U}']}\sum_{\varphi\in\mathcal{B}_{Q,\pi}} \Lambda_u^T\left(E(I_Q(\lambda,f)\varphi,\lambda)\theta(\phi_1)\right)(x)\overline{\Lambda_u^T\left(E(\varphi,\lambda)\theta(\phi_2)\right)}(y)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}\lambda \\
&= \int_{[\mathrm{U}']}\int_{[\mathrm{U}']}\Lambda_u^T\left(\int_{i\mathfrak{a}_Q^*}\sum_{\varphi\in\mathcal{B}_{Q,\pi}} E(I_Q(\lambda,f)\varphi,\lambda)\theta(\phi_1)\overline{E(\varphi,\lambda)\theta(\phi_2)}\,\mathrm{d}\lambda\right)\Lambda_u^T(x,y)\,\mathrm{d}x\,\mathrm{d}y
\end{aligned} \tag{11.13}$$

By Lemma 4.22 and the U-regularity of χ (see Remark 4.23), we have

$$(11.13) = \int_{[\mathrm{U}']}\int_{[\mathrm{U}']}\Lambda_u^T K_{f_+,\chi}\Lambda_u^T(x,y)\,\mathrm{d}x\,\mathrm{d}y.$$

This concludes.

11.5. **Spectral expansion of J_χ.** We now compute the spectral term J_χ from Section 8 in term of the relative character $J_{Q,\pi}$.

Let $\chi\in\mathfrak{X}(\mathrm{U})$ be a cuspidal datum. Recall that we defined a modified kernel function $K_{f_+,\chi}^T$ in Subsection 8.3, and that in Theorem 8.6 (2) we proved that the purely polynomial term of the exponential-polynomial

$$J_\chi^T(f_+) = \int_{[\mathrm{U}']}\int_{[\mathrm{U}']} K_{f_+,\chi}^T(x,y)\,\mathrm{d}x\,\mathrm{d}y,$$

is a constant which we denoted by $J_\chi(f_+)$. The map $f_+\mapsto J_\chi(f_+)$ is a continuous linear form on $\mathcal{S}(\mathrm{U}_+(\mathbb{A}))$.

Theorem 11.5. *Let $\in \mathfrak{X}(\mathrm{U})$ be a $(\mathrm{U},\mathrm{U}',\ ^{-1})$-regular cuspidal datum, represented by $(M_Q, \ )$. Then for all $f \in \mathcal{S}(\mathrm{U}(\mathbb{A}))$, $_1, _2 \in \mathcal{S}(L^-(\mathbb{A}))$, we have*

$$J\ (f \quad _1 \quad _2) = \int_{i\mathfrak{a}_Q^*} J_{Q,}\ (\ ; f \quad _1 \quad _2)\mathrm{d}\ . \tag{11.14}$$

Proof. Write $f_+ = f \quad _1 \quad _2$. By Lemma 11.4, for T_1 and T_2 positive enough, we have

$$\int_{[\mathrm{U}'] \ [\mathrm{U}']} (\Lambda_u^{T_1} K_{f_+,} \ \Lambda_u^{T_2})(x,y)\mathrm{d}x\mathrm{d}y = \int_{i\mathfrak{a}_Q^*} J_{Q,}\ (\ ; f_+)\mathrm{d}\ .$$

As the RHS is independent of T_1 and T_2, we conclude that so is the LHS. By Proposition 8.7 (2), we have

$$\lim_{d(T_1) \to \infty} \int_{[\mathrm{U}'] \ [\mathrm{U}']} (\Lambda_u^{T_1} K_{f_+,} \ \Lambda_u^{T_2})(x,y)\mathrm{d}x\mathrm{d}y = \int_{[\mathrm{U}'] \ [\mathrm{U}']} K_{f_+,} \ \Lambda_u^{T_2}(x,y)\mathrm{d}x\mathrm{d}y$$

The theorem now follows from the second assertion of Proposition 8.8.

Part 4. Comparison of the relative trace formulae

12. Infinitesimal geometric distributions

12.1. **General linear groups: simplification.** Let us retain the notation from Section 7. Recall that we have geometric distributions as follows. Take $f^+ \in \mathcal{S}(G^+(\mathbb{A}))$ and $\alpha \in \mathcal{A}(F)$. For $T \in \mathfrak{a}_{n+1}$ sufficiently positive, we have defined

$$i^T_\alpha(f^+) = \int_{[H]}\int_{[G']} k^T_{f^+,\alpha}(h,g')\eta_{G'}(g')\mathrm{d}g'\mathrm{d}h,$$

where $\eta_{G'}(g_1,g_2) = \eta(\det g_1)^n\eta(\det g_2)^{n+1}$ (cf. (6.1)). The goal of this subsection is to simplify it.

Let us introduce the following additional spaces and group actions.

- We put
$$S_n = \{g \in G_n \mid gg^{\mathrm{c},-1} = 1\},$$
which is a closed subvariety of G_n over F. The group G'_n acts on S_n from the right by the usual conjugation, and we have an isomorphism of G'_n-varieties $S_n \simeq G_n/G'_n$. Let
$$\nu : G_n \to S_n, \quad \nu(g) = gg^{\mathrm{c},-1}$$
be the natural projection.
- Put $X = S_n \times L^{-,\vee} \times L$, on which G'_n acts from the right by
$$(\delta,w,v)\cdot g' = (g'^{-1}\delta g', wg', g'^{-1}v).$$
- Put $G^+_n = G_n \times L^{-,\vee} \times L$, on which G' acts on the right by
$$(g,w,v)\cdot(g'_1,g'_2) = (g'^{-1}_1 gg'_2, wg'_1, g'^{-1}_1 v).$$
- We extend the map ν to a map $G^+_n \to X$ by setting $\nu(g,u,v) = (\nu(g),u,v)$. By composing this with the map $G_n \times G_n \to G_n$, $(g_1,g_2) \mapsto g_1^{-1}g_2$, we further obtain to a map $G^+ \to X$. Note that $G^+(\mathbb{A}) \to X(\mathbb{A})$ is surjective.
- Note that the quotient morphism $G/H \to G_n$, $(g_1,g_2) \mapsto g_1^{-1}g_2$ and ν are both principal homogeneous spaces, the categorical quotients $G^+_n/\!/G'$, $X/\!/G'_n$ and $G^+/\!/(H \times G')$ are canonically identified, and are all denoted by $\mathcal{A}$. The categorical quotient $q : G^+ \to \mathcal{A}$ factors through the map $\nu : G^+ \to X$, and we denote again by $q : X \to \mathcal{A}$ the induced map.
- If $\alpha \in \mathcal{A}(F)$, we let X_α be the inverse image of α in X (as a closed subscheme).
- Recall that $\mathcal{F}$ denotes the set of standard D-parabolic subgroups of J_n, c.f. Subsection 5.1.
- Let $P = MN \in \mathcal{F}$, write $P_n = M_nN_n$. We put
$$M^+_n = M_n \times M_{L_-} \times M_L; \quad N^+_n = N_n \times N^{-,\vee}_{L_-} \times N_L.$$
- The right action of G'_n on G^+_n restricts to an action of M'_n on M^+_n.
- Composing the embedding $M^+_n \subset G^+_n$ with the map $G^+_n \to \mathcal{A}$ we obtain a map $M^+_n \to \mathcal{A}$, and for any $\alpha \in \mathcal{A}(F)$ we denote by $M^+_{n,\alpha}$ the inverse image of α in M^+_n (as a closed subscheme).

For $\varphi \in \mathcal{S}(X(\mathbb{A}))$, define a function $k_{\varphi,P,\sigma}$ on $[G'_n]_{P'_n}$ by

$$k_{\varphi,P,\sigma}(h) = \sum_{m \in M_n^+(F) \cap X_\sigma(F)} \int_{N_n^+(\mathbb{A})/N'_n(\mathbb{A})} \varphi(h^{-1}\iota(xn)h)\mathrm{d}n,$$

where x is any element in M_n^+ such that $\iota(x) = m$, and N_n^+/N'_n stands for $N_n/N'_n \times N^-_{P,L^-} \times N_{P,L}$. The same argument as Lemma 7.1 gives that

$$\sum_{\sigma \in \mathcal{A}(F)} \sum_{m \in M^+_{n,\sigma}(F)} \int_{N_n^+(\mathbb{A})/N'_n(\mathbb{A})} |\varphi(h^{-1}\iota(xn)h)|\mathrm{d}n$$

is convergent.

Let us define an integral transform

$$\mathcal{S}(G^+(\mathbb{A})) \to \mathcal{S}(X(\mathbb{A})), \quad f^+ \mapsto \varphi_{f^+}$$

as follows. For any $(\nu, w, v) \in X(\mathbb{A})$ we put

$$\varphi_{f^+}(\nu, w, v) = \begin{cases} \displaystyle\int_{G_n(\mathbb{A})}\int_{G'_n(\mathbb{A})} f^+((h^{-1}, h^{-1}xg'_n), w, v)\mathrm{d}h\mathrm{d}g'_n, & n \text{ odd};\\ \displaystyle\int_{G_n(\mathbb{A})}\int_{G'_n(\mathbb{A})} f^+((h^{-1}, h^{-1}xg'_n), w, v)\eta(\det xg'_n)\mathrm{d}h\mathrm{d}g'_n, & n \text{ even}. \end{cases} \tag{12.1}$$

where $x \in G_n(\mathbb{A})$ is any element such that $\iota(x) = \nu$. We note that because the map $G^+ \to X$ is a smooth morphism, the integral transform $f^+ \mapsto \varphi_{f^+}$ is surjective, cf. [AG08, Theorem B.2.4]. If $\mathtt{S}$ is a finite set of places, and $f^+ \in \mathcal{S}(G^+(F_{\mathtt{S}}))$, then we define $\varphi_{f^+} \in \mathcal{S}(X(F_{\mathtt{S}}))$ in the same way, integrating over $F_{\mathtt{S}}$-points of the groups.

Lemma 12.1. *For all $\sigma \in \mathcal{A}(F)$ and $f^+ \in \mathcal{S}(G^+(\mathbb{A}))$, the expression*

$$\int_{[H]}\int_{[G'_n]} \sum_{\substack{\gamma \in P_H(F)\backslash H(F)\\ \delta_2 \in P'_2(F)\backslash G'_n(F)}} \sum_{m \in M^+_\sigma(F)} \int_{N^+(\mathbb{A})} |f^+(mn\iota(\gamma h, \delta_2 g'))|\mathrm{d}n\mathrm{d}g'_2\mathrm{d}h$$

is convergent and we have

$$\int_{[H]}\int_{[G'_n]} \sum_{\substack{\gamma \in P_H(F)\backslash H(F)\\ \delta_2 \in P'_2(F)\backslash G'_n(F)}} k_{f^+,P,\sigma}(\gamma h, \delta_2 g')\eta_{G'}(g')\mathrm{d}g'_2\mathrm{d}h = k_{\varphi_{f^+},P,\sigma}(g'_1)\eta(g'_1). \tag{12.2}$$

Here we recall that $g' = (g'_1, g'_2) \in G'(\mathbb{A})$, $g'_1, g'_2 \in G'_n(\mathbb{A})$, and $\delta_2 g'$ stands for the element $(g'_1, \delta_2 g'_2) \in G'(\mathbb{A})$.

Proof. We will prove (12.2). The absolute convergence can be obtained by the same computation, with only obvious minor modifications.

We first unfold the sum over γ and δ to conclude that (12.2) equals

$$\int_{[H]_{P_H}}\int_{[G'_n]_{P'_n}} k_{f^+,P,\sigma}(h, g')\eta_{G'}(g')\mathrm{d}g'_2\mathrm{d}h.$$

Recall that we have fixed maximal compact subgroups K_H and K_2' of $H(\mathbb{A})$ and $G_2'(\mathbb{A})$ respectively. By the Iwasawa decomposition the left hand side of (12.2) equals

$$(\det g_1')^n \sum_{m\in M^+(F)} \int_{K_H}\int_{K_2'}\int_{[M_{P_H}]}\int_{[M_n']}\int_{N^+(\mathbb{A})} e^{\langle 2\rho_{P_H}, H_{P_H}(m_H)\rangle + \langle 2\rho_{P_n'}, H_{P_n'}(m')\rangle}$$
$$f^+(mn\,(m_Hk_H,(g_1',m'k')))\ (\det m'k')^{n+1}\,dn\,dm'\,dm_H\,dk'\,dk_H.$$

Unfolding the sum over $m\in M^+(F)$ we conclude that this equals

$$(\det g_1')^n \sum_{m\in M^+(F)/(M_{P_H}(F)\ M_2'(F))} \int_{K_H}\int_{K_2'}\int_{M_{P_H}(\mathbb{A})}\int_{M_n'(\mathbb{A})}\int_{N^+(\mathbb{A})}$$
$$e^{\langle 2\rho_{P_H}, H_{P_H}(m_H)\rangle + \langle 2\rho_{P_n'}, H_{P_n'}(m')\rangle} f^+\ nm\ (m_Hk_H,(g_1',m'k'))\quad (\det m'k')^{n+1}\,dn\,dm'\,dm_H\,dk'\,dk_H.$$

We have

$$M^+(F)/(M_{P_H}(F)\quad M_n'(F)) = M^+_{n,}(F)/M_n'(F);\quad N^+(\mathbb{A}) = N_H(\mathbb{A})N_n^+(\mathbb{A}).$$

Note that element in $N_n^+(F)$ is considered as an element in $N^+(F)$ via $(g,x,y)\mapsto((1,g),x,y)$. Here the multiplication $N_HN_n^+$ means $N_HN_n\quad N^{-,}_{P,L_-}\quad N_{P,L}$. Then the above integral equals

$$(\det g_1')^n \sum_{m\in M^+_{n,}(F)/M_n'(F)} \int_{K_H}\int_{K_2'}\int_{M_{P_H}(\mathbb{A})}\int_{M_n'(\mathbb{A})}\int_{N_H(\mathbb{A})}\int_{N_n^+(\mathbb{A})}$$
$$e^{\langle 2\rho_{P_H}, H_{P_H}(m_H)\rangle + \langle 2\rho_{P_n'}, H_{P_n'}(m')\rangle} f^+\ n_Hmn_2\ (m_Hk_H,(g_1',m'k'))$$
$$(\det m'k')^{n+1}\,dn_2\,dn_H\,dm'\,dm_H\,dk'\,dk_H.$$

We further break up the integral over $N_n^+(\mathbb{A})$ into $N_n^+(\mathbb{A})/N_n'(\mathbb{A})$ and $N_n'(\mathbb{A})$ and arrive at

$$(\det g_1')^n \sum_{m\in M^+_{n,}(F)/M_n'(F)} \int_{K_H}\int_{K_2'}\int_{M_{P_H}(\mathbb{A})}\int_{M_n'(\mathbb{A})}\int_{N_H(\mathbb{A})}\int_{N_n^+(\mathbb{A})/N_n'(\mathbb{A})}\int_{N_n'(\mathbb{A})}$$
$$e^{\langle 2\rho_{P_H}, H_{P_H}(m_H)\rangle + \langle 2\rho_{P_n'}, H_{P_n'}(m')\rangle} f^+\ n_Hmn_2n_2'\ (m_Hk_H,(g_1',m'k'))$$
$$(\det m'k')^{n+1}\,dn_2'\,dn_2\,dn_H\,dm'\,dm_H\,dk'\,dk_H. \tag{12.3}$$

Using the Iwasawa decomposition, we combine the integrals for m', n' and k' as an integral over $G_n'(\mathbb{A})$, and the integrals over m_H, n_H and k_H as an integral over $H(\mathbb{A})$. Note that $(\det mn_2) = 1$ for $m\in M_n'(F)$ and $n\in N_n(\mathbb{A})$. We thus conclude that (12.3) equals

$$(\det g_1') \sum_{m\in M^+_{n,}(F)/M_n'(F)} \int_{N_{n,+}(\mathbb{A})/N_n'(\mathbb{A})} \varphi_{f^+}\ g_1'^{-1}(mn_2)g_1'\ dh\,dn_2.$$

Note that there is a slight difference in the definition of φ_{f^+} for n even or odd, which eventually simplifies to a uniform expression. Finally, the map induces a natural bijection between $M^+_{n,}(F)/M_n'(F)$ and $M_n^+(F)\cap X\ (F)$. This concludes the proof.

12.2. **General linear groups: infinitesimal variant.** Recall that we use gothic letters to denote the Lie algebra of the corresponding group. We slightly extend this convention to the case of an algebraic variety with a distinguished point e where we use the corresponding gothic letter to denote its tangent space at e. For instance, $G^+ = G \quad L^{-;} \quad L$ is a variety with a distinguished point $(1; 0; 0)$, and $\mathfrak{g}^+ = \mathfrak{g} \quad L^{-;} \quad L$ stands for the tangent space at this point.

We now introduce an infinitesimal variant of the distributions i, and relate i to it. This infinitesimal variant in fact is essentially the same as the one that arises from the Jacquet–Rallis relative trace formulae, which are used to study the Bessel periods on unitary groups.

We define $\mathfrak{s}_n := M_n(E)$, the subspace of $M_n(E)$ with pure imaginary entries, it can be identified with the tangent space of S_n at 1. Let $\mathfrak{x} = \mathfrak{s}_n \quad E_n \quad E^{n;}$, viewed as an algebraic variety over F. We write an element in it as $(\; ; w; v)$. It admits a right action of G_n^0 given by

$$(\; ; w; v) \; g^0 = (g^{0\; 1} \; g^0; wg^0; g^{0\; 1}v); \quad (\; ; w; v) \in \mathfrak{x}; \quad g^0 \in G_n^0(F). \tag{12.4}$$

Let $\mathfrak{x} \to B = \mathfrak{x} /\!/ G_n^0$ be the categorical quotient. A concrete description of B is given by [Zha14a, Lemma 3.1]. The categorical quotient B is isomorphic to the affine space over F of dimension $2n$. We will identify B with a closed subspace of the affine space over E of dimension $2n$, consisting of elements

$$(a_1; \dots; a_n; b_1; \dots; b_n)$$

satisfying

$$a_i^c = (\; 1)^i a_i; \quad b_i^c = (\; 1)^{i \; 1} b_i; \quad i = 1; \dots; n.$$

The quotient map $q : \mathfrak{x} \to B$ is given by

$$(\; ; w; v) \mapsto (a_1; \dots; a_n; b_1; \dots; b_n)$$

where

$$a_i = \mathrm{Trace} \wedge^i \; ; \quad b_i = w \; ^{i \; 1} v; \quad i = 1; \dots; n.$$

For any $\in B(F)$, we denote by $\mathfrak{x}$ the inverse image of in $\mathfrak{x}$ (as a closed subscheme).

Let $P = MN \in F$, we put $\mathfrak{x}_M = \mathfrak{m}_n^+ \cap \mathfrak{x}$ and $\mathfrak{x}_N = \mathfrak{n}_n^+ \cap \mathfrak{x}$. For $\in B(F)$, we put $\mathfrak{x}_{M;} = \mathfrak{x}_M \cap \mathfrak{x}$. Here we view all spaces as subspaces of $\mathrm{Res}_{E=F}\, \mathfrak{gl}_{n;E} \quad E_n \quad E^{n;}$ and take the intersections in it.

Take $' \in S(\mathfrak{x}(\mathbb{A}))$ and $\in B(F)$. For $g^0 \in [G_n^0]_{P_n^0}$ we define a kernel function

$$k_{';P;}(g^0) = \sum_{m \in \mathfrak{x}_{M;}(F)} \int_{\mathfrak{x}_N(\mathbb{A})} '((m+n) \; g^0) \mathrm{d}n.$$

For $T \in \mathfrak{a}_{n+1}^0$, we define a modified kernel function

$$k^T_{';}(g^0) = \sum_{P \in F} \quad _P \sum_{\in P_n^0(F) \backslash G_n^0(F)} \widehat{\;}_{P_{n+1}^0}(H_{P_{n+1}^0}(\; g^0) \quad T_{P_{n+1}^0}) k_{';P;}(\; g^0).$$

As in the case of other modified kernels, for a fixed g^0 the sum over is finite.

The next proposition summarizes the main results of [Zyd18], see also Théorèmes 5.1.5.1 and 5.2.1.1 of [CZ21].

Proposition 12.2. *Assume T is sufficiently positive. We have the following assertions.*

(1) *The expression*

$$\sum_{\xi\in B(F)}\int_{[G'_n]} |k^T_{\varphi,\xi}(g')|\,\mathrm{d}g'$$

is finite.

(2) *We put*

$$i^T(\varphi)=\sum_{\xi\in B(F)}\int_{[G'_n]} k^T_{\varphi,\xi}(g')\,\eta(\det g')\mathrm{d}g';$$

then as a function of T, $i^T(\varphi)$ is the restriction of a exponential-polynomial whose purely polynomial term is a constant which we denote by $i(\varphi)$.

(3) *The distribution $\varphi\mapsto i(\varphi)$ is a continuous linear form on $S(\mathfrak{x}(\mathbb{A}))$ that is (G'_n,η) invariant. This means*

$$i(g'\cdot\varphi)=\eta(\det g')\,i(\varphi)$$

for all $g'\in G'_n(\mathbb{A})$ where $g'\cdot\varphi(x)=\varphi(g'^{-1}xg)$.

We now introduce the Cayley transform. Let $E^1=\{x\in E \mid N_{E/F}x=xx^c=1\}$ and take a $\tau\in E^1$. We put

$$X_\tau=\{(\delta;w;v)\in X \mid \tau+\delta \text{ is invertible}\}$$

Then X_τ is an open subscheme of X and the action of G'_n preserves it. We also put

$$\mathfrak{x}'=\{(A;w;v)\mid A^2-4 \text{ is invertible}\}.$$

Then $\mathfrak{x}'$ is an open subscheme of $\mathfrak{x}$ and the action of G'_n preserves it.

We define a Cayley transform

$$\mathfrak{c}_\tau:\mathfrak{x}'\to X_\tau,\quad (A;w;v)\mapsto\left(\tau(1+A/2)(1-A/2)^{-1};w;v\right).$$

Note that $\mathfrak{c}_\tau$ is G'_n-equivariant and there are $\tau_1,\dots,\tau_{n+1}\in E^1$, such that the images of $\mathfrak{c}_{\tau_i}$ form an open cover of X. Let B' be the image of $\mathfrak{x}'$ in B and let A_τ be the image of X_τ in A. Then the Cayley transform $\mathfrak{c}_\tau$ induces an isomorphism $B'\to A_\tau$, which we still denote by $\mathfrak{c}_\tau$.

Let S be a finite set of places. For $\varphi\in S(X_\tau(F_{\mathsf{S}}))$ we define a function $\varphi_\natural\in S(\mathfrak{x}'(F_{\mathsf{S}}))$ as follows. If n is odd, we put

$$\varphi_\natural(A;w;v)=\eta(-1)^{(n-1)^2}\varphi(\mathfrak{c}_\tau(X);w;v).$$

If n is even, we put

$$\varphi_\natural(X;v;w)=\eta(-1)^{n(n-2)}\eta(\det(1-A/2))\varphi(\mathfrak{c}_\tau(A);w;v).$$

Then $\varphi_\natural\in S(\mathfrak{x}'(F_{\mathsf{S}}))$. If $f^+\in S(G^+(F_{\mathsf{S}}))$ then we write $f^+_\natural=(\varphi_{f^+})_\natural$. The various extra factors $\eta(\det(1-X/2))$ and powers of $\eta(-1)$ will be posteriori justified by Lemma 13.5. They all come from the comparison between the transfter factors on $G^+(F_{\mathsf{S}})$ and on $\mathfrak{x}(F_{\mathsf{S}})$.

Proposition 12.3. *Fix an $\in B^0(F)$. Let S be a finite set of places including all Archimedean places, such that for $v \notin S$ we have the following properties.*

- *E/F is unramified at v.*
- *is unramified at any places w of E above v.*
- *$2, \ ; \ ^{-1}$ are in O_{E_w} for any place of w of E above v.*
- *$\in B^0(O_{F_v})$.*

Then for any $f_S \in S(G^+(F_S))$ and $T \in \mathfrak{a}_0$ sufficiently positive, we have

$$i^T_{\mathfrak{c}(\ )}\ f_S \quad 1_{G^+(O^S_F)} = \mathrm{i}^T \ (f_S)_\natural \quad 1_{\mathfrak{g}^+(O^S_F)} \ .$$

Proof. Let us note that

$$' f^+_S \ \mathbf{1}_{G^+(O^S_F)} = \ ' f^+_S \quad \mathbf{1}_{X(O^S_F)}$$

By Lemma 12.1, we are reduced to showing that

$$k_{'} \quad \mathbf{1}_{X(O^S_F)};P;\mathfrak{c}(\ ) = k_{'_\natural} \quad \mathbf{1}_{\mathfrak{x}(O^S_F)};P;$$

for all $\in B(F)$, $P \in F_{\mathrm{RS}}$ and $' \in S(X(F_S))$. This follows from the discussion in [Zyd20, Section 5], cf. [Zyd20, Corollaire 5.8].

We also recall from Section 7 that we have distributions I^T and I on $S(G_+(\mathbb{A}))$. For $f_+ \in S(G_+(\mathbb{A}))$, we have defined a partial Fourier transform

$$^y : S(G_+(\mathbb{A})) \ \to \ S(G^+(\mathbb{A}));$$

cf. (7.4). We also have put

$$I^T(f_+) = i^T(f^+); \quad I \ (f_+) = i \ (f^+).$$

Let S be a finite set of places and $f_{+;S} \in S(G_+(F_S))$ then we put

$$f_{+;S;\natural} = \ f^y_{+;S} \ _\natural.$$

Corollary 12.4. *Let S be a finite set of places satisfying the conditions in Proposition 12.3, and $f_+ = \mathbf{1}_{G_+(O^S_F)} \quad f_{+;S}$ where $f_{+;S} \in S(G_+(F_S))$, then*

$$I^T_{\mathfrak{c}(\ )}\ f_{+;S} \quad 1_{G_+(O^S_F)} = \mathrm{i}^T \ f_{+;S;\natural} \quad 1_{\mathfrak{g}^+(O^S_F)}$$

12.3. **Unitary groups.** The geometric distributions J^T can be simplified and related to their infinitesimal invariants as in the case of general linear groups. We only state the results but omit the proofs, as they are essentially the same as the general linear group case.

Let V be an n-dimensional nondegenerate skew-Hermitian space. We put $\mathfrak{u}^+_V = \mathfrak{u}_V \quad V$, $Y^V = \mathrm{U}(V) \quad V$, and $\mathfrak{y}^V = \mathfrak{u}(V) \quad V$. Recall that F_V denotes the set of standard D-parabolic subgroup of $J(V)$ introduced in Subsection 5.3. For $P \in F_V$, we define subspaces of $\mathfrak{y}^V$, $\mathfrak{p}^+ = \mathfrak{p} \quad P_V$, $\mathfrak{m}^+ = \mathfrak{m} \quad M_{P_V}$, $\mathfrak{n}^+ = \mathfrak{n} \quad N_{P_V}$. They are the Lie algebras for $P^+; M^+; N^+$ introduced in Section 9 respectively.

The group $\mathrm{U}(V)$ has a right action on Y^V or $\mathcal{Y}^V$ by $(A,v)\cdot g=(g^{-1}Ag,g^{-1}v)$. The categorical quotient $q_V:\mathrm{U}_V^+\to \mathcal{A}$ factors through Y^V and identifies the categorical quotient $Y^V/\!/\mathrm{U}(V)$ with $\mathcal{A}$. By [Zha14a, Lemma 3.1] the categorical quotient $\mathcal{Y}^V/\!/\mathrm{U}'_V$ is identified with $\mathcal{B}$. The natural maps $Y^V\to \mathcal{A}_V$ and $\mathcal{Y}^V\to\mathcal{B}$, both denoted by q_V, are given by (the same formula)

$$(A,b)\mapsto (a_1,\dots,a_n;b_1,\dots,b_n),$$

where

$$a_i=\mathrm{Tr}\wedge^i A,\quad b_i=2(-1)^{n-1}\tau^{-1}q_V(A^{i-1}b,b),\quad i=1,\dots,n.$$

For $\xi\in\mathcal{A}(F)$ (resp. $\mathcal{B}(F)$), we denote by Y^V_ξ (resp. $\mathcal{Y}^V_\xi$) the inverse image of ξ in Y^V (resp. $\mathcal{Y}^V$) (as closed subvarieties). The choice of the extra factor $2(-1)^{n-1}\tau^{-1}$ in b_i's will be clear later when we compare the relative trace formulae.

For standard parabolic $P\in\mathcal{F}_V$ and $f\in\mathcal{S}(\mathcal{Y}^V(\mathbb{A}))$, define a kernel function on $[\mathrm{U}(V)]_P$ by

$$k_{f,P}(g)=\sum_{m\in\mathfrak{m}^+(F)}\int_{\mathfrak{n}_+(\mathbb{A})} f((m+n)\cdot g)\mathrm{d}n.$$

For $\xi\in\mathcal{A}(F)$, we define

$$k_{f,P,\xi}(g)=\sum_{m\in\mathfrak{m}^+(F)\cap\mathfrak{u}^+_{V,\xi}(F)}\int_{\mathfrak{n}_+(\mathbb{A})} f((m+n)\cdot g)\mathrm{d}n.$$

For $f\in\mathcal{S}(\mathfrak{u}^+(\mathbb{A}))$ and $T\in\mathfrak{a}_0$, we put

$$k_f^T(g)=\sum_{P\in\mathcal{F}_V}\epsilon_P\sum_{\delta\in P(F)\backslash\mathrm{U}(V)(F)}\hat{\tau}_P(H_P(\delta g)-T_P)k_{f,P}(\delta g),$$

where $g\in[\mathrm{U}(V)]$. Similarly, for $\xi\in\mathcal{B}(F)$, put

$$k_{f,\xi}^T(g)=\sum_{P\in\mathcal{F}_V}\epsilon_P\sum_{\delta\in P(F)\backslash\mathrm{U}(V)(F)}\hat{\tau}_P(H_P(\delta g)-T_P)k_{f,P,\xi}(\delta g).$$

The following theorem summarizes [Zyd18, Theorem 3.1,Theorem 4.5].

Theorem 12.5. *For T sufficiently positive, we have the following assertions.*

(1) *The expression*

$$\sum_{\xi\in\mathcal{B}(F)}\int_{[\mathrm{U}(V)]}\left|k_{f,\xi}^T(g)\right|\mathrm{d}g$$

is finite.

(2) *For $\xi\in\mathcal{B}(F)$ and $f\in\mathcal{S}(\mathfrak{u}_V^+(\mathbb{A}))$, the function of T*

$$\mathcal{I}_\xi^{V,T}(f)=\int_{[\mathrm{U}(V)]}k_{f,\xi}^T(g)\mathrm{d}g$$

is a restriction of an exponential polynomial, and its purely polynomial term is constant, denoted by $\mathcal{I}_\xi^V(f)$.

(3) *The distribution $f\mapsto\mathcal{I}_\xi(f)$ is a continuous linear form on $\mathcal{S}(\mathfrak{u}_V^+(\mathbb{A}))$ that is H-invariant.*

We note relate the geometric distributions on U_V^+ and on y^V. If $f^+ \in S(\mathrm{U}_V^+(\mathbb{A}))$ we put

$$\varphi_{f^+}(\; ; b) = \int_{\mathrm{U}(V)(\mathbb{A})} f^+((g^{-1} \; ; g^{-1} \;); v) \mathrm{d}g. \tag{12.5}$$

Then $\varphi_{f^+} \in S(Y^V(\mathbb{A}))$.

Let Y_V be the open subscheme of Y_V^+ consists of $(g; v)$ such that $\det(g \quad) \neq 0$. Denote by $\mathrm{U}_{V;}$ the open subvariety of U_V^+ consisting of elements $(g; v)$ such that $\det(g \quad) \neq 0$, and by y_V^0 the open subvariety of y_V consists of $(A; b)$ such that $\det(A \quad 2) \neq 0$. Then the (scheme-theoretic) images of Y_V and y_V^0 in A and B respectively are denoted by A and B^0.

We define the Cayley transform

$$\mathfrak{c} : \mathrm{y}_V^0 \to Y_V; \quad (A; b) \mapsto (1 + A/2)(1 - A/2)^{-1}; b \in \mathrm{U}_V^{+;} .$$

It induces a map $\mathfrak{c} : B^0 \to A$. We use the same notation $\mathfrak{c}$ for Cayley transforms on general linear groups and unitary groups, it nevertheless should cause no confusion.

Let S be a finite set of places. For $\varphi \in S(Y_V(F_\mathrm{S}))$, define $\varphi_\natural \in S(\mathrm{y}_V^0(F_S))$ by

$$\varphi_\natural(X) = \varphi(\mathfrak{c}(X)).$$

If $f^+ \in S(\mathrm{U}_V^+(F_\mathrm{S}))$, then we write $f_\natural^+ = (\varphi_{f^+})_\natural$.

Proposition 12.6. *Fix $\in B(F)$. If S is a set of places satisfying the same conditions as Proposition 12.3. Then for any $f_\mathrm{S} \in S(\mathrm{U}_V^{+;}(F_\mathrm{S}))$ and $T \in \mathfrak{a}_0$ sufficiently positive, we have*

$$j^T_{\mathfrak{c}(\;);V} \; f_\mathrm{S} \quad 1_{\mathrm{U}_V^+(O^S)} = \mathrm{j}^T_{;V} \; (f_\mathrm{S})_\natural \quad 1_{\mathrm{y}^V(O^S)} \; .$$

As the proof of Proposition 12.6, this follows from the discussion in [Zyd20, Section 5], cf. [Zyd20, Corollaire 5.8].

We also recall from Section 9 that we have distributions J^T and J on $S(\mathrm{U}_{V;+}(\mathbb{A}))$. For $f_+ \in S(\mathrm{U}_{V;+}(\mathbb{A}))$, we have defined a partial Fourier transform

$$^z : S(\mathrm{U}_{V;+}(\mathbb{A})) \to S(\mathrm{U}_V^+(\mathbb{A})).$$

We also have put

$$J^T(f_+) = j^T(f_+^z); \quad J(f_+) = j(f_+^z).$$

Let S be a finite set of places and $f_{+;\mathrm{S}} \in S(\mathrm{U}_{V;+}(F_\mathrm{S}))$ then we put

$$f_{+;\mathrm{S};\natural} = \left(f^z_{+;\mathrm{S}}\right)_\natural.$$

Corollary 12.7. *Let S be a finite set of places satisfying the conditions in Proposition 12.3, and $f_+ = \mathbf{1}_{\mathrm{U}_{V;+}(O_F^\mathrm{S})} \; f_{+;\mathrm{S}}$ where $f_{+;\mathrm{S}} \in S(\mathrm{U}_{V;+}(F_\mathrm{S}))$, then*

$$J^T_{\mathfrak{c}(\;)} \; f_{+;\mathrm{S}} \quad \mathbf{1}_{\mathrm{U}_{V;+}(O_F^\mathrm{S})} = \mathrm{j}^T \; f_{+;\mathrm{S};\natural} \quad \mathbf{1}_{\mathrm{y}^V(O_F^\mathrm{S})}$$

13. Matching of test functions

13.1. **Regular semisimple orbits and orbital integrals.** Let $(\ ; w; v)$ be an element in either $X(F)$ or $\mathrm{x}(F)$. We say that $(\ ; w; v)$ is regular semisimple if and only if

$$\det(w; w\ ; : : : ; w\ ^{n\ 1}); \quad \det(v;\ v; : : : ;\ ^{n\ 1}v)$$

are both nonzero. Let X_{rs} and x_{rs} be the open subvariety of X and x corresponding to the regular semisimple elements. The actions of $G_n^{\emptyset}$ on X_{rs} and x_{rs} are free. Let $(g; w; v)$ be an element in $G^{+}(F)$. We say it is regular semisimple if its image in $X(F)$ is regular semisimple. Let G^{+}_{rs} be the open subscheme of regular semisimple elements.

Let $(\ ; v)$ be an element in $Y^{V}(F)$ or $\mathrm{y}^{V}(F)$. We say that it is regular semisimple if

$$v;\ v; : : : ;\ ^{n\ 1}v$$

are linearly independent in V. Let Y^{V}_{rs} and $\mathrm{y}^{V}_{\mathrm{rs}}$ be the open subvarieties of Y^{V} and y^{V} respectively corresponding to the regular semisimple elements. The actions of $\mathrm{U}(V)$ on Y^{V}_{rs} and $\mathrm{y}^{V}_{\mathrm{rs}}$ are free. Let $(g; v)$ be an element in $\mathrm{U}^{+}_{V}(F)$ and $g = (g_1; g_2)$, $g_i\ 2\ \mathrm{U}(V)(F)$. We say it is regular semisimple if its image in $Y^{V}(F)$ is regular semisimple. Let $\mathrm{U}^{+}_{V;\mathrm{rs}}$ be the open subvariety of regular semisimple elements.

We say a regular semisimple element in $G^{+}(F)$ and a regular semisimple element in $\mathrm{U}^{+}_{V}(F)$ match if their images in $A(F)$ are coincide. Similarly we say a regular semisimple element in $\mathrm{x}_{n;\mathrm{rs}}(F)$ and a regular semisimple element in $\mathrm{u}^{+}_{V;\mathrm{rs}}(F)$ math if their images in $B(F)$ coincide. The match of regular semisimple elements depends only on the orbits of the corresponding elements, and hence we can speak of the matching of orbits.

Remark 13.1. We denote by $V\ $ the Hermitian space whose underline space is V and the Hermitian form is

$$q_{V\ } = 2(\ 1)^{n\ 1\ \ 1}q_{V}:$$

Then $\mathrm{U}(V\ )$ and $\mathrm{U}(V)$ are physically the same group. Define

$$\mathrm{U}^{+}_{V\ } = \mathrm{U}_{V}\ \ V\ ; \quad \mathrm{y}^{V\ } = \mathrm{u}(V\ )\ \ V\ :$$

They are physically the same as U^{+}_{V} and y^{V} respectively. We define regular semisimple elements in $\mathrm{U}^{+}_{V\ }$ and $\mathrm{y}^{V\ }$ as in the case of V. The spaces $\mathrm{U}^{+}_{V\ }$ and $\mathrm{y}^{V\ }$ are the ones appearing in the Jacquet–Rallis relative trace formulae that are used to study the Bessel periods, cf. [CZ21, Zha14a] for instance. While the identification of y^{V} and $\mathrm{y}^{V\ }$, our notion of matching is exactly the same as that in [BP21b, CZ21, Zha14a]. More precisely regular semisimple elements $(\ ; v; w)\ 2\ \mathrm{x}(F)$ and $(\ ; b)\ 2\ \mathrm{y}^{V}$ match if and only if they match in the sense of [BP21b, CZ21, Zha14a] when $(\ ; b)$ is viewed as an element in $\mathrm{y}^{V\ }$. This identification will make the local calculations later easier to trace.

Proposition 13.2. *The matching of regular semisimple orbits gives bijections*

$$
\begin{aligned}
G^+_{\mathrm{rs}}(F)/H(F)\times G'(F) &\leftrightarrow \coprod_{V\in\mathcal{H}} \mathrm{U}^+_{V,\mathrm{rs}}(F)/\mathrm{U}'_V(F)\times \mathrm{U}'_V(F);\\
\mathcal{X}_{\mathrm{rs}}(F)/G'_n(F) &\leftrightarrow \coprod_{V\in\mathcal{H}} \mathcal{Y}^V_{rs}(F)/\mathrm{U}(V)(F).
\end{aligned}
\tag{13.1}
$$

Let $\mathtt{S}$ be a finite set of places of F. Then we can define the matching of regular semisimple orbits in $G^+_{\mathrm{rs}}(F_{\mathtt{S}})$ and in $\mathrm{U}^+_{V,\mathrm{rs}}(F_{\mathtt{S}})$, and in $\mathcal{X}(F_{\mathtt{S}})$ and $\mathcal{Y}^V(F_{\mathtt{S}})$ in the same way. This again gives the bijection (13.1), with F replaced by $F_{\mathtt{S}}$. We record it again for further references.

$$
\begin{aligned}
G^+_{\mathrm{rs}}(F_{\mathtt{S}})/H(F_{\mathtt{S}})\times G'(F_{\mathtt{S}}) &\leftrightarrow \coprod_{V\in\mathcal{H}_{\mathtt{S}}} \mathrm{U}^+_{V,\mathrm{rs}}(F_{\mathtt{S}})/\mathrm{U}'_V(F_{\mathtt{S}})\times \mathrm{U}'_V(F_{\mathtt{S}});\\
\mathcal{X}_{\mathrm{rs}}(F_{\mathtt{S}})/G'_n(F_{\mathtt{S}}) &\leftrightarrow \coprod_{V\in\mathcal{H}_{\mathtt{S}}} \mathcal{Y}^V_{\mathrm{rs}}(F_{\mathtt{S}})/\mathrm{U}(V)(F_{\mathtt{S}}).
\end{aligned}
\tag{13.2}
$$

Recall that the actions appearing in (13.2) are all free, as we consider only the regular semisimple elements. Then the measures on $G^+_{\mathrm{rs}}(F_{\mathtt{S}})$ and etc. descend naturally to quotient measures on the various quotients in (13.2). The same argument as [BP21b, Lemma 5.11] gives the following lemma.

Lemma 13.3. *The bijections* (13.2) *are measure preserving.*

Let $(g,w,v)\in G^+(F)$ be regular semisimple which maps to $\gamma\in\mathcal{A}(F)$. Let $f_+\in\mathcal{S}(G_+(\mathbb{A}))$. Then the geometric distribution $I_\gamma(f_+)$ simplifies to

$$
I_\gamma(f_+)=\int_{H(\mathbb{A})}\int_{G'(\mathbb{A})} f_+^{y}(h^{-1}gg', wg'_2, g_2'^{-1}v)\,\eta_{G'}(g')\,\mathrm{d}g'\,\mathrm{d}h. \tag{13.3}
$$

The integral on the right hand side depends only on γ but not the specific element (g,w,v) that maps to it. This is the usual regular semisimple orbital integral of f_+.

The global orbital integral factorizes. If $f_{+,v}\in\mathcal{S}(G_+(F_v))$ and (g,w,v) be a regular semisimple element in $G^+(F_v)$, we define the local orbital integral $O((g,w,v),f_{+,v})$ the same way by the same formula (13.3), integrating over $H(F_v)$ and $G'(F_v)$ instead. Then if $f_+=\otimes f_{+,v}$ where $f_{+,v}\in\mathcal{S}(G_+(F_v))$, then

$$
I_\gamma(f_+)=\prod_v O((g,w,v),f_{+,v}).
$$

Let $\mathtt{S}$ be a finite set of places of F and $f_{+,\mathtt{S}}\in\mathcal{S}(G_+(F_{\mathtt{S}}))$, then we can define the regular semisimple orbital integrals $O((g,w,v),f_{+,\mathtt{S}})$ in the same way.

For each $\gamma\in\mathcal{A}(F)$ we have a distribution J_γ on $\mathcal{S}(\mathrm{U}^+_V(\mathbb{A}))$. We will consider this distribution for all V at the same time, so we add a supscript V and write J^V_γ to emphasize the dependance on V. Let $(g,v)\in\mathrm{U}^+_V(F)$ be regular semisimple which maps to $\gamma\in\mathcal{A}(F)$. Let $f^V_+\in\mathcal{S}(\mathrm{U}_{V,+}(\mathbb{A}))$. Then the geometric distribution $J^V_\gamma(f^V_+)$ simplifies to

$$
J^V_\gamma(f^V_+)=\int_{\mathrm{U}'_V(\mathbb{A})}\int_{\mathrm{U}'_V(\mathbb{A})} f_+^{V,y}(x^{-1}gy, y^{-1}v)\,\mathrm{d}x\,\mathrm{d}y. \tag{13.4}
$$

The integral on the right hand side depends only on but not the specific element $(\,; b)$ that maps to it. This is the usual regular semisimple orbital integral of f_+^V.

The global orbital integral factorizes. If $f^V_{+,v} \in \mathcal{S}(\mathrm{U}_{V,+}(F_v))$ and $(\,; b)$ be a regular semisimple element in $\mathrm{U}_{V,+}(F_v)$, we define the local orbital integral $O((\,; b); f^V_{+,v})$ the same way by the same formula (13.4), integrating over $\mathrm{U}(V)(F_v)$ instead. Then if $f^V_+ = \ f^V_{+,v}$ where $f^V_{+,v} \in \mathcal{S}(\mathrm{U}_{V,+}(F_v))$, then

$$J^V(f^V_+) = \prod_v O((\,; b); f^V_{+,v}).$$

Let S be a finite set of places of F and $f_{+,S} \in \mathcal{S}(\mathrm{U}_{V,+}(F_S))$, then we can define the regular semisimple orbital integrals $O((\,; b); f^V_{+,S})$ in the same way.

We now turn to the infinitesimal invariant. Let $(X, w, v) \in \mathfrak{x}(F)$ be regular semisimple which maps to $\ \in \mathcal{B}(F)$. Let $f^+ \in \mathcal{S}(\mathfrak{x}(\mathbb{A}))$. Then the geometric distribution $\mathfrak{i}\ (f^+)$ simplifies to

$$\mathfrak{i}\ (f^+) = \int_{G_n'(\mathbb{A})} f^+(g^{-1}Xg, wg, g^{-1}v) \ (\det g)\mathrm{d}g. \tag{13.5}$$

The integral on the right hand side depends only on but not the specific element (X, w, v) that maps to it. This is the usual regular semisimple orbital integral of f^+ that appeared in [CZ21, Zha14a]. This orbital integral factorize. If $f_{+,v} \in \mathcal{S}(\mathfrak{x}(F_v))$ and (X, w, v) be a regular semisimple element in $\mathfrak{x}(F_v)$, we define the local orbital integral $O((X, w, v); f_{+,v})$ the same way by the same formula (13.5), integrating over $G_n'(F_v)$ instead. Then if $f_+ = \ f_{+,v}$ where $f_{+,v} \in \mathcal{S}(\mathfrak{x}(F_v))$, then

$$\mathfrak{i}\ (f_+) = \prod_v O((X, w, v); f_{+,v}).$$

Let S be a finite set of places of F and $f_{+,S} \in \mathcal{S}(\mathfrak{x}(F_S))$, then we can define the regular semisimple orbital integrals $O((X, w, v); f_{+,S})$ in the same way.

For each $\ \in \mathcal{B}(F)$ we have a distribution $\mathfrak{j}\ $ on $\mathcal{S}(\mathfrak{y}^V(\mathbb{A}))$. Again we will consider this distribution for all V at the same time, so we add a superscript V and write $\mathfrak{j}^V$ to emphasize the dependance on V. Let $(\,; b) \in \mathfrak{y}^V(F)$ be regular semisimple which maps to $\ \in \mathcal{B}(F)$. Let $f^{V,+} \in \mathcal{S}(\mathfrak{y}^V(\mathbb{A}))$. Then the geometric distribution $\mathfrak{j}^V(f^{+,V})$ simplifies to

$$\mathfrak{j}^V(f^{+,V}) = \int_{\mathrm{U}(V)(\mathbb{A})} f^{+,V}(g^{-1}\ g, g^{-1}v)\mathrm{d}g.$$

The integral on the right hand side depends only on but not the specific element $(\,; b)$ that maps to it. This is the usual regular semisimple orbital integral of $f^{+,V}$ that appeared in [CZ21, Zha14a].

The global orbital integral factorizes. If $f^V_{+,v} \in \mathcal{S}(\mathfrak{y}^V(F_v))$ and $(\,; b)$ be a regular semisimple element in $\mathfrak{y}^V(F_v)$, we define the local orbital integral $O((\,; b); f^V_{+,v})$ the same way by the same formula (13.4), integrating over $\mathrm{U}(V)(F_v)$ instead. Then if $f^{+,V} = \ f_v^{+,V}$ where $f_v^{+,V} \in \mathcal{S}(\mathfrak{y}^V(F_v))$, then

$$\mathfrak{j}^V(f^{+,V}) = \prod_v O((\,; b); f_v^{+,V}).$$

Let S be a finite set of places of F and $f_{\mathsf{S}}^{+,V} \in \mathcal{S}(\mathrm{y}^V(F_{\mathsf{S}}))$, then we can define the regular semisimple orbital integrals $O((\ ,b), f_{\mathsf{S}}^{+,V})$ in the same way.

13.2. **Transfer.** We now compare the geometric distributions. Let H be the set of all isomorphism classes of nondegenerate skew-Hermitian spaces over E of dimension n. If S is a finite set of places, then we let H_{S} be the set of all isomorphism classes of nondegenerate skew-Hermitian spaces over E_{S} of dimension n. If we assume furthermore that S contains all Archimedean places and ramified places, then we denote by H^{S} the set of all isomorphism classes of nondegenerate skew-Hermitian spaces V over E of dimension n, such that $V_v = V \otimes E_v$ contains a self-dual O_{E_v} lattice for all $v \notin \mathsf{S}$.

13.2.1. *Local transfer.* We start from the local situation. Fix a place v of F. Let us introduce the transfer factors. For regular semisimple $x = (X, w, v) \in X(F_v)$ or $\mathrm{x}(F_v)$ we put

$$\Delta_+(x) = \det(w, wX, wX^2, \dots, wX^{n-1}).$$

Define a transfer factor Ω_v for $X(F_v)$ or $\mathrm{x}(F_v)$ by $\Omega_v(x) = \eta((-1)^n\Delta_+(x))$. The sign $(-1)^n$ is included to make it compatible with the transfer factor in [BP21b]. The function $x \mapsto \Omega_v(x)O(x,\varphi')$ where $x \in \mathrm{x}_{rs}(F_v)$ and $\varphi' \in \mathcal{S}(\mathrm{x}(F_v))$ descends to a function on $B_{rs}(F_v)$, which we denote by the same notation.

We define a transfer factor Ω_v on $G^+(F_v)$ as follows. Let $((g_1, g_2), w, v) \in G^+(F_v)$. Put $\nu = \nu(g_1^{-1}g_2) \in S_n(F_v)$. We put

$$\Omega_v((g_1, g_2), w, v) = \eta\left((-1)^n(\det g_1^{-1}g_2)^{-n+1}\det\Delta_+(\nu, w, v)\right).$$

The function $x \mapsto \Omega_v(x)O(x, f_+)$ where $x \in G_+(F_v)$ is regular semisimple and $f_+ \in \mathcal{S}(G_+(F_v))$ descends to a function on $A_{\mathrm{rs}}(F_v)$, which we also denote it by $x \mapsto \Omega_v(x)O(x, f_+)$.

A function $f_+ \in \mathcal{S}(G_+(F_v))$ and a collection of functions $\{f_+^V\}_{V \in H_v}$ where $f_+^V \in \mathcal{S}(\mathrm{U}_{V,+}(F_v))$ match if for all matching regular semisimple elements $x \in G^+(F_v)$ and $y \in \mathrm{U}_V^+(F_v)$, we have

$$\Omega_v(x)O(x, f_+) = O(y, f_+^V).$$

We say a function on $G_+(F_v)$ is transferable if there is a collection of functions on $\mathrm{U}_{V,+}(F_v)$ that matches it. We say a collection of functions on $\mathrm{U}_{V,+}(F_v)$ is transferable if there is a function on $G_+(F_v)$ that matches it. We say a single function φ^V on $\mathrm{U}_{V,+}(F_v)$ for a fixed V is transferable if the collection of functions $(\varphi^V, 0, \dots, 0)$ is transferable.

We say $\varphi' \in \mathcal{S}(\mathrm{x}(F_v))$ and a collection of functions $\{\varphi^V\}_{V \in H_v}$ where $\varphi^V \in \mathcal{S}(\mathrm{y}^V(F_v))$ match if for all matching regular semisimple elements $x \in \mathrm{x}_n(F_v)$ and $y \in \mathrm{y}^V(F_v)$, we have

$$\Omega_v(x)O(x, \varphi') = O(y, \varphi^V).$$

We say a function on $\mathrm{x}(F_v)$ is transferable if there is a collection of functions on $\mathrm{y}^V(F_v)$ that matches it. We say a collection of functions on $\mathrm{y}^V(F_v)$ transferable if there is a function on $\mathrm{x}(F_v)$

that matches it. We say a single function φ^V on $\mathfrak{y}^V(F)$ for a fixed V is transferable if the collection of functions $(\varphi^V, 0, \ldots, 0)$ is transferable.

Theorem 13.4. *If F is non-Archimedean, all functions in*

$$\mathcal{S}(G_+(F_v)),\ \mathcal{S}(\mathfrak{x}(F_v)),\ \mathcal{S}(G_+^V(F_v)),\ \ \mathcal{S}(\mathfrak{y}^V(F_v))$$

are transferable. If F is Archimedean, transferable functions form dense subspaces of these spaces.

Proof. If F is non-Archimedean, then the cases of $\mathcal{S}(\mathfrak{x}(F_v))$ and $\mathcal{S}(\mathfrak{y}^V(F_v))$ are proved in [Zha14a]. The cases of $\mathcal{S}(G_+(F_v))$ and $\mathcal{S}(\mathrm{U}_{V,+}(F_v))$ are explained in [Xue14]. It follows from the fact that the integral transforms (12.1) and (12.5) are surjective.

If F_v is Archimedean, the cases of $\mathcal{S}(\mathfrak{x}(F_v))$ and $\mathcal{S}(\mathfrak{y}^V(F_v))$ are proved in [Xue19]. The cases of $\mathcal{S}(G_+(F_v))$ and $\mathcal{S}(\mathrm{U}_{V,+}(F_v))$ follow from the surjectivity of (12.1) and (12.5) and the open mapping theorem, cf. [Xue19, Section 2]. The surjectivity is a little more complicated than its non-Archimedean counterpart. Let us explain the case of $\mathcal{S}(G_+(F_v))$. First the Fourier transform $\dagger$ is a continuous bijection. The map $f^+ \mapsto \varphi_{f^+}$ is surjective since $G^+(F_v) \to X(F_v)$ is surjective and submersive, cf. [AG08, Theorem B.2.4]. The open mapping theorem then ensures that the inverse image of a dense subset in $\mathcal{S}(X(F_v))$ is again dense in $\mathcal{S}(G_+(F_v))$.

Recall that we have defined maps $f_+ \mapsto f_{+,\natural}$ and $f_+^V \mapsto f_{+,\natural}^V$ in Subsections 12.2 and 12.3. Such maps depend on an element $\tau \in E^1$. Fix such a τ. The following lemma follows from the definition and a short calculation of the transfer factors.

Lemma 13.5. *If f_+ and the collection $\{f_+^V\}_{V \in \mathcal{H}_v}$ match, then so do $f_{+,\natural}$ and the collection $\{f_{+,\natural}^V\}_{V \in \mathcal{H}_v}$.*

There is a pairing on $\mathfrak{x}(F_v)$ given by

$$\langle (X_1, w_1, v_1), (X_2, w_2, v_2) \rangle = \mathrm{Trace}\, X_1 X_2 + w_1 v_2 + w_2 v_1. \tag{13.6}$$

If $\varphi \in \mathcal{S}(\mathfrak{x}(F_v))$ we define its Fourier transform

$$\widehat{\varphi}(y) = \int_{\mathfrak{x}(F_v)} \varphi(x) \overline{\psi(\langle x, y \rangle)} \mathrm{d}x.$$

Let V be a nondegenerate skew-Hermitian space. Define a bilinear form on $\mathfrak{y}^V(F_v)$ by

$$\langle (X, v), (Y, w) \rangle = \mathrm{Trace}\, XY + 2(-1)^{n-1} \mathrm{Tr}_{E/F}\, \tau^{-1} q_V(v, w),$$

and a Fourier transform

$$\mathcal{F}_V \varphi^V(y) = \int_{\mathfrak{y}^V(F_v)} \varphi^V(x) \overline{\psi(\langle x, y \rangle)} \mathrm{d}x.$$

Remark 13.6. This essentially means that we consider φ^V as an element in $\mathcal{S}(\mathfrak{y}^V(F_v))$ an take the Fourier transform as in [BP21b, CZ21, Zha14a], with ψ replaced by our $\overline{\psi}$.

Let v be a place of F. Set $(\ _v) = (\frac{1}{2}; \ _v; \ _v)$ where $(s; \ _v; \ _v)$ is the local root number. This is a fourth root of unity and satisfies the property that

$$(\overline{\ _v}) = \ _v(\ 1) \ (\ _v):$$

In the case $E_v{=}F_v = \mathrm{C}{=}\mathrm{R}$ and $\ _v(x) = e^{2 \ p\overline{\ \ 1}x}$ we have $(\ _v) = \ p\overline{\ \ 1}$.

Theorem 13.7. *Suppose that ' 2 S(x(F_v)) and a collection of functions '^{V} 2 S(y^V(F_v)) match. Then so do b and the collection* $(\ _v)^{\frac{n(n+1)}{2}} \ _v((2 \ )^n \operatorname{disc} V)^n F_V \ '^V$*.*

This is proved in [Zha14a] if v is non-Archimedean and [Xue19] if v is Archimedean. See [BP21b, Theorem 5.32, Remark 5.33] for corrections to [Zha14a].

Remark 13.8. Note that $\operatorname{disc} V \ = (2 \ )^n \operatorname{disc} V$. We also note that we used $\overline{\ _v}$ in the Fourier transform. So $(\ _v)$ in [BP21b] is replaced by $(\overline{\ _v})$ and this cancels the factor $\ _v(\ 1)^{\frac{n(n+1)}{2}}$ that appeared in [BP21b].

Finally we turn to the matching of test functions in the unramified situation, i.e. the fundamental lemma.

Theorem 13.9. *Assume the following conditions.*

(1) *The place v is odd and unrami ed. The element 2 o_{E_v}. The characters $\ _v$ and $\ _v$ are unrami ed.*
(2) *The skew-Hermitian space V is split and contains selfdual o_{E_v} lattice.*

Then the functions

$$\Delta_{H(F_v)}\Delta_{G^0(F_v)} \mathbf{1}_{G_+(O_{F_v})} \quad \textit{and} \quad \Delta^2_{\mathrm{U}(V)(F_v)} \mathbf{1}_{\mathrm{U}_{V;+}(O_{F_v})}$$

match. The functions

$$\Delta_{H(F_v)}\Delta_{G^0(F_v)} \mathbf{1}_{\mathrm{x}(O_{F_v})} \quad \textit{and} \quad \Delta^2_{\mathrm{U}(V)(F_v)} \mathbf{1}_{\mathrm{y}^V(O_{F_v})}$$

also match. Recall that $\Delta_{H(F_v)}$, $\Delta_{G^0(F_v)}$ and $\Delta_{\mathrm{U}(V)(F_v)}$ are the volumes de ned in Subsection 3.4.

Proof. The infinitesimal version was first proved by [Yun11] when the residue field characteristic is large. The group version is deduced from the infinitesimal version in [Liu14, Theorem 5.15] under the same assumption. The case of small residue characteristic in the infinitesimal case was later proved by [BP21c, Zha21]. The only reason [Liu14] needs the large residue characteristic is because [BP21c, Zha21] were not available that time. Once this is available, the result in [Liu14] holds without the assumption on the residue characteristic (apart from being odd). Indeed the group version of the fundamental lemma in the theorem is called the "semi-Lie" case in [Zha21].

13.2.2. *Global transfer.* We now turn to the global situation, so that F is a number field. Recall that for any reductive group G over F and finite set of places S of F, we have defined in Subsection 3.4 the number $\Delta_{\mathrm{G}}^{\mathrm{S},}$ to be the leading coefficient in the Laurent expansion at $s = 0$ of the partial Artin–Tate L function $L_{\mathrm{G}}^{\mathrm{S}}$.

We say a test function $f_+ \in S(G_+(\mathrm{A}))$ and a collection of test functions $\{f_+^V\}_{V \in H}$ where $f_+^V \in S(\mathrm{U}_V^+(\mathrm{A}))$ match, if there exists a finite set of places S of F containing all Archimedean place and ramified place in E, such that we have the following conditions.

- $f_+^V = 0$ for $V \notin H^{\mathrm{S}}$.
- For each $V \in H^{\mathrm{S}}$, $f_+^V = (\Delta_{\mathrm{U}(V)}^{\mathrm{S},})^2 f_{+,\mathrm{S}}^V \quad \mathbf{1}_{\mathrm{U}_V^+(O^{\mathrm{S}})}$ where $f_{+,\mathrm{S}}^V \in S(\mathrm{U}_V^+(F_{\mathrm{S}}))$.
- $f_+ = \Delta_H^{\mathrm{S},} \Delta_{G'}^{\mathrm{S},} f_{+,\mathrm{S}} \quad \mathbf{1}_{G_+(O^{\mathrm{S}})}$, where $f_{+,\mathrm{S}} \in S(G_+(F_{\mathrm{S}}))$.
- $f_{+,\mathrm{S}}$ and $\{f_{+,\mathrm{S}}^V\}_{V \in H^{\mathrm{S}}}$ match.

Similarly, we say $f \in S(\mathrm{x}(\mathrm{A}))$ and the collection $\{f^V\}_{V \in H}$ where $f^V \in S(\mathrm{u}_V^+(\mathrm{A}))$ match if there exists a finite set of places S of F, such that we have the following conditions.

- $f^V = 0$ for $V \notin H^{\mathrm{S}}$.
- For each $V \in H^{\mathrm{S}}$, $f^V = \Delta_{\mathrm{U}(V)}^{\mathrm{S},} f_{\mathrm{S}}^V \quad \mathbf{1}_{\mathrm{u}_V^+(O^{\mathrm{S}})}$ where $f_{\mathrm{S}}^V \in S(\mathrm{u}_V^+(F_{\mathrm{S}}))$.
- $f = \Delta_H^{\mathrm{S},} \Delta_{G'}^{\mathrm{S},} f_{\mathrm{S}} \quad \mathbf{1}_{\mathrm{x}(O^{\mathrm{S}})}$, where $f_{\mathrm{S}} \in S(\mathrm{x}(F_{\mathrm{S}}))$.
- f_{S} and $\{f_{\mathrm{S}}^V\}_{V \in H^{\mathrm{S}}}$ match.

Note that both in the group case and Lie algebra case, f and $\{f^V\}_{V \in H}$ are transfer for the set of places S as above, then they are also transfer for any set of places containing S.

The following theorem is proved in [CZ21, Theorem 13.3.4.1]. This is what is referred to as the singular transfer in [CZ21].

Theorem 13.10. *If $f \in S(\mathrm{x}(\mathrm{A}))$ and the collection $\{f^V\}_{V \in H}$ where $f^V \in S(\mathrm{y}^V(\mathrm{A}))$ match, then for each $\in B(F)$, we have*

$$\mathrm{i}(f) = \sum_{V \in H} \mathrm{j}^V(f^V).$$

From this we deduce our group version of singular transfer.

Theorem 13.11. *If $f_+ \in S(G_+(\mathrm{A}))$ and $\{f^V\}_{V \in H}$ where $f_+^V \in S(\mathrm{U}_{V,+}(\mathrm{A}))$ are transfer, then for each $\in A(F)$, we have*

$$I(f_+) = \sum_{V \in H} J^V(f_+^V).$$

Proof. Fix an $\in A(F)$. Choose a large finite set of places S as in the definition of matching of global test function. Then

$$f_+ = \mathbf{1}_{G_+(O_F^{\mathrm{S}})} \quad f_{+,\mathrm{S}}, \quad f_+^V = \mathbf{1}_{\mathrm{U}_{V,+}(O_F^{\mathrm{S}})} \quad f_{+,\mathrm{S}}^V.$$

We may enlarge S such that the conditions of Proposition 12.3 hold.

Choose $\tau \in E^-$ with norm 1 such that $\pi \in \mathcal{A}_\pi(F)$. Then we have the functions $f_{+,S,\natural}$ and $f^V_{+,S,\natural}$ and they still match by Lemma 13.5. By Proposition 12.3 and 12.6, we have

$$I_\pi(f_+) = i_\pi(\Delta^{S,*}_{G'_n} \mathbf{1}_{X(\mathcal{O}^S)} \otimes f_{+,S,\natural}), \quad J^V_\pi(f^V_+) = j^V_\pi(\Delta^{S,*}_{\mathrm{U}(V)} \mathbf{1}_{Y^V(\mathcal{O}^S)} \otimes f^V_{+,S,\natural}).$$

By definition, $f_{+,S,\natural}$ and $f^V_{+,S,\natural}$ match. The theorem then follows from Theorem 13.10.

13.3. **Nilpotent orbital integrals.** This section studies matching of the nilpotent orbital integrals. This is a local problem, and we will fix a place v of F. We will also speak of the functions $f_{+,\natural}$ where f_+ is a test function either on $G_+(F_v)$ or $\mathrm{U}_{V,+}(F_v)$. For this we mean that we fix $\tau = 1$ when defining these functions.

Let us start from the case of unitary groups. Let $V \in \mathcal{H}_v$ be a nondegenerate skew-Hermitian space of dimension n. Let $f_+ \in \mathcal{S}(\mathrm{U}_{V,+}(F_v))$. Put

$$O(1, f_+) = f_{+,\natural}(0).$$

We thus have by the Fourier inversion formula

$$O(1, f_+) = \int_{y^V} \mathcal{F}_V f_{+,\natural}(y) \mathrm{d}y. \tag{13.7}$$

Clearly the map $f_+ \mapsto O(1, f_+)$ is a continuous linear form on $\mathcal{S}(\mathrm{U}_{V,+}(F_v))$.

Let us move to the more complicated case of linear groups. Set

$$\nu_+ = \left((-1)^n \begin{pmatrix} 0 & 1 & & \\ & 0 & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{pmatrix}, (0, \dots, 0), \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \right) \in \mathsf{X}(F_v),$$

and

$$\nu_- = \left((-1)^n \begin{pmatrix} 0 & & & \\ 1 & 0 & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{pmatrix}, (0, \dots, 0, \tau), \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \right) \in \mathsf{X}(F_v).$$

Define for $\varphi \in \mathcal{S}(\mathsf{X}(F_v))$

$$O(s, \nu_+, \varphi) = \int_{G'_n(F_v)} \varphi(\nu_+ h) \eta(\det h) |\det h|^s \mathrm{d}h.$$

By [BP21b, Proposition 5.7.1] this integral of $O(s, \nu_+, \varphi)$ is absolutely convergent if $\mathrm{Re}\, s > 1 - \frac{1}{n}$. Moreover the function $s \mapsto O(s, \nu_+, \varphi)$ has a meromorphic continuation to $\mathbb{C}$ which is holomorphic at $s = 0$. Setting $O(\nu_+, \varphi) = O(s, \nu_+, \varphi)|_{s=0}$, we have

$$\kappa_v \eta_v((-1)^{n-1}\tau)^{\frac{n(n+1)}{2}} O(\nu_+, \varphi) = \int_{\mathsf{X}(F_v)} \Omega_v(x) \hat{\varphi}(x) \mathrm{d}x = \int_{B_{\mathrm{rs}}(F_v)} \Omega_v(x) O(x, \hat{\varphi}) \mathrm{d}x, \tag{13.8}$$

where

$$(13.9) \qquad {}_{v} = \prod_{i=1}^{n} (1 \quad i; \ {}^{i}_{v}; \ \overline{\ {}_{v}}).$$

Remark 13.12. Strictly speaking, the results in [BP21b] describe the nilpotent orbital integrals for the conjucation action of G_n^{0} on $\mathfrak{s}_{n+1}$ where G_n^{0} is viewed as a subgroup of G_{n+1}^{0} on the left right corner. This action is essentially the same as ours, since the lower right corner element in $\mathfrak{s}_{n+1}$ is invariant under the action of G_n^{0}. Our choice of   is made so that they corresponds exactly to the ones used in [BP21b].

If $f_+ \in S(G_+(F_v))$ we put

$$O_+(f_+) = O(\ {}_+; f_{+;\natural});$$

where in the map $f_+ \mapsto f_{+;\natural}$, we pick $= 1$ and an arbitrary small U. The maps $' \mapsto O(\ {}_+; ')$ and $f_+ \mapsto f_{+;\natural}$ are continuous. Thus the map $f_+ \mapsto O_+(f_+)$ is also continuous.

Lemma 13.13. *Let $f_+ \in S(G_+(F_v))$ and the collection $f_+^V \in S(\mathrm{U}_{V;+}(F_v))$ be matching test functions. Then*

$$ {}_{v}O_+(f_+) = \sum_{V \in H_v} {}_{v}((2 \ )^n \operatorname{disc} V)^n \ {}_{v}((\ 1)^{n \ 1} \ )^{\frac{n(n+1)}{2}} (\ {}_{v})^{\frac{n(n+1)}{2}} O(1; f_+^V).$$

Proof. By (13.8) we have

$$ {}_{v} \ {}_{v}((\ 1)^{n \ 1} \ )^{\frac{n(n+1)}{2}} O_+(f_+) = \int_{B_{\mathrm{rs}}(F_v)} \Omega_v(x) O(x; \hat{f}_{+;\natural}) \mathrm{d}x.$$

By Lemma 13.3, matching of orbits preserves measures, and by Theorem 13.7, $\hat{f}_{\natural}$ and the collection of functions $(\ {}_{v})^{\frac{n(n+1)}{2}} \ {}_{v}((2 \ )^n \operatorname{disc} V)^n F_V f_{+;\natural}^V$ match. Thus we have

$$\begin{aligned} & {}_{v} \ {}_{v}((\ 1)^{n \ 1} \ )^{\frac{n(n+1)}{2}} O_+(f_+) \\ = & \sum_{V} (\ {}_{v})^{\frac{n(n+1)}{2}} \ {}_{v}((2 \ )^n \operatorname{disc} V)^n \int_{B_{\mathrm{rs}}(F_v)} O(y; F_V f_{+;\natural}^V) \mathrm{d}y. \end{aligned}$$

By (13.7), the right hand side equals

$$\sum_{V} (\ {}_{v})^{\frac{n(n+1)}{2}} \ {}_{v}((2 \ )^n \operatorname{disc} V)^n O(1; f_+^V).$$

This proves the lemma.

14. Preliminaries on the spectral comparison

From now on till the end of Section 17, we will work at a single place v of F. So we fix a place v of F and suppress it all notation. So F is a local field of characteristic zero, and E a quadratic etale algebra over F. We will assume that E is a field in this and the next two sections. The case $E = F \quad F$ will be treated in Section 17.

14.1. **General notation and conventions.** Let us introduce some general notation.

Let X be an algebraic variety over F. We usually just write X for the F-points of X. All algebraic groups are F-groups. If V is a vector space over E, it is viewed as an affine variety over F via restriction of scalars.

Let G be a reductive group over F. Let Ξ^G be the Harish-Chandra Xi function on G. It depends on the choice of a maximal compact subgroup K of G. Different choices lead to equivalent functions. Since we use Ξ^G only for the purpose of estimates, the choice of K does not matter. Its properties we will make use of are recorded in [BP20, Proposition 1.5.1]. We also fix a logarithmic height function ς^G on $G(F)$, cf. [BP20, Section 1.5].

Let X be a smooth algebraic variety over a local field F of characteristic zero. Let $\mathcal{S}(X)$ be the Schwartz space. If F is non-Archimedean this is $C_c^\infty(X)$, the space of locally constant compactly supported functions. If F is Archimedean, this is the space of Schwartz function on X where X is viewed as a Nash manifold, cf. [AG08].

Let $\mathcal{C}(G)$ and $\mathcal{C}^w(G)$ be the spaces of Harish-Chandra Schwartz functions and tempered functions on G respectively, cf. [BP21b, Section 2.4]. Assume G is quasi-split, and let N be the unipotent radical of a Borel subgroup, be a generic character of N. We also have the spaces $\mathcal{S}(N\backslash G, \ )$ and $\mathcal{C}^w(N\backslash G, \ )$ as defined in [BP21b, Section 2.4]. These are nuclear Fréchet spaces or LF spaces. An important property we will use is that $\mathcal{S}(G)$ is dense in $\mathcal{C}^w(G)$.

14.2. **Measures.** We fix self-dual measures on F and E with respect to , and on E with respect to $_E$. We have

$$\mathrm{d}_{E} \ (\ ^{1}y) = | \ |_E^{\frac{1}{2}} \mathrm{d}_F y, \quad \mathrm{d}_E z = 2\mathrm{d}_F x \mathrm{d}_{E} \ y, \text{ if } z = x + y.$$

The (normalized) absolute values on E and F are denoted by $|\ |_E$ and $|\ |$ respectively. They satisfy the properties that $\mathrm{d}(ax) = |a|\mathrm{d}x$ and $|x|_E = |x|^2$ if $x \in F$.

We equip $\mathrm{GL}_n(F)$ (resp. $G_n = \mathrm{GL}_n(E)$) with the Haar measure

$$\mathrm{d}g = \frac{\prod_{i,j} \mathrm{d}_F g_{i,j}}{|\det g|^n}, \qquad \text{resp. } \mathrm{d}g = \frac{\prod_{i,j} \mathrm{d}_E g_{i,j}}{|\det g|_E^n}.$$

By taking products, we obtain Haar measures on G, G' and H.

The tangent space of S_n at $s = 1$ is $\mathfrak{s}_n$. There is a bilinear form on $\mathfrak{s}_n$ given by

$$\langle X, Y \rangle = \operatorname{Tr} XY.$$

Then, using the character , we put a self-dual measure on $\mathfrak{s}_n$. This is the same as the measure obtained by identifying $\mathfrak{s}_n$ with $(E \ )^{n^2}$. Recall that we have defined the Cayley transform (where we consider only the case $\ = 1$ now and we omit the subscript)

$$\mathfrak{c} : \mathfrak{s}_n \dashrightarrow S_n, \quad X \mapsto \frac{1 + X/2}{1 - X/2}.$$

We equip S_n with the unique G_n'-invariant measure such that $\mathfrak{c}$ is locally measure preserving near $X = 0$. If F is non-Archimedean, this means that $\mathfrak{c}(U)$ and U have the same volume if U is a small enough neighbourhood of zero. If F is Archimedean, this means that the ratio of the volumes of $c(U)$ and U tends to 1 as we shrink the neighbourhood U to 0.

Let V be a Hermitian or skew-Hermitian space. There is a bilinear form on the Lie algebra $\mathfrak{u}(V)$ of $\mathrm{U}(V)$, given by

$$\langle X, Y\rangle = \mathrm{Tr}\, XY.$$

Then using the character ψ, we put a self-dual measure on $\mathfrak{u}(V)$. We have a Cayley transform

$$\mathfrak{c} : \mathfrak{u}(V) \dashrightarrow \mathrm{U}(V), \quad X \mapsto \frac{1 + X/2}{1 - X/2}.$$

We equip $\mathrm{U}(V)$ with the Haar measure $\mathrm{d}h$ normalized so that the Cayley transform is locally measure preserving near $X = 0$, which is interpreted in the same way as in the case of $\mathfrak{s}_n$. This yields measures on U^0_V and U_V.

Remark 14.1. Let us temporarily switch back to the global setting, so that F is a number field. It follows from the discussion in [BP21a, § 2.5] that for $\mathsf{G} \in \{\mathrm{GL}_n, G_n, \mathrm{U}(V)\}$ the normalized products over the places v of F of our local measures

$$\mathrm{d}g = (\Delta^*_{\mathsf{G}})^{-1} \prod_v \Delta_{\mathsf{G},v} \mathrm{d}g_v$$

are factorizations of the Tamagawa measures defined in Subsection 3.4. Moreover, for $\mathsf{G} = \mathrm{GL}_n$ or G_n, we even have $\mathrm{d}g_v = \mathrm{d}_{\psi_v} g_v$ for every v.

14.3. **Representations.** Let G be a reductive group over a local field F of characteristic zero. If F is non-Archimedean, a representation of G means a smooth representation. If F is Archimedean, a representation of G means a smooth representation of moderate growth. For $\lambda \in \mathfrak{a}^*_{G,\mathbb{C}}$ and a representation π of G, we put

$$\pi_\lambda(g) = \pi(g) e^{\langle \lambda, H_G(g)\rangle}.$$

We denote by $\Pi_2(G)$ (resp. $\mathrm{Temp}(G)$) the set of isomorphism classes of irreducible square integrable (resp. tempered) representations of $G(F)$.

Let $\mathcal{X}_{\mathrm{temp}}(G)$ be the set of isomorphism classes of the form $\mathrm{Ind}_P^G \sigma$, where $P = MN$ ranges over all parabolic subgroups of G and $\sigma \in \Pi_2(M)$. Since each $\pi \in \mathrm{Temp}(G)$ is a subrepresentation of an element in $\mathcal{X}_{\mathrm{temp}}(G)$. We therefore have a map $\mathrm{Temp}(G) \to \mathcal{X}_{\mathrm{temp}}(G)$. The set $\mathcal{X}_{\mathrm{temp}}(G)$ has a structure of a topological space with infinitely many connected components, and the connected components are of the shape

$$\mathcal{O} = \{\mathrm{Ind}_P^G \sigma_\lambda \mid \lambda \in \sqrt{-1}\mathfrak{a}^*_M\}.$$

Let $W(G, M) = N_G M / M$ be the Weyl group and

$$W(G, \sigma) = \{w \in W(G, M) \mid w\sigma \simeq \sigma\}.$$

Then the map

$$\mathrm{Ind}_P^G : \sqrt{-1}\mathfrak{a}^*_M / W(G, \sigma) \to \mathcal{O}, \quad \lambda \mapsto \mathrm{Ind}_P^G \sigma_\lambda,$$

is surjective, and a local homeomorphism at 0.

Let V be a Fréchet space or more generally an LF space. We shall frequently use the notion of Schwartz functions on $\mathcal{X}_{\mathrm{temp}}(G)$ valued in V. This space is denoted by $\mathcal{S}(\mathcal{X}_{\mathrm{temp}}(G), V)$. If $V = \mathbb{C}$ we simply write $\mathcal{S}(\mathcal{X}_{\mathrm{temp}}(G))$. This notion is defined and discussed in detail in [BP21b, Section 2.9]. In particular $\mathcal{S}(\mathcal{X}_{\mathrm{temp}}(G))$ is a Fréchet space if F is Archimedean and an LF space if F is non-Archimedean. The most important property of a Schwartz function on $\mathcal{X}_{\mathrm{temp}}(G)$ valued in V is that it is absolutely integrable with respect the measures on $\mathcal{X}_{\mathrm{temp}}(G)$ that we define below.

If $f \in \mathcal{S}(G)$, we put for each $\pi \in \mathcal{X}_{\mathrm{temp}}(G)$

$$f_\pi(g) = \mathrm{Trace}(\pi(g^{-1})\pi(f)), \quad g \in G.$$

Then $f_\pi \in \mathcal{C}^w(G)$ and by [BP21b, Proposition 2.13.1] the map

$$\mathcal{X}_{\mathrm{temp}}(G) \to \mathcal{C}^w(G), \quad \pi \mapsto f_\pi \tag{14.1}$$

is Schwartz.

Let $\widehat{A_M}$ be the set of unitary characters of A_M. It admits a normalized measure $\mathrm{d}\chi$, cf. [BP21b, (2.4.2)]. The map

$$\sqrt{-1}\mathfrak{a}_M^* \to \widehat{A_M}, \quad \lambda \mapsto (a \mapsto \chi(a)|a|^{\lambda})$$

is locally a homeomorphism. Let $\mathrm{d}\lambda$ be the measure on $\sqrt{-1}\mathfrak{a}_M^*$ such that this map is locally measure preserving.

We now assign measures on $\mathcal{X}_{\mathrm{temp}}(G)$ following [BP21b, Section 2.7]. Consider first $\Pi_2(M)$. We assign the unique measure $\mathrm{d}\sigma$ to $\Pi_2(M)$ such that the map

$$\Pi_2(M) \to \widehat{A_M}, \quad \sigma \mapsto \omega_\sigma|_{A_M}$$

is locally measure preserving. Here ω_σ stands for the central character of σ. Next we consider the map

$$\mathrm{Ind}_P^G : \Pi_2(M) \to \mathcal{X}_{\mathrm{temp}}(G), \quad \sigma \mapsto \mathrm{Ind}_P^G \sigma.$$

This map is quasi-finite and proper, and the image is a collection of some connected components of $\mathcal{X}_{\mathrm{temp}}(G)$. We thus equip the image with the pushforward measure

$$\frac{1}{|W(G,M)|}\mathrm{Ind}^G_{M,*}\,\mathrm{d}\sigma.$$

The measure on $\mathcal{X}_{\mathrm{temp}}(G)$ whose restriction to the image of Ind_P^G equals this one is denoted by $\mathrm{d}\pi$. Near a point $\pi_0 = \mathrm{Ind}_P^G \sigma \in \mathcal{X}_{\mathrm{temp}}(G)$, this measure can be described more explicitly as follows. Let V be a sufficiently small $W(G,\sigma)$-invariant neighbourhood of $\sqrt{-1}\mathfrak{a}_M^*$ such that the map Ind_P^G induces a topological isomorphism between $V/W(G,\sigma)$ and a small neighbourhood U of π_0. Then for every $\varphi \in C_c^\infty(U)$ we have

$$\int_U \varphi(\pi)\,\mathrm{d}\pi = \frac{1}{|W(G,\sigma)|}\int_V \varphi(\mathrm{Ind}_P^G(\sigma_\lambda))\,\mathrm{d}\lambda.$$

There is unique Borel measure on $X_{\mathrm{temp}}(G)$ called the Plancherel measure ([HC76], [Wal03]) and denoted by $\mathrm{d}\mu_G$. It is characterized by that

$$f = \int_{X_{\mathrm{temp}}(G)} f_\pi \, \mathrm{d}\mu_G(\pi), \tag{14.2}$$

for all $f \in S(G)$. Here the integration of the right hand side is taken in $C^w(G)$, and is absolutely convergent in the sense of [BP21b, Section 2.3].

There is a function μ_G such that $\mathrm{d}\mu_G(\pi) = \mu_G(\pi)\mathrm{d}\pi$ for all $\pi \in X_{\mathrm{temp}}(G)$. If $G = \mathrm{U}(V)$ where V is a nondegenerate Hermitian or skew-Hermitian space, then by [BP21b, Theorem 5.53], we have

$$\mu_G(\pi) = \frac{|\gamma^*(0, \pi, \mathrm{Ad}, \psi)|}{|S_\pi|}. \tag{14.3}$$

Here S_π is a certain finite abelian group (the group of centralizers of the L-parameter of π), cf. [BP21b, Section 2.11], and $\gamma^*(0, \pi, \mathrm{Ad}, \psi)$ is the normalized value of the adjoint gamma factor to be defined in Subsection 14.4.

14.4. **Local Langlands correspondences.**

14.4.1. *Formulation of the correspondence for arbitrary connected reductive groups.* Let G be a connected reductive group over a local field of characteristic zero F. Let W_F be the Weil group of F and

$$WD_F = \begin{cases} W_F \times \mathrm{SL}_2(\mathbb{C}), & F \text{ non-Archimedean;} \\ W_F, & F \text{ Archimedean.} \end{cases}$$

the Weil–Deligne group. Let $\widehat{G}$ be the Langlands dual group of G, and ${}^L G = \widehat{G} \rtimes W_F$ the (Weil form) of the L-group of G where the Weil group W_F acts on $\widehat{G}$ by a fixed pinning. A Langlands parameter is a $\widehat{G}$-conjugacy of group homomorphisms

$$\varphi : WD_F \to {}^L G$$

which are algebraic when restricted to $\mathrm{SL}_2(\mathbb{C})$ (if F is non-Archimedean), send W_F to semisimple elements, and commute with the natural projections $WD_F \to W_F$ and ${}^L G \to W_F$. A Langlands parameter φ is called tempered if the projection of $\varphi(W_F)$ in $\widehat{G}$ is bounded in $\widehat{G}$. Let $\Phi(G)$ the set of Langlands parameters of G and $\Phi_{\mathrm{temp}}(G)$ be the set of tempered Langlands parameters.

Let $\mathrm{Irr}(G)$ be the set of all irreducible representations of G. The local Langlands correspondence postulates that there is a finite-to-one map

$$\mathrm{Irr}(G) \to \Phi(G), \quad \pi \mapsto \varphi_\pi$$

satisfying various properties. The fibers of this map are usually called the L-packets. We are interested in the following cases where the Langlands correspondence has been established: F is Archimedean, cf. [Lan89], $G = G_n$, cf. [HT01], G is unitary group, cf. [KMSW]. The Langlands correspondence in these cases also natural extends to the product of groups, or restrictions of scalars, or quotient by the split center. Some useful properties of the local Langlands correspondence is listed in [BP21b, p. 194]. One notable point is that the correspondences sends $\mathrm{Temp}(G)$ to $\Phi_{\mathrm{temp}}(G)$ and this map factors through $X_{\mathrm{temp}}(G)$. If $G = G_n$ or its variant, the correspondence is a bijection.

14.4.2. *Local Langlands for unitary groups and base change.* Let us now assume that $G = \mathrm{U}(V)$ where V is a nondegenerate skew-Hermitian space (or a product of these groups). Let $\pi, \pi' \in \mathrm{Temp}(G)$. Then we define $\pi \sim \pi'$ if $\varphi_\pi = \varphi_{\pi'}$. We then have a well-defined surjective maps

$$\mathrm{Temp}(G) \to \mathrm{Temp}(G)/\sim, \quad X_{\mathrm{temp}}(G) \to \mathrm{Temp}(G)/\sim.$$

There is a base change morphism $\mathrm{BC} : {}^L G \to {}^L G_n$ given by $(g, \sigma) \mapsto (g, J_n {}^t g^{-1} J_n^{-1}, \sigma)$ where $J_n = \begin{pmatrix} & & & 1 \\ & & -1 & \\ & \iddots & & \\ (-1)^{n-1} & & & \end{pmatrix}$. This induces a map $\Phi_{\mathrm{temp}}(G) \to \Phi_{\mathrm{temp}}(G_n)$ between Langlands-parameters and, by composing with the local Langlands correspondence, an assignment

$$\mathrm{BC} : \mathrm{Temp}(G) \to (\mathrm{Temp}(G)/\sim) \to \mathrm{Temp}(G_n). \tag{14.4}$$

Fix a split skew-Hermitian space V_{qs} and $G = \mathrm{U}(V_{qs})$ (or a product of them). By [BP21b, Lemma 2.101], there is a unique topology on $\mathrm{Temp}(G)/\sim$ such that the map $X_{\mathrm{temp}}(G) \to \mathrm{Temp}(G)/\sim$ is locally an isomorphism, and for every connected component $O \subset X_{\mathrm{temp}}(G)$, the map induces an isomorphism between O and a connected component of $\mathrm{Temp}(G)/\sim$. It follows that there is a unique measure on $\mathrm{Temp}(G)/\sim$ such that the map $X_{\mathrm{temp}}(G) \to \mathrm{Temp}(G)/\sim$ is locally measure preserving when $X_{\mathrm{temp}}(G)$ is given the measure $d\pi$. We denote this measure on $\mathrm{Temp}(G)/\sim$ again by $d\pi$.

There is an abelian group S_π attached to any $\pi \in \mathrm{Temp}(G)$, called the group of centralizer. We do not need the precise definition of it, but only refer to [BP21b, Section 2.11] for a brief description of its cardinality.

14.4.3. *Local γ-factors.* We write $\zeta_F(s)$ for the local zeta function of F, i.e.

$$\zeta_F(s) = \begin{cases} (1 - q_F^{-s})^{-1}, & F \text{ is non-Archimedean;} \\ \pi^{-s/2}\Gamma(\frac{s}{2}), & F = \mathbb{R}. \end{cases}$$

Let M be a finite dimensional vector space and $\varphi : WD_F \to \mathrm{GL}(M)$ a representation of WD_F that is algebraic when restricted to $\mathrm{SL}_2(\mathbb{C})$ (if F is non-Archimedean). We can define the local L-factor $L(s, \varphi)$, and a local root number $\epsilon(s, \varphi, \psi)$, cf. [Tat79, Section 3]. Define the local gamma factor

$$\gamma(s, \varphi, \psi) = \epsilon(s, \varphi, \psi)\frac{L(1-s, \varphi^\vee)}{L(s, \varphi)}$$

where $\varphi^\vee$ stands for the dual representation of φ. Assume that φ is tempered, i.e. the image $\varphi(W_F)$ is bounded in $\mathrm{GL}(M)$, then we set

$$\gamma^*(0, \varphi, \psi) = \lim_{s \to 0}(\zeta_F(s))^{n_\varphi}\gamma(s, \varphi, \psi)$$

where n_φ stands for the order of poles of $\gamma(s, \varphi, \psi)$ at $s = 0$ ($n_\varphi < 0$ means there is a zero).

Assume that the local Langlands correspondence is known for the group G. Let $r : {}^L G \to \mathrm{GL}(M)$ be a semisimple finite dimensional representation which is algebraic when restricted to $\widehat{G}$ and bounded when restricted to $\mathrm{SL}_2(\mathbb{C})$. One major example we will encounter is the adjoint representation

$$\mathrm{Ad} : {}^L G \to \mathrm{Lie}(\widehat{G}).$$

For $\pi \in \mathrm{Irr}(G)$ we set

$$L(s,\pi,r) = L(s, r\circ\phi_\pi), \quad \epsilon(s,\pi,r,\psi) = \epsilon(s, r\circ\phi_\pi,\psi), \quad \gamma(s,\pi,r,\psi) = \gamma(s,r\circ\phi_\pi,\psi).$$

If $\pi \in \mathrm{Temp}(G)$ we also set

$$\gamma^*(0,\pi,r,\psi) = \gamma^*(0,r\circ\phi_\pi,\psi).$$

15. Local spherical characters

15.1. **Spherical character on the unitary groups.** Let V be a nondegenerate n-dimensional skew-Hermitian space. Recall that ω and $\omega^\vee$ are the Weil representation of $\mathrm{U}(V)$ with respect to the characters ψ, μ and ψ^{-1}, μ^{-1} respectively. We have $\omega^\vee \simeq \overline{\omega}$. They are realized on $\mathcal{S}(L^\vee)$ where $\mathrm{Res}\, V = L + L^\vee$ is a polarization of the symplectic space $\mathrm{Res}\, V$.

Let π be a finite length tempered representation of U_V. We put

$$m(\pi) = \dim \mathrm{Hom}_{\mathrm{U}'_V}(\pi \widehat{\otimes} \overline{\omega}, \mathbb{C}).$$

If π is irreducible we have $m(\pi) \le 1$ by [Sun12, SZ12].

As every $\pi \in \mathrm{Temp}(\mathrm{U}_V)$ embeds in a unique representation in $X_{\mathrm{temp}}(\mathrm{U}_V)$, there is a map $\mathrm{Temp}(\mathrm{U}_V) \to X_{\mathrm{temp}}(\mathrm{U}_V)$. Let $\mathrm{Temp}_{\mathrm{U}'_V}(\mathrm{U}_V) \subset \mathrm{Temp}(\mathrm{U}_V)$ be the subspace of π with $m(\pi) \neq 0$. By [Xue, Proposition 3.4], the natural map

$$\mathrm{Temp}_{\mathrm{U}'_V}(\mathrm{U}_V) \to X_{\mathrm{temp}}(\mathrm{U}_V)$$

is injective, and the image is a collection of connected components of $X_{\mathrm{temp}}(\mathrm{U}_V)$. We thus give $\mathrm{Temp}_{\mathrm{U}'_V}(\mathrm{U}_V)$ the topology induced from $X_{\mathrm{temp}}(\mathrm{U}_V)$.

For $f_+ \in \mathcal{S}(\mathrm{U}_{V,+})$, we put

$$f_+^\natural((g_1,g_2),v) = \int_{L^\vee} \omega_{\overline{(1)}}(g_1) f_+((g_1,g_2), x+l', x-l')\, \psi(-2\,\mathrm{Tr}_{E/F}\, q_V(x,l))\,\mathrm{d}x, \tag{15.1}$$

where $g_i \in \mathrm{U}(V)$ and we write $v = l + l'$ where $l \in L$ and $l' \in L^\vee$. This is is local counterpart of (9.3) and the notation is interpreted in the same way there. It define a continuous isomorphism

$$\mathcal{S}(\mathrm{U}_{V,+}) \to \mathcal{S}(\mathrm{U}_V^+).$$

The defining expression makes sense for the functions in $C^w(\mathrm{U}_V) \widehat{\otimes} \mathcal{S}(L^\vee \times L^\vee)$, as the integration takes place only in the variables in L. We thus end up with a map

$$C^w(\mathrm{U}_V) \widehat{\otimes} \mathcal{S}(L^\vee \times L^\vee) \to C^\infty(\mathrm{U}_V^+),$$

which we still denote by $\natural$. Note that the image of the map is no longer $C^w(\mathrm{U}_V) \widehat{\otimes} \mathcal{S}(V)$.

Lemma 15.1. *Let $f_+ \in C^w(\mathrm{U}_V)\widehat{\otimes} S(L^- \oplus L^-)$. The integral*

$$\int_{\mathrm{U}_V^0} f_+^z(h,0)\,\mathrm{d}h$$

is absolutely convergent and defines a continuous linear form on $C^w(\mathrm{U}_V)\widehat{\otimes} S(L^- \oplus L^-)$.

Proof. This is essentially [Xue23, Lemma 3.3]. Strictly speaking, [Xue23] deals with only the case F being Archimedean, but the non-Archimedean case goes through by the same argument.

We denote the linear form in the lemma by L. If $f \in C^w(\mathrm{U}_V)$ and $\phi_1, \phi_2 \in S(L^-)$ then L takes a more familiar form

$$L(f \otimes \phi_1 \otimes \phi_2) = \int_{\mathrm{U}_V^0} f(h^{-1})\langle \omega^-(h)\phi_1, \overline{\phi_2}\rangle_{L^2}\,\mathrm{d}h, \tag{15.2}$$

where we recall that $\langle\ ,\ \rangle_{L^2}$ stands for the L^2-inner product.

Recall that if $f \in S(\mathrm{U}_V)$ and π is a tempered representation of U_V of finite length, then we put $f_\pi(g) = \mathrm{Trace}(\pi(g^{-1})\pi(f))$ and $f_\pi \in C^w(\mathrm{U}_V)$. We may extend this to a continuous map

$$S(\mathrm{U}_{V,+}) = S(\mathrm{U}_V)\widehat{\otimes} S(L^- \oplus L^-) \to C^w(\mathrm{U}_V)\widehat{\otimes} S(L^- \oplus L^-), \quad f_+ \mapsto f_{+,\pi}.$$

Let $f_+ \in S(\mathrm{U}_{V,+})$. We defined a linear form

$$J_\pi(f_+) = L(f_{+,\pi}).$$

The above lemma ensures that this makes sense and J_π defines a continuous linear form on $S(\mathrm{U}_{V,+})$. If $f_+ = f \otimes \phi_1 \otimes \phi_2$ where $f \in S(\mathrm{U}_V)$ and $\phi_1, \phi_2 \in S(L^-)$, then the linear form take a more familiar form

$$J_\pi(f \otimes \phi_1 \otimes \phi_2) = \int_{\mathrm{U}_V^0} \mathrm{Trace}(\pi(h)\pi(f))\langle \omega^-(h)\phi_1, \overline{\phi_2}\rangle_{L^2}\,\mathrm{d}h.$$

Lemma 15.2. *For a fixed, $f_+ \in S(\mathrm{U}_{V,+})$ the function on $X_{\mathrm{temp}}(\mathrm{U}_{V,+})$ given by $\pi \mapsto J_\pi(f_+)$ is Schwartz. Moreover the map*

$$S(\mathrm{U}_{V,+}) \to S(X_{\mathrm{temp}}(\mathrm{U}_{V,+})), \quad f_+ \mapsto (\pi \mapsto J_\pi(f_+))$$

is continuous.

Proof. For a fixed $f_+ \in S(\mathrm{U}_{V,+})$, the map

$$X_{\mathrm{temp}}(\mathrm{U}_V) \to C^w(\mathrm{U}_V)\widehat{\otimes} S(L \oplus L), \quad \pi \mapsto f_{+,\pi}$$

is a Schwartz function valued in $C^w(\mathrm{U}_V)\widehat{\otimes} S(L \oplus L)$ by (14.1), and this map depends continuously on f_+ by [BP21b, (2.6.1)]. The lemma then follows from Lemma 15.1.

One main result of [Xue] is the following.

Proposition 15.3. *If $\pi \in \mathrm{Temp}(\mathrm{U}_V)$, then $m(\pi) \neq 0$ if and only if J_π is not identically zero.*

We will also need the following lemma.

Lemma 15.4. *For all $f_+ \in S(\mathrm{U}_{V,+})$ we have*

$$O(1, f_+) = \int_{X_{\mathrm{temp}}(\mathrm{U}_V)} J_\pi(f_+)\mathrm{d}\mu_{\mathrm{U}_V}(\pi). \tag{15.3}$$

The right hand side is absolutely convergent and defines a continuous linear form on $S(\mathrm{U}_{V,+})$.

Proof. The absolute convergence and the continuity of the right hand side follow from Lemma 15.2. Clearly the left hand side also defines a continuous linear form. We may prove the lemma under the additional assumption $f_+ = f \otimes \phi_1 \otimes \phi_2$ where $f \in S(G(F))$ and $\phi_1, \phi_2 \in S(L^\vee)$.

By definition we have

$$O(1, f \otimes \phi_1 \otimes \phi_2) = \int_{\mathrm{U}_V^0} f(g^{-1})\langle \omega^\vee(g)\phi_1, \overline{\phi_2}\rangle_{L^2}\mathrm{d}g.$$

By the Plancherel formula for the group U_V, we have

$$f = \int_{X_{\mathrm{temp}}(\mathrm{U}_V)} f_\pi \,\mathrm{d}\mu_{\mathrm{U}_V}(\pi),$$

where the integral is absolutely convergent in $C^w(\mathrm{U}_V)$, cf. (14.2). By Lemma 15.1 we have

$$O(1, f \otimes \phi_1 \otimes \phi_2) = \int_{X_{\mathrm{temp}}(\mathrm{U}_V)} \int_{\mathrm{U}_V^0} f_\pi(h^{-1})\langle \omega^\vee(h)\phi_1, \overline{\phi_2}\rangle_{L^2}\mathrm{d}h\mathrm{d}\mu_{\mathrm{U}_V}(\pi).$$

This proves the lemma.

15.2. **Spherical character on the linear groups.** We will use the following notation.

- Recall that $G_n = \mathrm{Res}_{E/F}\mathrm{GL}_{n,E}$, $G_n' = \mathrm{GL}_{n,F}$, $G = G_n \times G_n$, $G' = G_n' \times G_n'$, and $H = G_n$ which diagonally embeds in G.
- Let B_n the minimal parabolic subgroup of G_n consisting of upper triangular matrices. Let T_n be the diagonal torus, and N_n the unipotent radical.
- Let $e_n = (0, \dots, 0, 1) \in E_n$ and let P_n be the mirabolic subgroup of G_n which consists of matrices whose last row is e_n.
- Define subgroups of G by $B = B_n \times B_n$, $T = T_n \times T_n$, $N = N_n \times N_n$, and $P = P_n \times P_n$.
- We fix a maximal compact subgroup K of G.
- If X is a subgroup of G, we put $X_H = X \cap H$.

Let ψ is the character of N_n given by

$$\psi(u) = \psi^E((-1)^n(u_{1,2} + \dots + u_{n-1,n})), \quad u \in N_n.$$

Put $\psi_N = \psi \boxtimes \overline{\psi}$. As a note for the notation, μ is used in the previous section to denote an element in E^1. But that element in E^1 is always taken to be 1 for the rest of this part, and μ will stand for the character we define here.

For $W \in C^w(N\backslash G, \psi_N)$ and $\Phi \in S(E_n)$, we define a linear form

$$\lambda(W \otimes \Phi) = \int_{N_H\backslash H} W(h)\Phi(e_n h)\mu(\det h)^{-1}|\det h|_E^{\frac{1}{2}}\mathrm{d}h. \tag{15.4}$$

Lemma 15.5. *The integral is absolutely convergent and extends to a continuous linear form on* $C^w(N\backslash G, \psi_N)\widehat{\otimes} S(E_n)$*.*

Proof. By the Iwasawa decomposition, the integral is bounded by

$$\int_{T_H}\int_{K_H} |W(ak)\Phi(e_n ak)||a_1\cdots a_n|_E^{\frac{1}{2}}\delta_n^{-1}(a)\mathrm{d}k\mathrm{d}a,$$

where $a=(a_1,\ldots,a_n)\in T_H$, $a_i\in E^\times$ and δ_n the modulus character of the B_H. By [BP21b, Lemma 2.4.3], there is a $d>0$, such that for all $N>0$ we can find a continuous seminorm $\nu_{d,N}$ on $C^w(N\backslash G,\psi_N)$, with

$$|W(ak)|\ll \prod_{i=1}^{n-1}(1+|a_ia_{i+1}^{-1}|)^{-N}\delta_n(a)^{\frac{1}{2}}\varsigma(a)^d\nu_{d,N}(W).$$

Here $\varsigma(a)$ is a fixed logarithmic height function on T_H. Since $\Phi\in S(E_n)$, for any $N>0$ there is a continuous seminorm ν'_N on $S(E_n)$ such that we also have

$$\Phi(e_nak)\ll(1+|a_n|)^{-N}\nu'_N(\Phi).$$

The lemma then reduces to [BP21b, Lemma 2.4.4].

For $W\in C^w(N\backslash G,\psi_N)$, define the linear forms

$$\lambda(W)=\int_{N'\backslash P'}W(h)\mu_{n+1}(h)\mathrm{d}h, \tag{15.5}$$

where $\mu_{n+1}(g_1,g_2)=\mu^{n+1}(\det g_1g_2)$. By [BP21b, Lemma 2.15.1] this integral is absolutely convergent and define a continuous linear form on $C^w(N\backslash G,\psi_N)$.

Let Π be an irreducible tempered representation of G. Let $W(\Pi,\psi_N)$ be its Whittaker model. Then $W(\Pi,\psi_N)\subset C^w(N\backslash G,\psi_N)$ by [BP21b, Lemma 2.8.1]. We fix an inner product on $W(\Pi,\psi_N)$ by

$$\langle W_1,W_2\rangle_G^{\mathrm{Wh}}=\int_{N\backslash P}W_1(p)\overline{W_2(p)}\mathrm{d}p. \tag{15.6}$$

Here $\mathrm{d}p$ stands for the right invariant measure on P.

In what follows we will use the following construction. For $f\in S(G)$, we put

$$W_f(g_1,g_2)=\int_N f(g_1^{-1}ug_2)\psi_N(u)\mathrm{d}u,\quad g_1,g_2\in G. \tag{15.7}$$

Then by [BP21b, Section 2.14], $W_f(g_1,\cdot)\in S(N\backslash G,\overline{\psi_N})$ for any $g_1\in G$. As explained in [BP21b, Section 2.14], this construction actually extends to all $f\in C^w(G)$ by replacing $\int_N$ by a regularized version $\int_N^*$. By [BP21b, Lemma 2.14.1] the resulting function W_f lies in $C^w(N\backslash G\times N\backslash G,\psi_N\boxtimes\overline{\psi_N})$ and the map

$$C^w(G)\to C^w(N\backslash G\times N\backslash G,\psi_N\boxtimes\overline{\psi_N}),\quad f\mapsto W_f$$

is continuous.

For $f \in \mathcal{S}(G)$ and $\Pi \in \mathcal{X}_{\mathrm{temp}}(G)$ (note that $\mathrm{Temp}(G) = \mathcal{X}_{\mathrm{temp}}(G)$ as topological spaces, cf. [BP21b, Subsection 4.1, p. 245-246]), we put

$$W_{f,\Pi} = W_f \in C^w(N\backslash G \times N\backslash G, \psi_N \boxtimes \overline{\psi_N}). \tag{15.8}$$

Then the map $f \mapsto W_{f,\Pi}$ is continuous. Moreover by [BP21b, Proposition 2.14.2] the map

$$\mathcal{X}_{\mathrm{temp}}(G) \to C^w(N\backslash G \times N\backslash G, \psi_N \boxtimes \overline{\psi_N}), \quad \Pi \mapsto W_{f,\Pi}$$

is Schwartz.

We have the Plancherel formula for Whittaker functions, cf. [BP21b, Proposition 2.14.2]. For any $f \in \mathcal{S}(G)$, we have

$$W_f = \int_{\mathcal{X}_{\mathrm{temp}}(G)} W_{f,\Pi} \, \mathrm{d}\mu_G(\Pi). \tag{15.9}$$

where the right hand side is absolutely convergent in $C^w(N\backslash G \times N\backslash G, \psi_N \boxtimes \overline{\psi_N})$.

Finally by [BP21b, (2.14.3)]

$$W_{f,\Pi} = |\tau|_E^{n(n-1)} \sum_W \Pi(f)W \boxtimes \overline{W}, \tag{15.10}$$

where the sum runs over an orthonormal basis of $\mathcal{W}(\Pi, \psi_N)$ and the right hand side is absolutely convergent in $C^w(N\backslash G \times N\backslash G, \psi_N \boxtimes \overline{\psi_N})$.

Let $f \in \mathcal{S}(G)$ and $\Phi \in \mathcal{S}(E_n)$. Define

$$I_\Pi(f \otimes \Phi) = \sum_W \lambda(\Pi(f)W, \Phi)\overline{\beta(W)}, \tag{15.11}$$

where the sum runs over an orthonormal basis of $\mathcal{W}(\Pi, \psi_N)$.

Lemma 15.6. *For fixed f and Φ, the function on $\mathcal{X}_{\mathrm{temp}}(G)$ given by*

$$\Pi \mapsto I_\Pi(f \otimes \Phi)$$

is Schwartz. Moreover the map

$$\mathcal{S}(G) \times \mathcal{S}(E_n) \to \mathcal{S}(\mathcal{X}_{\mathrm{temp}}(G)), \quad f \otimes \Phi \mapsto (\Pi \mapsto I_\Pi(f \otimes \Phi))$$

extends to a continuous map

$$\mathcal{S}(G_+) \to \mathcal{S}(\mathcal{X}_{\mathrm{temp}}(G)).$$

Proof. The map

$$\mathcal{S}(G) \to C^w(N\backslash G \times N\backslash G, \psi_N \boxtimes \overline{\psi_N}), \quad f \mapsto W_{f,\Pi}$$

extends to a map

$$\mathcal{S}(G_+) = \mathcal{S}(G) \widehat{\otimes} \mathcal{S}(E_n) \to C^w(N\backslash G \times N\backslash G, \psi_N \boxtimes \overline{\psi_N}) \widehat{\otimes} \mathcal{S}(E_n)$$

in an obvious way, which we again denote by $f_+ \mapsto W_{f_+,\Pi}$ where $f_+ \in \mathcal{S}(G_+)$.

With this notation, the identity (15.10) also extends and holds when $f_+ \in S(G_+)$. Here the action of f_+ on $W \in W(\Pi, \ _N)$ is given by

$$\int_G f_+(g, x)\Pi(g)W \mathrm{d}g$$

and is an element in $W(\Pi, \ _N) \widehat{\otimes} S(E_n)$. Since the sum (15.10) is absolutely convergent, we conclude that

$$I_\Pi(f_+) = |\ |_E^{n(n-1)} (\ \widehat{\ }\ )(W_{f_+,\Pi}). \tag{15.12}$$

Here $\widehat{\ }$ is the linear form defined in [BP21b, Lemma-Definition 2.41]. The rest of the lemma follows from the fact that $\Pi \mapsto W_{f_+,\Pi}$ is Schwartz and the map $f_+ \mapsto W_{f_+,\Pi}$ is continuous.

Recall that V_{qs} is a fixed split n-dimensional skew-Hermitian space. Also recall that is the constant defined in (13.9).

Proposition 15.7. *Let $f_+ \in S(G_+)$. We have*

$$O_+(f_+) = |\ |_E^{\frac{n(n-1)}{4}} \int_{\mathrm{Temp}(\mathrm{U}_{V_{qs}})/} I_{\mathrm{BC}(\ )}(f_+) \frac{|\ (0, \ , \mathrm{Ad}, \ )|}{|S\ |} \mathrm{d}\ . \tag{15.13}$$

The right hand side is absolutely convergent.

The rest of this subsection is devoted to the proof of this proposition. We need some preparations.

Put N_S be the image of N_n in S_n and $\mathfrak{n}_S$ be its tangent space at 1. They are given the quotient measure. Then the Cayley transform preserves the measures on N_S and $\mathfrak{n}_S$. Put $\mathfrak{n}_X = \mathfrak{n}_S \quad \{0\} \quad E^{-,n}$ which is given the obvious measure. Then we have $\mathfrak{n}_X = N_n/N_n^0 \quad \{0\} \quad E^{-,n}$, which is measure preserving. Recall that we have a right action of G_n^0 on $\mathfrak{x}$ given by (12.4). It is proved in [BP21b, Lemma 5.73] (cf. Remark 13.12 after (13.8)) that for any $\varphi \in S(\mathfrak{x}(F))$ we have

$$O(\ _+, \varphi) = |\ |_E^{\frac{n(n+1)}{4}} \int_{G_n^0/N_n^0} \int_{\mathfrak{n}_X} \varphi(x \ h) \overline{\ (h \ , x)} \ (\det h)\mathrm{d}x\mathrm{d}h. \tag{15.14}$$

This is convergent as an iterated integral.

Lemma 15.8. *Let $f \in S(G)$ and $\Phi \in S(E_n)$. Then*

$$O_+(f \quad \Phi) = |\ |_E^{\frac{n(n-1)}{4}} \int_{N_H \backslash H} \int_{N^0 \backslash G^0} W_f(h, g)\Phi(e_n h) \ (\det h) \ ^{-1}|\det h|_E^{\frac{1}{2}} \ _{n+1}(g)\mathrm{d}g\mathrm{d}h. \tag{15.15}$$

The right hand side is absolutely convergent.

Proof. The absolute convergence of the right hand side follows from the fact that

$$W_f \in C^w(N \backslash G \quad N \backslash G, \ _N \quad \overline{\ _N}),$$

and the absolute convergence of and . Unwinding the definitions, we see that the right hand side of (15.15) equals

$$\int_{N_n'\backslash G_n'}\int_{N_n'\backslash G_n'}\int_{N_H\backslash H}\int_{N_n}\int_{N_n} f(h^{-1}u_1g_1, h^{-1}u_2g_2)\,(u_1u_2^{-1})\Phi(e_nh)|\det h|_E^{\frac{1}{2}}\,(\det h)^{-1}\,(\det g_1g_2)^{n+1}\mathrm{d}u_1\mathrm{d}u_2\mathrm{d}h\mathrm{d}g_2\mathrm{d}g_1.$$

Making a change of variables $u_2 \mapsto u_1u_2$ and combine the integration over u_1 and h we obtain

$$\int_{N_n'\backslash G_n'}\int_{N_n'\backslash G_n'}\int_{H}\int_{N_n} f(h^{-1}g_1, h^{-1}u_2g_2)\,(u_2)^{-1}\Phi(e_nh)|\det h|_E^{\frac{1}{2}}\,(\det h)^{-1}\,(\det g_1g_2)^{n+1}\mathrm{d}u_2\mathrm{d}h\mathrm{d}g_2\mathrm{d}g_1.$$

Make another change of variable $h \mapsto g_1h$, split the integral of $u \in N_n$ into N_n' and N_n/N_n' and absorb N_n' into the integration over $g_2 \in N_n'\backslash G_n'$. We obtain

$$\int_{N_n'\backslash G_n'}\int_{G_n'}\int_{G_n}\int_{N_n/N_n'} f(h^{-1}, h^{-1}g_1^{-1}u_2g_2)\,(u_2)^{-1}(\mathrm{R}_{-1}(h)\Phi)(e_ng_1)|\det g_1|\,(\det g_1)^{n}\,(\det g_2)^{n+1}\mathrm{d}u_2\mathrm{d}h\mathrm{d}g_2\mathrm{d}g_1.$$

Recall that for any $\Phi \in S(E_n)$ we have defined the partial Fourier transform

$$\Phi^{\dagger}(x, y) = \int_{F_n}\Phi(x+x)\,((-1)^n xy)\mathrm{d}x.$$

Then we have an inversion formula

$$\Phi(x+x) = |\,|_E^{\frac{n}{2}}\int_{E^{-,n}}\Phi^{\dagger}(x, y)\,((-1)^{n+1}xy)\mathrm{d}y. \tag{15.16}$$

It follows that

$$(\mathrm{R}_{-1}(h)\Phi)(e_ng_1) = |\,|_E^{\frac{n}{2}}|\det g_1|^{-1}\int_{E^{-,n}}(\mathrm{R}_{-1}(h)\Phi)^{\dagger}(0, g_1^{-1}y)\,((-1)^{n+1}e_ny)\mathrm{d}y.$$

Plugging this back into the previous integral we obtain that the right hand side of (15.15) equals

$$\begin{aligned}|\,|_E^{\frac{n}{2}}\int_{N_n'\backslash G_n'}\int_{G_n'}\int_{G_n}\int_{N_n/N_n'}\int_{E^{-,n}} & f(h^{-1}, h^{-1}g_1^{-1}u_2g_2) \\ & (u_2)^{-1}(\mathrm{R}_{-1}(h)\Phi)^{\dagger}(0, g_1^{-1}y)\,((-1)^{n+1}e_ny)\,(\det g_1)^{n}\,(\det g_2)^{n+1}\mathrm{d}y\mathrm{d}u_2\mathrm{d}h\mathrm{d}g_2\mathrm{d}g_1.\end{aligned} \tag{15.17}$$

The inner four integrals (the integrals apart from g_1) are absolutely convergent. This can be seen as follows. Recall that $\mathrm{R}^{\dagger}_{-1}$ is the unique representation of G_n on $S(E_n \oplus E^{-,n})$ such that $\mathrm{R}^{\dagger}_{-1}(h)\Phi^{\dagger} = (\mathrm{R}_{-1}(h)\Phi)^{\dagger}$ for all $\Phi \in S(E_n \oplus E^{-,n})$. It is isomorphic to R_{-1} and thus of moderate growth. Thus the function

$$(g_1, g_2, x, y) \mapsto f(g_1^{-1}, g_1^{-1}g_2)(\mathrm{R}_{-1}(g_1)\Phi)^{\dagger}(x, y)$$

where $g_1, g_2 \in G_n$, $x \in E_n$ and $y \in E^{-,n}$, is again Schwartz function on $G_n \times G_n \times E_n \times E^{-,n}$. Moreover the map

$$G_n \times N_n/N_n' \times G_n' \to G_n \times G_n, \quad (g, u, h_2) \mapsto (g^{-1}, g^{-1}uh_2)$$

is a closed embedding. It follows that the inner four integrals of (15.17) are integrating a Schwartz function on $G_n(E) \times G_n(E) \times E_n \times E^{\ ,n}$ over a closed submanifold, and are thus convergent. Therefore we can switch the order of the inner four integrals to conclude

$$(15.17) = |\ |_E^{\frac{n}{2}} \int_{N_n^0 \backslash G_n^0} \int_{N_n / N_n^0} \int_{E^{\ ,n}} \varphi_{f \otimes \Phi}(g_1^{-1} u_2 \overline{u_2}^{-1} g_1, 0, g_1^{-1} y)$$
$$(u_2)^{-1} \ ((-1)^{n+1} e_n y) \mathrm{d}y \mathrm{d}u_2 \mathrm{d}h_1.$$

Here $\varphi_{f \otimes \Phi}$ is defined by (12.1) (or rather the local counterpart). By definition we have

$$(u_2)^{-1} \ ((-1)^{n+1} e_n y) = \overline{\ (\langle \ , (A, 0, y) \rangle)},$$

where $A \in \mathfrak{s}_n$ and $\mathrm{c}(A) = u_2 \overline{u_2}^{-1}$, and the pairing $\langle \ , \ \rangle$ on the right hand side is the one on $\mathfrak{X}$ defined in Subsection 12.2 by (13.6). By our choice of the measures the last integral equals

$$|\ |_E^{\frac{n}{2}} \int_{N_n^0 \backslash G_n^0} \int_{\mathfrak{n}_X} (f \otimes \Phi)_{\natural}(h_1^{-1} Y h_1) \overline{\ (\langle \ , Y \rangle)} \mathrm{d}Y \mathrm{d}h_1,$$

which equals

$$|\ |_E^{\frac{n}{2}} \ ^{\frac{n(n+1)}{4}} \ O_+(f \otimes \Phi),$$

by (15.14). This proves the lemma.

Proof of Proposition 15.7. The absolute convergence of the right hand side of (15.13) follows from Lemma 15.6, and the fact the function $\frac{|\ (0, \ , \mathrm{Ad}, \ )|}{|S \ |}$ is of moderate growth, cf. [BP21b, Lemma 2.45] and [BP21b, (2.7.4)]. The continuity also follows from Lemma 15.6.

Since both sides are continuous in f_+, we may additionally assume that the test function f_+ is of the form $f \otimes \Phi$, where $f \in S(G(F))$ and $\Phi \in S(E_n)$.

By Lemma 15.8 we have

$$O_+(f \otimes \Phi) = |\ |_E^{\frac{n(n-1)}{4}} \int_{N_H \backslash H} \int_{N^0 \backslash G^0} W_f(h, g) \Phi(e_n h) \ (\det h)^{-1} |\det h|_E^{\frac{1}{2}} \ _{n+1}(g) \mathrm{d}g \mathrm{d}h.$$

For any fixed $h \in H$, by [BP21b, Corollary 3.51, Theorem 4.22] the inner integral equals

$$|\ |_E^{\frac{n(n-1)}{2}} \int_{\mathrm{Temp}(\mathrm{U}_{V_{qs}})/\sim} \ (W_{f,\mathrm{BC}(\ )}(h, \ )) \frac{|\ (0, \ , \mathrm{Ad}, \ )|}{|S \ |} \mathrm{d} \ .$$

Here the linear form applies to the variable and the integral over is absolutely convergent. Thus

$$O_+(f \otimes \Phi) = |\ |_E^{\frac{n(n-1)}{4} + \frac{n(n-1)}{2}} \int_{N_H \backslash H} \int_{\mathrm{Temp}(\mathrm{U}_{V_{qs}})/\sim} \ (W_{f,\mathrm{BC}(\ )}(h, \ ))$$
$$\Phi(e_n h) \ (\det h)^{-1} |\det h|_E^{\frac{1}{2}} \frac{|\ (0, \ , \mathrm{Ad}, \ )|}{|S \ |} \mathrm{d} \ \mathrm{d}h.$$

For a fixed Φ, we denote the linear form $W \mapsto \ (W, \Phi)$ by $\ _{\Phi}$. Then we have

$$\begin{aligned} &O_+(f \otimes \Phi) \\ =& |\ |_E^{\frac{n(n-1)}{4} + \frac{n(n-1)}{2}} \int_{\mathrm{Temp}(\mathrm{U}_{V_{qs}})/\sim} (\ _{\Phi} \widehat{\ } \ ) \ W_{f,\mathrm{BC}(\ )} \ \frac{|\ (0, \ , \mathrm{Ad}, \ )|}{|S \ |} \mathrm{d} \ . \end{aligned}$$

Moreover by (15.12)

$$(\ _\Phi b\ \ )\ W_{f,\mathrm{BC}(\ )} = |\ |_E^{n(n-1)} I_{\mathrm{BC}(\ )}(f\ \ \Phi).$$

It follows that

$$O_+(f\ \ \Phi) = |\ |_E^{\frac{n(n-1)}{4}} \int_{\mathrm{Temp}(\mathrm{U}_{V_{qs}})/\ } I_{\mathrm{BC}(\ )}(f\ \ \Phi)\frac{|\ (0,\ ,\mathrm{Ad},\ )|}{|S\ |}\mathrm{d}\ .$$

This proves the proposition.

16. Local relative trace formulae

16.1. **Local trace formula on the unitary groups.** Let V be a nondegenerate n-dimensional skew-Hermitian space. We consider in this subsection $f_{1,+}, f_{2,+} \in S(\mathrm{U}_{V,+})$. Put

$$T(f_{1,+}, f_{2,+}) = \int_{\mathrm{U}_V^0}\int_{\mathrm{U}_V^0}\int_{\mathrm{U}_V}\int_V f_{1,+}^z(h_1 g h_2, v h_2)\overline{f_{2,+}^z(g,v)}\mathrm{d}v\mathrm{d}g\mathrm{d}h_1\mathrm{d}h_2. \tag{16.1}$$

Lemma 16.1. *The integral* (16.1) *is absolutely convergent and defines a continuous Hermitian form on* $S(\mathrm{U}_{V,+})$*.*

Proof. We will assume that F is Archimedean. The non-Archimedean case is similar and easier. As $f_{i,+}^z \in S(\mathrm{U}_V^+)$, Fatou's lemma implies that the lemma is deduced from the following fact. For any $f_i \in S(\mathrm{U}_V)$ and $\ _i \in S(V)$, $i = 1, 2$, there are continuous semi-norms $\ $ on $S(\mathrm{U}_V)$ and $\ ^0$ on $S(V)$ such that

$$\int_{\mathrm{U}_V^0}\int_{\mathrm{U}_V^0}\int_{\mathrm{U}_V}\int_V |f_1(h_1 g h_2)\ _1(v h_2) f_2(g)\ _2(v)|\mathrm{d}g\mathrm{d}v\mathrm{d}h_1\mathrm{d}h_2 \ \ (\varphi_1)\ (\varphi_2)\ ^0(\ _1)\ ^0(\ _2). \tag{16.2}$$

Let us fix a norm function on V as follows. We may choose a norm $\|\ \|_V$ on V that is invariant under the translation of K (a fixed maximal compact subgroup of $\mathrm{U}(V)$). The convergence of (16.2) then reduces to that there exists a $d > 0$ such that

$$\int_{\mathrm{U}_V^0}\int_{\mathrm{U}_V^0}\int_{\mathrm{U}_V}\int_V |f_1(h_1 g h_2) f_2(g)|(1+\|v\|)^{-d}(1+\|v h_2\|)^{-d}\mathrm{d}v\mathrm{d}g\mathrm{d}h_1\mathrm{d}h_2 \ \ (\varphi_1)\ (\varphi_2).$$

Integrate over h_1 and g_1 first and change of variables. We are reduced to prove that

$$\int_{\mathrm{U}_V^0}\int_{\mathrm{U}_V^0}\int_V f_3(h_2^{-1} g_2 h_2) f_4(g_2)(1+\|v\|)^{-d}(1+\|v h_2\|)^{-d}\mathrm{d}v\mathrm{d}h_2\mathrm{d}g_2$$

is absolutely convergent, where f_3, f_4 are positive Schwartz functions on $\mathrm{U}(V)$.

Since $S(\mathrm{U}(V))$ is contained in $C(\mathrm{U}(V))$ (the embedding is continuous), we need to prove

$$\int_{\mathrm{U}(V)}\int_{\mathrm{U}(V)}\int_V \Xi^{\mathrm{U}(V)}(h_2^{-1} g_2 h_2)\Xi^{\mathrm{U}(V)}(g_2)\varsigma(g_2)^{-d}(1+\|v\|)^{-d}(1+\|v h_2\|)^{-d}\mathrm{d}v\mathrm{d}h_2\mathrm{d}g_2$$

is convergent for sufficiently large d (this is a rather crude reduction, which however works). Using the doubling principle for $\Xi^{\mathrm{U}(V)}$, we are reduced to the convergence of

$$\int_{\mathrm{U}(V)} \Xi(g_2)^2 \varsigma(g_2)^{-d}\mathrm{d}g_2$$

and

$$\int_{U(V)}\int_V \Xi(h_2)^2(1+\|v\|)^{-d}(1+\|vh_2\|)^{-d}\,dv\,dh_2$$

when d is large. The first one is [Wal03, Lemma II.1.5]. The second one follows from the Cartan decomposition $h_2 = k_1 a k_2$, $a \in A^+$, $k_1, k_2 \in K$ and

$$\int_V (1+\|v\|)^{-d}(1+\|va\|)^{-d}\,dv \le C\,|a_1 \cdots a_r|_E$$

for some constant C.

Proposition 16.2. Let $f_{1,+}, f_{2,+} \in S(U_{V,+})$. We have

$$\int_{X_{temp}(U_V)} J_\pi(f_{1,+})\overline{J_\pi(f_{2,+})}\,d\mu_G(\pi) = \int_{A_{rs}} O(y, f_{1,+})\overline{O(y, f_{2,+})}\,dy.$$

Both sides are absolutely convergent and define continuous Hermitian form on $S(U_{V,+})$. Note that if $y \in A$ is not in the image of U_V^+ (recall by our convention this all mean their F-points), then we take the convention that $O(y, f_{i,+}) = 0$.

Proof. We compute the integral (16.1) in two ways. By Lemma 16.1 we may change the order of integration. We make the change of variables

$$h_1 \mapsto h_1 h_2^{-1} g_1^{-1}, \quad g_2 \mapsto g_1 g_2,$$

then integrate h_1 and g_1 first. We combine the variables $g_2 \in U(V)$ and $v \in V$ as a single variable $y \in Y^V$. We then end up with

$$T(f_1, f_2) = \int_{Y^V(F)}\int_{U(V)} \varphi_{f^z_{1,+}}(y \cdot h_2)\overline{\varphi_{f^z_{2,+}}(y)}\,dh_2\,dy.$$

Here we recall that $\varphi_{f^z_{i,+}}$ is the function defined by (12.5). As Y^V_{rs} has a measure zero complement in Y^V (recall the convention that they stands for the F-point of the underline algebraic varieties), this equals

$$\int_{Y^V_{rs}} O(y, f_{1,+})\overline{\varphi_{f^z_{2,+}}(y)}\,dy.$$

By the definition of the measure on A_{rs} this equals

$$\int_{A^V_{rs}(F)} O(y, f_{1,+})\overline{O(y, f_{2,+})}\,dy.$$

We now compute (16.1) spectrally and show that

$$(16.1) = \int_{X_{temp}(U_V)} J_\pi(f_{1,+})\overline{J_\pi(f_{2,+})}\,d\mu_G(\pi).$$

Since both sides are continuous in $f_{1,+}$ and $f_{2,+}$, we may assume that $f_{1,+} = f_1 \otimes \phi_1 \otimes \phi_2$, $f_{2,+} = f_2 \otimes \phi_3 \otimes \phi_4$ where $f_1, f_2 \in S(U_V)$ and $\phi_1, \ldots, \phi_4 \in S(L)$. First integrate over $v \in V$. Since L^2-norm is preserved under the Fourier transform, the integral (16.1) becomes

$$\int_{U_V^0}\int_{U_V^0}\int_{U_V} f_1(h_1 g h_2)\overline{f_2(g)}\langle\omega^\vee(h_1)\phi_3, \phi_1\rangle\langle\omega^\vee(h_2)\phi_2, \phi_4\rangle\,dg\,dh_1\,dh_2;$$

which equals

$$\int_{U_V^0}\int_{U_V^0} f_2 \quad L(h_1^{-1})f_1 \quad (h_2)\overline{\langle \omega_-(h_1) \; {}_3; \; {}_1\rangle}\langle \omega_-(h_2) \; {}_2; \; {}_4\rangle dh_2 dh_1:$$

Here $f_2(g) = \overline{f_2(g^{-1})}$ and stands for the usual convolution product in $S(G(F))$. Using the Plancherel formula (14.2), this integral equals

$$\int_{U_V^0}\int_{U_V^0}\int_{X_{temp}(U_V)} \text{Trace} \quad (h_2^{-1}) \; (f_2) \; (h_1^{-1}) \; (f_1)$$

$$\overline{\langle \omega_-(h_1) \; {}_3; \; {}_1\rangle}\langle \omega_-(h_2) \; {}_2; \; {}_4\rangle d \; {}_{U_V}(\;)dh_2 dh_1:$$

Here we need to invoke results from [Xue]. We de ned a linear form

$$L_J : (\quad \to) \, b \, \overline{\quad \to} \to \mathbb{C};$$

and a map

$$L^{;\,0}: \quad \to \quad \mathrm{Hom}(\bar{}; \mathbb{C})$$

there. Integrating over h_1 rst by [Xue, (3.2)], the above integral equals

$$\int_{U_V^0}\int_{X_{temp}(U_V)} \text{Trace} \quad (h_2^{-1}) \; (f_2)L^{1;\,3} \; (f_1) \; \langle \omega_-(h_2) \; {}_2; \; {}_4\rangle d \; {}_{U_V}(\;)dh_2:$$

Integrating over h_2 we get

$$\int_{X_{temp}(U_V)} L_J \quad (\; (f_2)L^{1;\,3} \; (f_1))^{4;\,2} \quad d \; {}_{U_V}(\;):$$

By [Xue, Lemma 3.6] we have

$$(\; (f_2)L^{1;\,3} \; (f_1))^{4;\,2} = \; (f_2)^{4;\,3}L_J \; (f_1)^{1;\,2};$$

and by [Xue, Lemma 3.5] we have

$$L_J(\; (f_2)^{4;\,3}L_J \; (f_1)^{1;\,2}) = L_J(\; (f_2)^{4;\,3})L_J(\; (f_1)^{1;\,2}):$$

By de nition

$$L_J(\; (f_2)^{4;\,3}) = \overline{J \; (f_2 \quad \bar{}_3 \quad {}_4)}; \quad L_J(\; (f_1)^{1;\,2}) = J \; (f_1 \quad \bar{}_1 \quad {}_2):$$

This proves the proposition.

16.2. Local trace formula on the linear groups. Let us now consider $f_{1;+}; f_{2;+} \in S(G_+(F))$. Put

(16.3) $$T^0(f_{1;+}; f_{2;+}) = \int_H \int_{G^0} \int_{G_+} f_{1;+}^y(x \quad (h; g^0)) \overline{f_{2;+}^y(x)} \; {}_{G^0}(g^0)dx\,dh\,dg^0:$$

Lemma 16.3. *The integral (16.3) is absolutely convergent. It de nes a continuous Hermitian form on $S(G^0(F) \quad E_n)$. Moreover*

(16.4) $$T^0(f_{1;+}; f_{2;+}) = \int_{A_{rs}(F)} O(x; f_{1;+})\overline{O(x; f_{2;+})}dx;$$

where the integral is absolutely convergent.

Proof. The proof of the absolute convergence is essentially the same as the convergence of (16.1). Similar calculation as in the proof of Proposition 16.2 gives that

$$T'(f_{1,+}, f_{2,+}) = \int_X \int_{G_n} \varphi_{f^\dagger_{1,+}}(x\cdot g)\overline{\varphi_{f^\dagger_{2,+}}(x)}\,dg\,dx;$$

and by definition this equals

$$\int_{A_{rs}(F)} O(x, f_{1,+})\overline{O(x, f_{2,+})}\,dx.$$

This proves the lemma.

We are going to calculate $T'(f_{1,+}, f_{2,+})$ spectrally. We will assume that $f_{1,+} = f_1 \otimes \phi_1$ and $f_{2,+} = f_2 \otimes \phi_2$ where $f_1, f_2 \in S(G)$ and $\phi_1, \phi_2 \in S(E_n)$. We need some preparations. The proof of the following lemma is analogous to Lemma 16.1 and [BP21b, Lemma 5.4.2 (ii)], which we omit.

Lemma 16.4. For every $\phi_1, \phi_2 \in S(E_n)$ and $\varphi \in C^w(G)$ the integral

$$\int_H \int_{E_n} \varphi(h)|\det h|_E^{\frac{1}{2}}\phi_1(vh)\overline{\phi_2(v)}\,dv\,dh$$

is absolutely convergent and defines a continuous linear form in $\varphi \in C^w(G)$. In particular, if $\varphi_2 \in C^w(G_n)$, then for every $\varphi_1 \in C^w(G_n)$ the integral

$$\int_{G_n} \int_{E_n} \varphi_1(h)\varphi_2(h)|\det h|_E^{\frac{1}{2}}\phi_1(vh)\overline{\phi_2(v)}\,dv\,dh$$

is absolutely convergent and defines a continuous linear form on $\varphi_1 \in C^w(G_n)$.

Following [BP21b], for any irreducible tempered representation π of G, we define a continuous Hermitian form on $S(G)$ by

$$\langle f_1, f_2\rangle_{X,\pi} = \sum_{W \in W(\pi, \psi_{N'})} \ell(\pi(f_1^-)W)\overline{\ell(\pi(f_2^-)W)}:$$

where $f^-(x) = f(x^{-1})$. Here the linear form ℓ is a variant of β, cf. (15.5), and is given by

$$\ell : C^w(N\backslash G, \psi_N) \to \mathbb{C}, \quad W \mapsto \ell(W) = \int_{N'\backslash P'} W(p)\,dp.$$

It is a continuous linear form on $C^w(N\backslash G, \psi_N)$ by [BP21b, Lemma 2.15.1]. The subscript X stands for the symmetric variety $G'\backslash G$. For our purposes we do not need the details, but only treat $\langle \cdot, \cdot \rangle_{X,\pi}$ as a single piece of notation.

Lemma 16.5. The function

$$\pi \mapsto \langle f_1, f_2\rangle_{X,\pi}$$

is Schwartz. The map

$$S(G)^2 \to S(\mathrm{Temp}(G)), \quad (f_1, f_2) \mapsto (\pi \mapsto \langle f_1, f_2\rangle_{X,\pi})$$

is continuous. Moreover

$$\int_H \int_{E_n} (L(h^{-1})f_1, f_2)_{X,\sigma}\, \eta(\det h)^{-1} |\det h|_E^{\frac{1}{2}} \phi_1(vh)\overline{\phi_2(v)}\, dv\, dh \tag{16.5}$$
$$= |\tau|_E^{\frac{n(n-1)}{2}} I_\sigma(f_1 \otimes \phi_1)\overline{I_\sigma(f_2 \otimes \phi_2)}.$$

Proof. Let us focus on the proof of the equality (16.5). The proof of the rest is exactly the same as those in [BP21b, Lemma 4.21, Proposition 5.43].

First as [BP21b, Proof of (5.4.2)] the sum of functions

$$g \mapsto \sum_{W_1, W_2 \in \mathcal{W}(\sigma, \psi_N)} \langle \sigma(g)W_1, W_2 \rangle_G^{Whitt}\, \beta((\sigma(f_1)W_1)\overline{\beta((\sigma(f_2)W_2)}$$

is absolutely convergent in $C^w(G)$. It follows that the expression

$$\int_H \int_{E_n} \sum_{W_1, W_2 \in \mathcal{W}(\sigma, \psi_N)} \langle \sigma(h)W_1, W_2 \rangle_G^{Whitt}\, \beta((\sigma(f_1)W_1)\overline{\beta((\sigma(f_2)W_2)}$$
$$\eta(\det h)^{-1} |\det h|_E^{\frac{1}{2}} \phi_1(vh)\overline{\phi_2(v)}\, dv\, dh$$

is absolutely convergent and equals the left hand side of (16.5). We can thus switch the order of the sum and the integrals. We will prove that for all $W_1, W_2 \in \mathcal{W}(\sigma, \psi_N)$ we have

$$\int_H \int_{E_n} \langle \sigma(h)W_1, W_2 \rangle_G^{Whitt}\, \eta(\det h)^{-1} |\det h|_E^{\frac{1}{2}} \phi_1(vh)\overline{\phi_2(v)}\, dv\, dh \tag{16.6}$$
$$= |\tau|_E^{\frac{n(n-1)}{2}} \beta(W_1, \phi_1)\overline{\beta(W_2, \phi_2)},$$

which implies (16.5) directly.

First both sides of (16.6) are continuous linear forms on W_1 and W_2 by [BP21b, (2.6.1)] and Lemma 16.4, and thus we may assume that $\sigma = \sigma' \boxtimes \sigma''$ where σ', σ'' are irreducible tempered representations of G_n, and $W_i = W_i' \otimes W_i''$ where

$$W_i' \in \mathcal{W}(\sigma', \psi); \quad W_i'' \in \mathcal{W}(\sigma'', \overline{\psi}).$$

We claim that for any $\varphi \in C^w(G_n)$ and any $W_1'', W_2'' \in \mathcal{W}(\sigma'', \overline{\psi})$, we have

$$\int_{G_n} \int_{E_n} \varphi(h) \langle \sigma''(h)W_1'', W_2'' \rangle_{G_n}^{Whitt}\, \eta(\det h)^{-1} |\det h|_E^{\frac{1}{2}} \phi_1(vh)\overline{\phi_2(v)}\, dv\, dh \tag{16.7}$$
$$= \int\int_{(N_n \backslash G_n)^2} W_\varphi(h, g) W_1''(h)\overline{W_2''(g)}\, \eta(\det hg)^{-1} |\det hg|_E^{\frac{1}{2}} \phi_1(e_n h)\overline{\phi_2(e_n g)}\, dh\, dg,$$

where we recall that W_φ was defined in (15.7).

Let us first explain that this implies (16.6). The left hand side of (16.6) equals

$$\int_{G_n} \int_{E_n} \langle \sigma'(h)W_1', W_2' \rangle_{G_n(E)}^{Whitt} \langle \sigma''(h)W_1'', W_2'' \rangle_{G_n(E)}^{Whitt}\, \eta(\det h)^{-1} |\det h|_E^{\frac{1}{2}} \phi_1(vh)\overline{\phi_2(v)}\, dv\, dh.$$

We obtain (16.6) by applying (16.7) to

$$\varphi(g) = \langle \sigma'(g)W_1', W_2' \rangle_{G_n}^{Whitt}$$

and using (15.10).

Finally we prove (16.7). Both sides are continuous linear forms in (this follows from Lemma 16.4 for the left hand side, and from Lemma 15.5 for the right), and therefore we may assume that $\in S(G_n)$, which makes everything absolutely convergent. We first replace the integral over $v \in E_n$ by $g \in P_n \backslash G_n$. The left hand side of (16.7) equals

$$\int_H \int_{P_n \backslash G_n} (h) h(h) \langle W_1, W_2 \rangle^{Whitt}_{G_n} (\det h)^{-1} |\det h|_E^{\frac{1}{2}} |\det g|_E \, _1(e_n g h) \overline{_2(e_n g)} \, dg dh.$$

Make a change of variable $h \mapsto g^{-1} h$ we end up with

$$\int_{P_n \backslash (G_n \times G_n)} (g^{-1} h) h(h) \langle W_1, (g) W_2 \rangle^{Whitt}_{G_n} (\det g^{-1} h)^{-1} |\det hg|_E^{\frac{1}{2}} \, _1(e_n h) \overline{_2(e_n g)} \, dg dh.$$

Here P_n embeds in $G_n \times G_n$ diagonally. Plugging in the definition of $\langle \, , \, \rangle^{Whitt}_{G_n}$ we obtain

$$\int_{N_n \backslash (G_n \times G_n)} (g^{-1} h) W_1(h) \overline{W_2(g)} (\det g^{-1} h)^{-1} |\det hg|_E^{\frac{1}{2}} \, _1(e_n h) \overline{_2(e_n g)} \, dg dh.$$

Finally we decompose the integration of over $N_n \backslash (G_n \times G_n)$ as a integral over N_n followed by a double integral over $(N_n \backslash G_n)^2$ and conclude that the above integral equals

$$\int_{(N_n \backslash G_n)^2} \int_{N_n} (g^{-1} u h) \overline{(u)} W_1(h) \overline{W_2(g)} (\det g^{-1} h)^{-1} |\det hg|_E^{\frac{1}{2}} \, _1(e_n h) \overline{_2(e_n g)} \, du dg dh.$$

By the definition of W, this equals the right hand side of (16.7). This finishes the proof of the lemma.

Proposition 16.6. We have

$$| |_E^{\frac{n(n-1)}{2}} \int_{Temp(U_{V_{qs}})/} I_{BC( )}(f_{1,+}) \overline{I_{BC( )}(f_{2,+})} \frac{| (0, , Ad,)|}{|S |} d$$
$$= \int_{A_{rs}(F)} O(x, f_{1,+}) \overline{O(x, f_{2,+})} dx.$$

Proof. By Lemma 16.3, we just need to prove that

$$T'(f_{1,+}, f_{2,+}) = | |_E^{\frac{n(n-1)}{2}} \int_{Temp(U_{V_{qs}})/} I_{BC( )}(f_{1,+}) \overline{I_{BC( )}(f_{2,+})} \frac{| (0, , Ad,)|}{|S |} d . \tag{16.8}$$

The absolute convergence of the integral follows from the fact that the function $\mapsto I (f'_i)$ is Schwartz, and the function $\frac{| (0, , Ad,)|}{|S |}$ is of moderate growth.

Since both sides of (16.8) are continuous in both $f_{1,+}$ and $f_{2,+}$ by Lemma 15.6, we may assume that they both lie in $S(G) \otimes S(E_n)$. Thus we are reduced to calculate

$$T'(f_1 \otimes _1, f_2 \otimes _2)$$

where $f_1, f_2 \in S(G)$ and $_1, _2 \in S(E_n)$.

Since partial Fourier transform preserves the L^2-norm, integrating over $E_n \times E^{-,n}$ first gives

$$\int_H \int_{G'} \int_G \int_{E_n} f_1(hxg') \overline{f_2(x)} \, _1(vh) \overline{_2(v)} (\det h)^{-1} |\det h|_E^{\frac{1}{2}} \, _{n+1}(g') dv dx dg' dh.$$

This expression is again absolutely convergent, and hence we can integrate g^0 and x first. By [BP21b, Proposition 5.61] the integrals over g^0 and x gives gives

$$\int_{\mathrm{Temp}(U_{V_{qs}})/\sim} \langle L(h_1^{-1}) f_1, f_2 \rangle_{X,BC(\ )}\ _1(vh_1)\ \overline{_2(v)}\ \frac{j\ (0;\ ;\ Ad;\ )j}{jS\ j} d\ .$$

Strictly speaking the statement in [BP21b] is not exactly the same as this. But as in [BP21b, Proposition 5.61], since both sides of continuous linear forms on f_1 and f_2 we may assume that $f_1, f_2 \in S(G_n) \ S\ (G_n)$, and thus the desired equality boils down to two equalities on G_n and both follow directly from [BP21b, Theorem 4.22] (the case in [BP21b] boils down to one on G_n and the other on G_{n+1}).

We thus obtain

$$\int_{H}\int_{E_n}\int_{\mathrm{Temp}(U_{V_{qs}})/\sim} \langle L(h^{-1}) f_1, f_2 \rangle_{X,BC(\ )}\ _1(vh)\ \overline{_2(v)}\ \frac{j\ (0;\ ;\ Ad;\ )j}{jS\ j} d\ dvdh.$$

As the case of [BP21b, (5.6.9)], this expression is absolutely convergent, and the identity (16.8) follows from Lemma 16.5.

16.3. Character identities. We are ready to compare the relative characters. We begin with the following proposition, which is a weaker version of it.

Proposition 16.7. *For any $\ \in \mathrm{Temp}_{U_V^0}(U_V)$, there is a constant $\ (\ )$ such that*

$$J\ (f_+^V) = \ (\ ) I_{BC(\ )}(f_+);$$

for matching test functions $f_+ \in S(G_+)$ and $f_+^V \in S(U_{V,+})$. Moreover the function

$$\in \mathrm{Temp}_{U_V^0}(U_V) \mapsto \ (\ )$$

is continuous. Here the topology on $\mathrm{Temp}_{U_V^0}(U_V)$ is discussed in Subsection 15.1.

Proof. The proof is the same as [BP21a, Proposition 4.2.1] and is left to the interested reader, cf. [BP21b, Proposition 5.81].

We now compute $\ (\ )$. In theory we could compute it directly using the spectral expansions of the unipotent orbital integrals. But when F is Archimedean, due to the lack the matching for all test functions, we need an a priori computation of $j\ (\ )j$, which we could only achieve using the local relative trace formulae.

Proposition 16.8. *For any $\ \in \mathrm{Temp}_{U_V^0}(U_V)$ we have $j\ (\ )j = j\ j_E^{\frac{n(n-1)}{4}}$. In particular it is independent of .*

Proof. Take $f^\vee_{1,+}, f^\vee_{2,+} \in S(U_{V,+})$ and $f_{1,+}, f_{2,+} \in S(G_+)$. Assume that they match, then by Proposition 16.2 and Proposition 16.6 we have

$$|\tau|_E^{\frac{n(n-1)}{2}} \int_{\mathrm{Temp}(U_{V_{qs}})/\sim} I_{BC(\sigma)}(f_{1,+})\overline{I_{BC(\sigma)}(f_{2,+})}\frac{|\gamma(0,\sigma,\mathrm{Ad},\psi)|}{|S_\sigma|}d\sigma$$
$$= \sum_V \int_{\mathrm{Temp}_{U_V^0}(U_V)} J_\pi(f^\vee_{1,+})\overline{J_\pi(f^\vee_{2,+})}d\mu_{U_V}(\pi);$$

where the sum runs over all isomorphism classes of nondegenerate n-dimensional skew-Hermitian spaces. It follows from Proposition 16.7 that

$$|\tau|_E^{\frac{n(n-1)}{2}} \int_{\mathrm{Temp}(U_{V_{qs}})/\sim} I_{BC(\sigma)}(f_{1,+})\overline{I_{BC(\sigma)}(f_{2,+})}\frac{|\gamma(0,\sigma,\mathrm{Ad},\psi)|}{|S_\sigma|}d\sigma$$
$$= \sum_V \int_{\mathrm{Temp}_{U_V^0}(U_V)} I_{BC(\pi)}(f_{1,+})\overline{I_{BC(\pi)}(f_{2,+})}|\alpha(\pi)|^2 d\mu_{U_V}(\pi).$$

Using the integration formula [BP21b, (2.10.1)], and the formal degree conjecture for unitary groups [BP21b, Theorem 5.53] (this is one of the main results of [BP21b]), we conclude

$$|\tau|_E^{\frac{n(n-1)}{2}} \int_{\mathrm{Temp}(U_{V_{qs}})/\sim} I_{BC(\sigma)}(f_{1,+})\overline{I_{BC(\sigma)}(f_{2,+})}\frac{|\gamma(0,\sigma,\mathrm{Ad},\psi)|}{|S_\sigma|}d\sigma$$
$$= \int_{\mathrm{Temp}(U_{V_{qs}})/\sim} \left(\sum_V \sum_{\pi \in \mathrm{Temp}_{U_V^0}(U_V),\, BC(\pi)=\sigma} |\alpha(\pi)|^2 \right) I_{BC(\sigma)}(f_{1,+})\overline{I_{BC(\sigma)}(f_{2,+})}\frac{|\gamma(0,\sigma,\mathrm{Ad},\psi)|}{|S_\sigma|}d\sigma. \tag{16.9}$$

A priori, this identity holds for all transferrable $f_{1,+}, f_{2,+}$. We now explain that it holds for all $f_{1,+}, f_{2,+}$. If F is non-Archimedean, there is nothing to prove as all test functions are transferable. Assume F is Archimedean. Since transferable functions form a dense subspace of $S(G_+)$, it is enough to explain that both sides define continuous linear forms in $f_{1,+}, f_{2,+}$. Since for any $\sigma \in \mathrm{Temp}(G)$, I_σ is a continuous linear form, it is enough to explain that both sides of (16.9) are absolutely convergent. This is clear for the left hand side, as the map $\sigma \mapsto I_\sigma$ and $\frac{|\gamma(0,\sigma,\mathrm{Ad},\psi)|}{|S_\sigma|}$ is of moderate growth in σ. It remains to explain the absolute convergence of the right hand side. For this we apply the Cauchy–Schwarz inequality to conclude that if $f_{1,+}, f_{2,+}$ are transferable, then the right hand side of (16.9), when $I_{BC(\sigma)}(f_{i,+})$'s are replaced by their absolute values, is bounded by

$$|\tau|_E^{\frac{n(n-1)}{2}} \nu(f_{1,+})^{\frac{1}{2}} \nu(f_{2,+})^{\frac{1}{2}}$$

where ν us the continuous seminorm on $S(G_+)$ given by

$$\nu(f_{i,+}) = \int_{\mathrm{Temp}(U_{V_{qs}})/\sim} |I_{BC(\sigma)}(f_{i,+})|^2 \frac{|\gamma(0,\sigma,\mathrm{Ad},\psi)|}{|S_\sigma|}d\sigma.$$

Then by Fatou's lemma and the density of transferable functions we conclude that the same holds for all $f_{1,+}, f_{2,+}$. This proves the absolute convergence of the right hand side.

Once we have that (16.9) holds for all $f_{1;+}; f_{2;+}$, we apply the spectrum-separating technique of [BP21b, Lemma 5.82] and conclude that

$$j \ j_E^{\frac{n(n\ 1)}{2}} I_{BC(\ )}(f_{1;+})\overline{I_{BC(\ )}(f_{2;+})} = \sum_{V} \sum_{2\,\mathrm{Temp}_{U_V^0}(U_V)} j\ (\ )j^2 I_{BC(\ )}(f_{1;+})\overline{I_{BC(\ )}(f_{2;+})};$$

for each $\ 2\ \mathrm{Temp}(U_{V_{qs}})=\ $. By the local Gan{Gross{Prasad conjecture [GI16, Xue], there is a unique in the double sum on the right hand side such that $\ 2\ \mathrm{Temp}_{U_V^0}(U_V)$. Since $I_{BC(\ )}$ is not identically zero, which can be deduced from Proposition 15.3, we conclude that

$$j\ (\ )j^2 = j\ j_E^{\frac{n(n\ 1)}{2}}$$

for this .

Theorem 16.9. For all $\ 2\ \mathrm{Temp}_{U_V^0}(U_V)$, we have

$$(\ ) = j\ j_E^{\frac{n(n\ 1)}{4}}\ ((2\ )^n \mathrm{disc}V)^n\ ((\ 1)^{n\ 1}\ )^{\frac{n(n+1)}{2}}\ (\ )^{\frac{n(n+1)}{2}}:$$

Proof. For each n-dimensional skew-Hermitian space V, we put

$$_V = \ ((2\ )^n \mathrm{disc}V)^n\ ((\ 1)^{n\ 1}\ )^{\frac{n(n+1)}{2}}\ (\ )^{\frac{n(n+1)}{2}}:$$

If f_+ and the collection f_+^V match, by Lemma 13.13 we have

$$O_+(f_+) = \sum_V {}_V O(1; f_+^V);$$

where the sum runs over all (isomorphism classes of) nondegenerate n-dimensional skew-Hermitian space. By Lemma 15.4 and Proposition 15.7, we have

$$j\ j_E^{\frac{n(n\ 1)}{4}} \int_{\mathrm{Temp}(U_{V_{qs}})=} I_{BC(\ )}(f_+)\frac{j\ \ (0;\ ;\ \mathrm{Ad};\ )j}{jS\ j}d\ = \sum_V {}_V \int_{\mathrm{Temp}_{U_V^0}(U_V)} J\ (f_+^V)d\ _{U_V}(\ ):$$

Using Proposition 16.7, the integration formula [BP21b, (2.10.1)] and the formal degree conjecture [BP21b, Theorem 5.53] we have

$$(16.10)\quad \begin{aligned} & j\ j_E^{\frac{n(n\ 1)}{4}} \int_{\mathrm{Temp}(U_{V_{qs}})=} I_{BC(\ )}(f^0)\frac{j\ \ (0;\ ;\ \mathrm{Ad};\ )j}{jS\ j}d \\ = & \int_{\mathrm{Temp}(U_{V_{qs}})=} \left(\sum_V {}_V \sum_{2\,\mathrm{Temp}_{H^V(F)}(G^V(F))} (\ )\ {}_{G^V}(\ ) \right) I_{BC(\ )}(f^0)\frac{j\ \ (0;\ ;\ \mathrm{Ad};\ )j}{jS\ j}d\ : \end{aligned}$$

This identity a priori holds for transferable f^0. We now explain that it holds for all f^0. If F is non-Archimedean, the all f^0 are transferable. If F is Archimedean, this holds for all transferable f^0. Assume that F is Archimedean. Then as in the proof of Proposition 16.8, we need to explain that both sides are absolutely convergent. The left hand side is clear as $\ 7!\ I\ $ is Schwartz. The absolute convergence of the right hand side follows from the additional fact that $j\ (\ )j$ is a constant for all $\in \mathrm{Temp}_{U(V)}(U_V)$ by Proposition 16.8, and that the inner double sum contains a unique nonzero term by the local Gan-Gross-Prasad conjecture.

Using the spectrum separating technique as explained in [BP21b, Lemma 5.82], thanks to Proposition 15.3, we conclude that

$$\sum_{V} \quad {}_{V} \sum_{\in \mathrm{Temp}_{U(V)}(U_V)} (\ ) = |\ |_E^{\frac{n(n-1)}{2}} .$$

By the local GGP conjecture again, there is a unique nonzero term on the right hand side. The theorem then follows.

16.4. Character identity and matching. Let V be a nondegenerate n-dimensional skew-Hermitian space. For any $\in \mathrm{Temp}(U_V)$, let $(\ )$ be as Theorem 16.9. The following theorem characterizes the matching of test functions via spherical character identities.

Theorem 16.10. Let $f_+^V \in S(U_{V,+})$, $f_+ \in S(G_+)$. The following are equivalent.

(1) For all $\in \mathrm{Temp}_{U_V^0}(U_V)$ we have

$$(\ ) I_{BC(\ )}(f_+) = J\ (f_+^V).$$

(2) For all matching $x \in G_+$ and $y \in U_{V,+}$ we have

$$(\ x) O(x; f_+) = O(y; f^V).$$

Proof. (2) $\Rightarrow$ (1) is Proposition 16.7. Let us prove that (1) $\Rightarrow$ (2). Let $f_{1,+}^V \in S(U_{V,+})$ and $f_{1,+} \in S(G_+)$ be matching test functions. By Proposition 16.2, Proposition 16.6 and Theorem 16.9, we have

$$\int_{A_{rs}(F)} O(x; f_+)\overline{O(x; f_{1,+})} dx = \int_{A_{rs}(F)} O(y; f_+^V)\overline{O(y; f_{1,+}^V)} dy. \tag{16.11}$$

Let $x \in A_{rs}$ be a regular semisimple point, and $U \subset A_{rs}$ a small neighbourhood of it. Since the maps $G^+ \to A$ and $U_V^+ \to A$ are locally a fibration near x, we see that for any $\varphi \in C_c^\infty(U)$ there are $f_{1,+}$ and $f_{1,+}^V$ such that

$$\varphi(x) = (\ x) O(x; f_{1,+}) = O(x; f_{1,+}^V).$$

By definition $f_{1,+}$ and $f_{1,+}^V$ match. Since φ is arbitrary, we conclude from (16.11) that

$$(\ x) O(x; f_+) = O(y; f_+^V).$$

This proves (1) $\Rightarrow$ (2).

17. The split case

17.1. Setup. We now make a few remarks on the case $E = F \times F$. The Galois conjugation c swaps the two factor of E. The objects that we have worked with in the previous sections make perfect sense. In particular, we can speak of regular semisimple elements, orbital integrals, and spherical characters.

We can find a $\tau_1 \in F^\times$ such that $\tau = (\tau_1, -\tau_1)$. We have a nontrivial additive character ψ of F and $\psi^E((x_1, x_2)) = \psi(\tau_1(x_1 - x_2))$. The character μ takes the form $\mu = (\mu_1, \mu_1^{-1})$ where μ_1 is a character of $F^\times$. We have the identification $G = G_n \times G_n = (G_n^0 \times G_n^0) \times (G_n^0 \times G_n^0)$, $H = G_n = G_n^0 \times G_n^0$ embedded in G via $(h_1, h_2) \mapsto ((h_1, h_2), (h_1, h_2))$, and $G' = G_n^0 \times G_n^0$ embedded by $(g_1, g_2) \mapsto ((g_1, g_1), (g_2, g_2))$. The symmetric space S_n consists of elements of the form (a, a^{-1}) where $a \in G_n^0$. It is identified with G_n^0 via the projection to the first factor. The projection $\rho : G_n \to S_n$ is given by $(a, b) \mapsto (ab^{-1}, a^{-1}b)$. The partial Fourier transform $\dagger$ is given by

$$\mathcal{S}(E_n) = \mathcal{S}(F_n \times F_n) \to \mathcal{S}(F_n \times F^n); \quad \phi^\dagger(x, y) = \int_{F_n} \phi(l + x, l - x)\, \psi((-1)^n \tau_1 l y)\, dl$$

Here we have made the identification that $E^- = \{(x, -x) \mid x \in F\} = F$. Recall that if $f = f_1 \otimes f_2 \in \mathcal{S}(G)$ where $f_i \in \mathcal{S}(G_n)$, and $\phi \in \mathcal{S}(F_n \times F_n)$, we have defined a function $\varphi_{(f \otimes \phi)^\dagger} \in \mathcal{S}(X)$ by (12.1). For $a \in G_n^0$, $x = ((a, a^{-1}), w, v) \in X = S_n \times F_n \times F^n$, we have

$$\varphi_{(f \otimes \phi)^\dagger}(x) = \int_{G_n^0} \int_H f_1(h^{-1}) f_2(h^{-1}(a, 1)g)(R_{\mu^{-1}}(h)\phi)^\dagger(w, v)\, dh\, dg. \tag{17.1}$$

The orbital integral equals

$$\begin{aligned} &O((((1, 1), (a, 1)), w, v), f \otimes \phi) \\ = &\int f_1(h_1^{-1} h_2^{-1}, h_2^{-1}) f_2(h_1^{-1} g_1^{-1} a g_2, g_1^{-1} g_2)\, R_{\mu^{-1}}((h_1, 1))\phi^\dagger(w g_1, g_1^{-1} v)\, dh_1\, dh_2\, dg_1\, dg_2. \end{aligned} \tag{17.2}$$

Here the integration is over $h_1, h_2, g_1, g_2 \in G_n^0$.

We identify the skew-Hermitian space with $V = F^n \times F^n$ with the skew-Hermitian form given by

$$q_V((x_1, x_2), (y_1, y_2)) = ({}^t x_1 y_2, -{}^t x_2 y_1).$$

A polarization of $\operatorname{Res} V$ is given by

$$\operatorname{Res} V = L + L^-, \quad L = \{(0, x) \mid x \in F^n\}, \quad L^- = \{(x, 0) \mid x \in F^n\}.$$

Identify $U(V)$ with the subgroup $\{(g, {}^t g^{-1}) \mid g \in G_n^0\}$ of G_n, which is isomorphic to G_n^0 via the projection to the first factor. Then U_V is identified with $G_n^0 \times G_n^0$ and $U(V) = G_n^0$ embeds in U_V diagonally. Identify both L and L^- with F^n. The Weil representation ω is realized on $\mathcal{S}(L^-)$ and is given by

$$\omega(g)\phi(x) = |\det g|^{\frac{1}{2}} \mu_1(\det g)\, \phi({}^t g x), \quad \phi \in \mathcal{S}(F^n).$$

We observe that the representation ! – ! of $G_n^0 \quad G_n^0$ is isomorphic to R $_1$. The partial Fourier transform ^{z} is

$$S(F^n \quad F^n) \,!\, S \;(F^n \quad F^n); \quad (\;_1 \quad _2)^z(x;y) = \int_{F^n} \;_1(u+x) \;_2(u \quad x) \;(\quad 2\,{}^t uy)dl;$$

cf. (15.1).

Recall that if $f^\vee = f_1^\vee \quad f_2^\vee \; 2\; S(G_n^0 \quad G_n^0)$, $f_1^\vee; f_2^\vee \; 2\; S(G_n^0)$, and $\;_1^\vee; \;_2^\vee \; 2\; S(F^n)$, we have defined a function ' $_{(f^\vee \quad _1^\vee \quad _2^\vee)^z}$ by (12.5). Though there is only one V, we write the supscript to distinguish the notation from the GL-side. If a 2 G_n^0 and y = (a; w; v) 2 Y = ($G_n^0 \quad G_n^0$) $F^n \quad F^n$ is regular semisimple, then we have

$$(17.3) \qquad '_{(f \quad _1 \quad _2)^z}(y) = \int_{G_n^0} f(g^{\;1}(1;a))(! - (g) \;_1 \quad _2)^z(x;y)dg:$$

The orbital integral equals

$$(17.4) \qquad \begin{aligned} &O(((1;a);(x;y)); f^\vee \quad {}_1^\vee \quad {}_2^\vee) \\ &= \int_{G_n^0}\int_{G_n^0} f_1^\vee(h^{\;1}) f_2^\vee(h^{\;1}g^{\;1}ag) \;\; ! - (h) \;{}_1^\vee \quad {}_2^\vee \;{}^z({}^t gx; g^{\;1}y)dhdg: \end{aligned}$$

17.2. Matching. Let us now consider the matching of orbital integrals. Let

$$x = ((g_1;g_2);w;v) \;2\; G^+; \quad y = ((a_1;a_2);x;y) \;2\; U_V^+;$$

be regular semisimple elements. Put $= \;(g_1^{\;1}g_2) \;2\; S_n \;'\; G_n^0$ and $\;2\; a_1^{\;1}a_2 \;2\; U(V) \;'\; G_n^0$. By definition they match if and are conjugate in G_n^0 and w ^{i}v = 2(1)$^{n \;1}$ $_1^{\;1}$ ^{t}x ^{i}y for all i = 0; : : : ; n 1. The latter condition is equivalent to (1)^n $_1$w ^{i}v = 2^{t}x ^{i}y.

Lemma 17.1. Take $f_{11}; f_{12}; f_{21}; f_{22} \;2\; S(G_n^0)$, $\;_1; \;_2 \;2\; S(F_n)$. Put

$$f_1 = f_{11} \quad f_{12}; \; f_2 = f_{21} \quad f_{22} \;2\; S(G_n); \quad f = \;_1(\;1)^{\frac{n(n+1)}{2}} f_1 \quad f_2 \;2\; S(G);$$

and

$$f_1^\vee = f_{11} \quad f_{\overline{12}}; \; f_2^\vee = f_{21} \quad f_{\overline{22}} \;2\; S(G_n^0); \quad f^\vee = f_1^\vee \quad f_2^\vee \;2\; S(G_n^0 \quad G_n^0);$$

and a function $\;2\; S(E_n)$ by

$$(\,{}^t x;y) = \;{}_1^\vee(x) \;{}_2^\vee(y); \quad x;y \;2\; F^n:$$

Then f and $f^\vee \quad {}_1^\vee \quad {}_2^\vee$ match.

Proof. This follows from the explicit form of the orbital integrals (17.2) and (17.4). The factor $_1(\;1)^{\frac{n(n+1)}{2}}$ comes from the transfer factor. Indeed, in this case, for any regular semisimple $((g_1;g_2);w;v) \;2\; G^+$, we have $(((g_1;g_2);w;v)) = \;_1(\;1)^{\frac{n(n+1)}{2}}$.

We can define the spherical characters as before. With the above matching, we have the same character identity. Let be the constant as defined in Theorem 16.9. Since E = F F, it simplifies to

$$= j \;_1 j^{\frac{n(n\;1)}{2}} \;_1(\;1)^{\frac{n(n+1)}{2}}:$$

The following is the split version of Theorem 16.9.

Proposition 17.2. Let $= {}_1 \quad {}_2$ be an irreducible tempered of $G_n^0 \quad G_n^0$. Let ${}_i = {}_i \quad {}_{\bar{i}}$ be the (split) base change of ${}_i$ to G_n. Put $= {}_1 \quad {}_2$, which is an irreducible tempered representation of G. Let f; ; $f^{\vee}$; ${}_1^{\vee}$; ${}_2^{\vee}$ be the functions de ned in Lemma 17.1. Then

$$J \ (f^{\vee} \quad {}_1^{\vee} \quad {}_2^{\vee}) = I \quad (f \quad) :$$

Proof. Let

$$_1(u) = \ (\ _1(u_{12} + \quad + u_{n \ 1;n})) ; \quad u \ 2 \ N_n^0$$

be a generic character of N_n^0 and we let $W_1 = W(\ _1; \ _1)$ and $W_2 = W(\ _2; \overline{\ _1})$ be Whittaker models of ${}_1$ and ${}_2$ respectively. The factor ${}_1$ appears for compatibility with the Whittaker model used in the nonsplit case. Let $W_{\bar{1}} = W(\ _{\bar{1}}; \overline{\ _1})$ and $W_{\bar{2}} = W(\ _{\bar{2}}; \ _1)$ be the Whittaker models for ${}_{\bar{1}}$ and ${}_{\bar{2}}$ respectively. If $W \ 2 \ W_i$, we de ne $W^- \ 2 \ W_{\bar{i}}$ by

$$W^-(g) = W(w_n \, {}^t g \ ^1)$$

where w_n is the longest Weyl group element in G_n^0 whose antidiagonal elements equal one. If W runs over an orthogonal basis of W_i, then W^- runs over an orthogonal basis of $W_{\bar{i}}$.

By de nition we have

$$J \ (f^{\vee} \quad {}_1^{\vee} \quad {}_2^{\vee}) = \sum_{W_1;W_2} \int_{G_n^0} h \ _1(h) \ _1(f_1^{\vee})W_1; W_1 i^{Wh} h \ _2(h) \ (f_2^{\vee})W_2; W_2 i^{Wh} h! \ ^-(h) \ _1; \overline{\ _2} i \, dh:$$

where $W_1; W_2$ range over orthonormal basis of W_1 and W_2 respectively.

We now compute I . The linear form in this case reduces to a pairing between W_i and $W_{\bar{i}}$, $i = 1; 2$. It follows that

$$I \ (f \quad) = \ _1(\ 1)^{\frac{n(n+1)}{2}} \sum_{W_1;W_2} (\ _1(f_{11})W_1 \quad _2(f_{21})W_2 \quad _1) \ (\ _1(f_{12})W_{\bar{1}} \quad _2(f_{22})W_{\bar{2}} \quad _2);$$

where W_1 and W_2 run over orthonormal bases of W_1 and W_2 respectively. It also equals

$$I \ (f \quad) = \ _1(\ 1)^{\frac{n(n+1)}{2}} \sum_{W_1;W_2} (\ _1(f_1)W_1 \quad _2(f_2)W_2 \quad _1) \ (W_{\bar{1}} \quad W_{\bar{2}} \quad _2):$$

We identify $W_{\bar{i}}$ with $\overline{W_i}$. Then the proposition follows from the next lemma. It is part of the proof of this proposition, but we list it as a separate lemma as it is maybe of some independent interest.

Lemma 17.3. Let the notation be as in the above proof. For any $W_i \ 2 \ W_i$, $i = 1; 2$, and $2 \ S(F_n)$, we have

$$j \ _1 j^{\frac{n(n-1)}{2}} \ (W_1; W_2; \ ) \ \overline{(W_1; W_2; \ )}$$
$$= \int_{G_n^0} h \ _1(h) W_1; W_1 i^{Wh} h \ _2(h) W_2; W_2 i^{Wh} h! \ ^-(h) \ ; \quad i \, dh:$$

Proof. For the proof of this lemma, it is more convenient to use the Whittaker models $W(\pi_i,\psi)$, $i = 1,2$ where

$$\psi(u) = \psi(u_{12} + \cdots + u_{n-1,n}), \quad u \in N_n^0$$

is a generic character of N_n^0, i.e. without the factor τ_1. Put temporarily

$$\delta = \mathrm{diag}[\tau_1^{n-1}, \tau_1^{n-2}, \ldots, 1] \in G_n(F).$$

If $W_i \in W(\pi_i,\psi)$, then $W_i^\delta \in W(\pi_i,\psi_{\tau_1})$ where

$$W_i^\delta(g) = W_i(\delta g).$$

With this change, for $W_i \in W(\pi_i,\psi)$, $i = 1,2$ we have

$$\Psi(W_1^\delta, W_2^\delta, \phi) = |\tau_1|^{\frac{n(n+1)(n-1)}{6}} \Psi(W_1, W_2, \phi), \quad \langle W_i^\delta, W_i^\delta\rangle^{Wh} = |\tau_1|^{\frac{n(n-1)(n-2)}{6}} \langle W_i, W_i\rangle^{Wh}.$$

The desired equality in the lemma thus reduces to

$$\Psi(W_1,W_2,\phi_1)\overline{\Psi(W_1,W_2,\phi_2)} = \int_{G_n(F)} \langle \pi_1(h)W_1, W_1\rangle^{Wh} \langle \pi_2(h)W_2, W_2\rangle^{Wh} \langle \omega_{\psi^{-1}}(h)\phi_1, \phi_2\rangle dh \tag{17.5}$$

for all $W_i \in W(\pi_i,\psi)$.

Both sides of (17.5) define nonzero elements in

$$\mathrm{Hom}_{G_n^0}(\pi_1 \hat\otimes \pi_2 \hat\otimes \mathcal{S}(F_n), \mathbb{C}) \otimes \overline{\mathrm{Hom}_{G_n^0}(\pi_1 \hat\otimes \pi_2 \hat\otimes \mathcal{S}(F_n), \mathbb{C})},$$

and this Hom space is one dimensional by [Sun12, SZ12]. It follows that (17.5) holds if we could find some $W_i \in W(\pi_i,\psi)$, $i = 1,2$, and a $\phi \in \mathcal{S}(F_n)$ such that (17.5) holds for this choice and is nonzero. To achieve this, we are going to reduce it to an analogues statement for the Rankin–Selberg integral for $G_n^0 \times G_{n+1}^0$, proved in [Zha14b, Proposition 4.10].

We will consider a G_n^0 naturally as a subgroup of G_{n+1}^0, by $g \mapsto \begin{pmatrix} g & \\ & 1 \end{pmatrix}$, $g \in G_n^0$. Recall that if π_3 is an irreducible tempered representation of G_{n+1}^0, then it is well-known that $C_c^\infty(N_n^0\backslash G_n^0, \psi)$ is contained in $W(\pi_i,\psi)$, cf. [GK75, Theorem 6] and [Kem15, Theorem 1]. Let $f \in C_c^\infty(N_{n-1}^0\backslash G_{n-1}^0, \psi)$, then there is an $W_f \in W(\pi_2,\psi)$ such that $W_f|_{G_n^0} = f$. Let $\phi \in C_c^\infty(F_{n-1} \times F^\times)$. Consider the function on G_n^0 given by

$$\varphi(g) = W_f(g)\,\omega(\det g)^{-1}|\det g|^{\frac12}\,\phi(e_n g).$$

By the above choices we conclude that $\varphi \in C_c^\infty(N_n^0\backslash G_n^0, \psi)$. Let π_3 be an irreducible tempered representation of G_{n+1}^0. Then there is a $W_\varphi \in W(\pi_3,\psi)$, such that its restriction to G_n^0 equals φ.

By [Zha14b, Proposition 4.10] for any $W_1 \in W(\pi_1,\psi)$ we have

$$\Psi^0(W_1, W_\varphi)\overline{\Psi^0(W_1, W_\varphi)} = \int_{G_n^0} \langle \pi_1(h)W_1, W_1\rangle^{Wh} \langle \pi_3(h)W_\varphi, W_\varphi\rangle^{Wh} dh. \tag{17.6}$$

Here Ψ^0 stands for the Rankin–Selberg integral for $G_n^0 \times G_{n+1}^0$, i.e.

$$\Psi^0(W_1, W_\varphi) = \int_{G_n^0} W_1(h) W_\varphi(h) dh.$$

By our choices we have

$$^{0}(W_1; W_{\cdot}) = \quad (W_1; W_f; \quad); \quad \langle W_{\cdot}; W_{\cdot} \rangle^{Wh} = \langle W_f; W_f \rangle^{Wh} \langle \quad; \quad \rangle.$$

Thus the equality (17.6) reduces to (17.5). It is clear from the above construction that we can choose W_1, f and such that the above integrals do not vanish. This finishes the proof of the lemma, and hence Proposition 17.2.

The local trace formulae Proposition 16.2 and 16.6 played important roles in the proof of the local character identity Theorem 16.9. Let us mention that the same local relative trace formula hold under the current assumption that $E = F \quad F$. The proof goes through with only obvious modification. Note that the counterpart of [BP20, Theorem 5.53], i.e. the formal degree conjecture, holds for general linear groups by [HII08].

18. Global comparison of relative characters for regular cuspidal data

18.1. Preliminaries.

We are back to the global situation, so E/F stands for a quadratic extension of number fields.

18.1.1. Levels. Throughout this section, we will work with Schwartz functions of a fixed level. Let $V_{F,\infty}$ be the set of Archimedean places of F. Let S be a finite set of places of F containing $V_{F,\infty}$ as well as the ramified places of E/F. We denote by H^S the finite set of nondegenerate skew c-Hermitian spaces of rank n that admit a selfdual O_{E_v}-lattice $\quad_v$ for every $v \notin S$. If $V \in H^S$, for every $v \notin S$ the groups G and U_V admit models over O_{F_v}, and we have open compact subgroups $K_v := G(O_{F_v})$ and $K_{V,v} := U_V(O_{F_v})$ (the stabiliser of $\quad_v$). For each finite place $v \in S \setminus V_{F,\infty}$, let $K_v^0 \quad G(F_v)$ and $K_{V,v}^0 \quad U_V(F_v)$ be some open compact subgroups. Set

$$K^{\infty} := \prod_{v \in S \setminus V_{F,\infty}} K_v^0 \prod_{v \notin S} K_v; \text{ and } K_V^{\infty} := \prod_{v \in S \setminus V_{F,\infty}} K_{V,v}^0 \prod_{v \notin S} K_{V,v}.$$

At each finite place v we have decompositions

$$S(G_+(F_v)) = S(G(F_v)) \quad S(E_{n,v}) \text{ and } S(U_{V,+}(F_v)) = S(U_V(F_v)) \quad S(L_{\overline{v}}) \quad S(L_{\overline{v}}).$$

We have left and right regular actions of $G(F_v)$ (resp. $U_V(F_v)$) on the first factors. Let $S(G_+(\mathbb{A}); K^{\infty})$ and $S(U_{V,+}(\mathbb{A}); K_V^{\infty})$ be the subsets of bi-K^{∞} invariant (resp. bi-K_V^{∞} invariant) functions of $S(G_+(\mathbb{A}))$ (resp. $S(U_{V,+}(\mathbb{A}))$). Denote by $S(G(\mathbb{A}); K^{\infty})$ and $S(U_V(\mathbb{A}); K_V^{\infty})$ the usual Schwartz spaces of bi-invariant functions. Note that $S(G(\mathbb{A}); K^{\infty}) \quad S(\mathbb{A}_{E,n})$ is dense in $S(G_+(\mathbb{A}); K^{\infty})$ (resp. $S(U_V(\mathbb{A}); K_V^{\infty}) \quad S(L_{-}(\mathbb{A})) \quad S(L_{-}(\mathbb{A}))$ is dense in $S(U_{V,+}(\mathbb{A}); K_V^{\infty})$).

18.1.2. Multipliers. Let T be the union of $S \setminus V_{F,\infty}$ and of the set of all finite places of F that are inert in E. Denote by $M^T(G(\mathbb{A}))$ the algebra of T-multipliers defined in [BPLZZ21, Definition 3.5] relatively to $\prod_{v \notin T} K_v$. Any $m \in M^T(G(\mathbb{A}))$ gives rise to a continuous linear operator $m \quad$ of

$S(G(\mathbb{A});K^{\infty})$. They satisfy the following property: for every irreducible admissible representation π of $G(\mathbb{A})$, there exists a complex number $m(\pi)$ such that for all $f \in S(G(\mathbb{A});K^{\infty})$ we have

$$\pi(m * f) = m(\pi)\,\pi(f). \tag{18.1}$$

At the level of restricted tensor products, there exists a finite set of places S^0 containing $V_{F;\infty}$ and disjoint from T such that, for any $f = f_{S^0} \otimes f^{S^0} \in S(G(\mathbb{A});K^{\infty})$, we have

$$m * f = m_{S^0} * f_{S^0} \otimes f^{S^0} \tag{18.2}$$

where m_{S^0} is a continuous linear operator of $S(G(F_{S^0});K^{\infty}_{S^0})$ the subalgebra of $S(G(F_{S^0}))$ of Schwartz functions which are $\prod_{v \in S^0 \setminus V_{F;\infty}} K_v$ bi-invariant.

We extend any multiplier $m \in \mathcal{M}^T(G(\mathbb{A}))$ to $S(G_+(\mathbb{A});K^{\infty})$ by acting trivially on the last factor of $S(G(\mathbb{A});K^{\infty}) \otimes S(A_{E;n})$ and using density.

For $V \in \mathcal{H}^S$, denote by $\mathcal{M}^T(U_V(\mathbb{A}))$ the algebra of T-multipliers of $U_V(\mathbb{A})$ relatively to $\prod_{v \notin T} K_{V;v}$. We have similar statements as above, and in particular we extend any $m_V \in \mathcal{M}^T(U_V(\mathbb{A}))$ to $S(U_{V;+}(\mathbb{A});K^{\infty}_V)$.

18.1.3. Cuspidal data and Arthur parameters. Let σ be an irreducible automorphic cuspidal representation of $M_P(\mathbb{A})$ the standard Levi of some standard parabolic subgroup P of G, and set $\pi = \mathrm{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})}\sigma$. Assume that π is a $(G;H;\infty)$-regular Hermitian Arthur parameter in the sense of section 1.1, and denote by $\chi_0 \in \mathfrak{X}(G)$ the cuspidal datum represented by the pair $(M_P;\sigma)$. It follows from Remark 10.1 that χ_0 is a $(G;H;\infty)$-regular Hermitian cuspidal datum in the sense of section 10.2. Moreover, let π_0 be the discrete component of π (see Subsection 1.1). Enlarging S and shrinking the K^0_v if necessary, we assume that π admits non-zero K^{∞}-fixed vectors.

Recall that L_π is a Levi subgroup of G containing M_P defined in section 10.2, and that we introduced in (1.2) the group S_π. Its order is

$$|S_\pi| = 2^{\dim(\mathfrak{a}_{L_\pi}) - \dim(\mathfrak{a}^{L_\pi}_{M_P})}.$$

Recall that the weak base change was defined in section 1.1. For $V \in \mathcal{H}^S$, we denote by $\mathfrak{X}^V \subset \mathfrak{X}(U_V)$ the set of cuspidal data represented by pairs $(M_{Q_V};\tau)$ such that we have the following conditions.

- π is a weak base change of $(Q_V;\tau)$.
- τ is $\prod_{v \notin S}(M_{Q_V}(F_v) \cap K_{V;v})$-unramified.
- with the identifications $M_Q = G^\flat \times U$ where $G^\flat$ is a product of linear groups, and $U = U(V_1) \times U(V_2)$ with V_1 and V_2 being two skew c-Hermitian spaces, we have $\tau = \tau^\flat \boxtimes \eta_0$, where η_0 is a cuspidal automorphic representation of $U(\mathbb{A})$ and for all $v \notin T \cup V_{F;\infty}$, the representation $\pi_{0;v}$ is the split base change of $\eta_{0;v}$.

It follows from the first point and the conditions on that such a $(\quad M_{Q_V}, \ )$ represents a $(U_V, U_V^0, \ ^{1})$-regular cuspidal datum in the sense of section 11.2. Moreover, we have a natural isomorphism

$$\text{(18.3)} \qquad bc: \mathfrak{a}_{M_P}^{L,} \to \mathfrak{a}_{M_{Q_V}}.$$

Indeed, if we write $G^{]} = \prod G_{m_i}$ where the m_i are integers, we have $\mathfrak{a}_{M_{Q_V}} = \bigoplus_i \mathfrak{a}_{G_{m_i}}$. But $\prod G_{m_i} \quad G_{m_i}$ is also a factor of M_P, so that $\mathfrak{a}_{M_{Q_V}} \quad \mathfrak{a}_{M_{Q_V}} = \bigoplus_i \mathfrak{a}_{G_{m_i}} \quad \mathfrak{a}_{G_{m_i}}$ is a subspace of $\mathfrak{a}_{M_P}$. The map (18.3) is now the inverse of $x \mapsto (x, \ x)$ whose image is $\mathfrak{a}_{M_P}^{L,}$. In particular, $\mathfrak{a}_{M_P}^{L,}$ is isomorphic to $\mathfrak{a}$ defined in (1.3). Via bc, the pullback of the measure on $i\mathfrak{a}_{M_{Q_V}}$ is $2^{\dim(\mathfrak{a}_{M_P}^{L,})}$ times the measure on $i\mathfrak{a}_{M_P}^{L,}$, where we recall that the measures on these spaces were defined in Subsection 4.5.2.

18.1.4. Relative characters. Let be as in the previous section. Recall that we have defined in (10.25) for every $\ \in i\mathfrak{a}_{M_P}^{L,}$ a distribution I , which we have extended by continuity to $S(G_+(\mathbb{A}))$ by Lemma 10.8.

On the unitary side, let $V \in \mathcal{H}^S$ and let $(M_{Q_V}, \ )$ be a representative of a class in $\mathfrak{X}^V$. Then we have defined in (11.5) for every $\ \in i\mathfrak{a}_{Q_V}$ the distribution $J_{Q_V, }(, \ )$, extended by continuity to $S(U_{V,+}(\mathbb{A}))$. We define for $\ \in i\mathfrak{a}_{M_P}^{L,}$

$$\text{(18.4)} \qquad J^V(, f_+^V) := \sum_{(M_{Q_V}, \ )} J_{Q_V, }(bc( ), f_+^V),$$

where the sum ranges over representatives of classes in $\mathfrak{X}^V$. Note that this is independent of the chosen representatives by Lemma 11.2. This sum is absolutely convergent with uniform convergence on compact subsets by a strengthening of [BPCZ22, Proposition 2.8.4.1] using [Mül89, Corollary 0.3] (see the discussion in [BPC, Section 7.1.5]). In particular, it yields an holomorphic function in .

18.2. A global identity.

Theorem 18.1. Let $f_+ \in S(G_+(\mathbb{A}), K^{1})$ be of the form $f_+ = \ _H^{S,} \ _{G^0}^{S,} f_{+,S} \prod_{v \notin S} 1_{G_+(\mathcal{O}_v)}$ where $f_{+,S} \in S(G_+(F_S))$, and let $f_+^V \in S(U_{V,+}(\mathbb{A}), K_V^{1})$ for every $V \in \mathcal{H}^S$. For $V \notin \mathcal{H}^S$, set $f_+^V = 0$. Assume that f_+ and $\{f_+^V\}_V$ match in the sense of Section 13.2.2. Then for every $\ _0 \in i\mathfrak{a}_{M_P}^{L,}$ we have

$$\text{(18.5)} \qquad \sum_{V \in \mathcal{H}^S} J^V(\ _0, f_+^V) = |S \ |^{1} I_{\ _0}(f_+).$$

Proof. The proof is very similar to [BPC, Theorem 7.1.6.1] and uses multipliers. We recall below the main steps. The following two lemmas are [BPC, Lemma 7.1.7.1] and [BPC, Lemma 7.1.7.2].

Lemma 18.2. Let $V \in \mathcal{H}^S$ and $\ _0 \in i\mathfrak{a}_{M_P}^{L,}$ be in general position. There exists a multiplier $m_V \in \mathcal{M}^{T}(U_V(\mathbb{A}))$ such that we have the following conditions.

(1) For all $f^V \in S(U_V(\mathbb{A}); K_V^{\infty})$, the right convolution $m_V * f^V$ sends $L^2([U_V])$ into

$$\bigoplus_{\chi \in \mathfrak{X}^V} L^2_{\chi}([U_V]).$$

(2) For $(M_{Q_V}, \tau) \in \mathfrak{X}^V$, we have $m_V(\mathrm{Ind}_{Q_V}^{U_V}(\tau_{bc(\lambda_0)})) = 1$.

Lemma 18.3. Let $\lambda_0 \in i\mathfrak{a}_{M_P}^{L,*}$. There exists a multiplier $m \in \mathcal{M}^T(G(\mathbb{A}))$ such that we have the following conditions.

(1) For all $f \in S(G(\mathbb{A}); K^{\infty})$, the right convolution $m * f$ sends $L^2(G(F)A_G(\mathbb{A})\backslash G(\mathbb{A}))$ into $L^2_{\chi_0}([G])$.

(2) We have $m(\lambda_0) = 1$.

Let $\lambda_0 \in i\mathfrak{a}_{M_P}^{L,*}$ in general position, and take the multipliers m and m_V given by Lemma 18.2 and Lemma 18.3 for $V \in \mathcal{H}^S$. By [BPLZZ21, Lemma 4.12], we may assume that the m_V are base changes of m in the sense of [BPLZZ21, (4.7)]. This implies that for every large enough finite set of places S' disjoint from T and every $\sigma_{S'} \in \mathrm{Temp}_{U(V)(F_{S'})}(U_V(F_{S'}))$ we have

$$m_{S'}(BC(\sigma_{S'})) = m_{V,S'}(\sigma_{S'}); \tag{18.6}$$

where the local base change BC was defined in (14.4).

Lemma 18.4. Let $f_+ \in S(G_+(\mathbb{A}); K^{\infty})$ and $f_+^V \in S(U_{V,+}(\mathbb{A}); K_V^{\infty})$ for $V \in \mathcal{H}^S$. Set $f_+^V = m_V * f_+^V = 0$ for $V \notin \mathcal{H}^S$, and assume that f_+ and $\{f_+^V\}_V$ match in the sense of section 13.2.2. Then $m * f_+$ and $\{m_V * f_+^V\}_V$ also match.

Proof. For every $v \notin S$ that is inert we have $(m * f_+)_v = 1_{G_+(\mathcal{O}_v)}$. It follows from the fundamental lemma (Theorem 13.9) that for every $V \notin \mathcal{H}^S$ the function $m * f_+$ matches with $0 = m_V * f_+^V \in S(U_{V,+}(\mathbb{A}))$. Assume now that $V \in \mathcal{H}^S$. Let S' be a finite set of places such that m and the m_V act trivially outside of S' as in (18.2). It is enough to show that m and m_V preserve the matching at the places in S'. By Theorem 16.10 and its split counterpart Proposition 17.2, if $f_+ \in S(G_+(F_{S'}); K_{S'}^{\infty})$ and $f_+^V \in S(U_{V,+}(F_{S'})); K_{V,S'}^{\infty})$ match, then they have matching local relative characters. By (18.1) and (18.6), $m_{S'} * f_+$ and $m_{V,S'} * f_+^V$ also have matching relative characters. It follows from Theorem 16.10 and Proposition 17.2 again that they match, which concludes the proof.

By Lemma 18.4 and Theorem 13.11 we have for each $\chi \in \mathfrak{X}(F)$

$$I_{\chi}(m * f_+) = \sum_{V \in \mathcal{H}^S} J_{\chi}^V(m_V * f_+^V).$$

By Theorems 7.7 and 9.7 we obtain the spectral identity

$$\sum_{\chi \in \mathfrak{X}(G)} I_{\chi}(m * f_+) = \sum_{V \in \mathcal{H}^S} \sum_{\chi \in \mathfrak{X}(U_V)} J_{\chi}^V(m_V * f_+^V); \tag{18.7}$$

where both sides are absolutely convergent. By Theorem 6.8 and Lemma 18.2 for the LHS, and by Proposition 8.8 and Lemma 18.3 for the RHS, (18.7) reduces to

$$I_{\ 0}(m \quad f_+) = \sum_{V \in H^S} \sum_{\ \in X^V} J^V(m_V \quad f_+^V):$$

By Theorems 10.10 and 11.5, and an easy change of variables, (18.7) reads

$$\begin{aligned} 2^{\ \dim(\mathfrak{a}_L \ )} \int_{i\mathfrak{a}^L_{M_P}\ ;} I \quad (m \quad f_+)d \quad &= \sum_{V \in H^S} \sum_{\ \in X^V} \int_{i\mathfrak{a}_Q} J_{Q;} \ (\ ; m_V \quad f_+^V)d \\ &= 2^{\ \dim \mathfrak{a}^L_{M_P}\ ;} \sum_{V \in H^S} \int_{i\mathfrak{a}^L_{M_P}\ ;} J^V(\ ; m_V \quad f_+^V)d \ ; \end{aligned}$$

where $(M_Q; \ )$ is a representative of $\ \in X^V$. Note that all the integrals are absolutely convergent by Theorem 10.10 and Proposition 11.3. We now use the same spectral separation techniques as in [BPC, Theorem 7.1.6.1] to conclude.

19. Proof of the Gan{Gross{Prasad conjecture: Eisenstein case

In this section we prove the Gan{Gross{Prasad conjecture for $(G; H; \ ^1)$-regular Hermitian Arthur parameters stated in Theorem 1.2, and its re nement Theorem 1.5. We will use the notations of Section 10, Section 11 and Section 18.

19.1. Proof of Theorem 1.2. The proof follows from the comparison of trace formulae established in Section 18 and is very similar to [BPC, Theorem 1.2.3.1]. We recall the main steps.

Let $= \mathrm{Ind}_{P(A)}^{G(A)}$ be a $(G; H; \ ^1)$-regular Hermitian Arthur parameter in the sense of Section 1.1, and let $2\ i a_{M_P}^{L\ ;} = i a$. By continuity, the relative character I de ned in (10.25) is non-zero if and only if it is non-zero on pure tensors, which happens if and only if the linear forms $Z^{RS}(\ ;\ ;\ ;0)$ and are non zero on $W(\ ;\ _N)\ S\ (A_{E;n})$ and $W(\ ;\ _N)$ respectively, where we recall that $W(\ ;\ _N)$ is the Whittaker model of . We have the following properties.

By [JPSS83] and [Jac04], $Z^{RS}(\ ;\ ;\ ;0)$ is non zero if and only if $L(\frac{1}{2};\ \ ^1) \neq 0$.

By [GK75], [Jac10, Proposition 5] and [Kem15], is always non zero on W .

On the unitary side, let V be a nondegenerate skew-Hermitian space of dimension n. Let be an automorphic cuspidal representation of $M_{Q_V}(A)$ a Levi subgroup of a parabolic subgroup Q_V of U_V. Assume that is the weak base change of $(M_{Q_V};\ )$ (see Section 1.1). Recall that we have de ned in (18.3) a linear map bc. It follows from (11.5) that $J_{Q_V;}\ (bc(\ );\ )$ is non zero if and only if the period $P(\ ;\ ; bc(\ ))$ is non zero on $A_{Q_V;}\ (U_V)\ S\ (L^{-}(A))$. Therefore, Theorem 1.2 amounts to the equivalence of the following two statements.

(A) The distribution I is nonzero.

(B) There exists a V, a parabolic Q_V of U_V and a cuspidal representation of M_{Q_V} such that is the weak base change of $(M_{Q_V};\ )$ and $J_{Q_V;}\ (bc(\ );:)$ is non-zero.

19.1.1. Proof of (A) =) (B). Let S be a su ciently large nite set of places of F containing the Archimedean ones. For each $v\ 2\ S n V_{F;1}$ take $K_v^0\ \ G(F_v)$ to be an open compact subgroup, and for each $v\ 2\!\!=\ S$ let K_v be a maximal open compact subgroup of $G(F_v)$ as in Subsection 18.1. De ne $K^1 = \prod_{v 2 S n V_{F;1}} K_v^0 \prod_{v 2\!= S} K_v$ and assume that I is non zero on $S(G_+(A); K^1)$ (which implies that has non-zero K^1- xed vectors). As I is continuous, by the existence of transfer in the non-Archimedean case and its approximation in the Archimedean case (Theorem 13.4) and the fundamental Lemma (Theorem 13.9), up to enlarging S and shrinking K^1 we may choose $f_+\ 2\ S(G_+(A); K^1)$ and a collection of $f_+^V\ 2\ S(U_{V;+}(A); K_V^1)$ for $V\ 2\ H^S$ which satisfy the hypotheses of Theorem 18.1 and such that $I\ (f_+) \neq 0$. The result now follows from the comparison of global relative trace formulae in Theorem 18.1.

19.1.2. Proof of (B) =) (A). Choose

S to a nite set of places of F containing the Archimedean ones,

$V \in H^S$, $K^0_{V,v} \subset U_V(F_v)$ an open compact subgroup for each $v \in S \setminus V_{F,\infty}$, $K_{V,v} \subset U_V(F_v)$ a maximal open compact subgroup for each $v \notin S$ as in Subsection 18.1,

a standard parabolic subgroup Q_V of U_V and a cuspidal representation of $M_{Q_V}(\mathbb{A})$,

$g^V_+ = g^V_\infty \otimes 1_{K^0_\infty} \otimes 2 \in S(U_{V,+}(\mathbb{A}); K^1_V)$, where $K^1_V := \prod_{v \in S \setminus V_{F,\infty}} K^0_{V,v} \prod_{v \notin S} K_{V,v}$,

such that

the cuspidal datum represented by $(M_{Q_V}; \ )$ belongs to X^V (see section 18.1.3),

$J_{Q_V,}(bc(\ ); g^V_+) \neq 0$.

Set $f^V_+ = g^V \ast g^{V,}_{1} \ast \overline{\ }_{1}$, where $g^{V,}(h) = \overline{g^V(h^{-1})}$. Then we have $J_{Q^0_V, {}^0}(bc(\ ); f^V_+) \geq 0$ for all pairs $(Q^0_V; {}^0)$ in X^V, and moreover $J_{Q_V,}(bc(\ ); f^V_+) > 0$. This yields $J^V(\ ; f^V_+) > 0$. For $V' \in H^S$ different from V, set $f^{V'}_+ = 0$. Then we have

$$\sum_{V' \in H^S} J^{V'}(\ ; f^{V'}_+) > 0.$$

As the LHS is continuous, by the existence of non-Archimedean transfer and the approximation of smooth-transfer (Theorem 13.4) and the fundamental Lemma (Theorem 13.9), we may assume that there exists an open compact subgroup K^1 of $G(\mathbb{A}_f)$ and $f_+ \in S(G_+(\mathbb{A}); K^1)$ such that f_+ and $\{f^V_+\}$ satisfy the hypotheses of Theorem 18.1 and $\sum_{V' \in H^S} J^{V'}(\ ; f^{V'}_+) \neq 0$. The result now follows again from the comparison of global relative trace formulae in Theorem 18.1.

19.2. Proof of Theorem 1.5.

19.2.1. Setting. Let V be a nondegenerate skew-c-Hermitian space of dimension n. Let $Q_V = M_{Q_V} N_{Q_V}$ be a standard parabolic subgroup of U_V, and $= {}^0_V \ {}_V$ be a cuspidal representation of $M_{Q_V}(\mathbb{A})$ which is tempered everywhere. The group $P = \mathrm{Res}_{E/F}(Q_V \otimes_F E)$ can be identified with a parabolic subgroup $P = MN$ of G. By [Mok15] and [KMSW], admits a strong base change to M, i.e. for all place v we have ${}_v = BC({}_v)$ where the local base change map was defined in (14.4). In particular, $:= \mathrm{Ind}^{G(\mathbb{A})}_{P(\mathbb{A})}$ is a weak base change of $(Q_V; \ )$. It follows that and are tempered everywhere. We now assume that is a $(G; H; {}^{-1})$-regular Hermitian Arthur parameter, and choose $\in i\mathfrak{a}^{L,}_{M_P} = i\mathfrak{a}$.

We pick S a finite set of places of F containing all the Archimedean ones and such that E/F is unramified outside of S, $V \in H^S$, and ${}_v$, ${}_v$ and ${}_v$ are unramified outside of S.

For $G \in \{GL_n; G_n; U(V)\}$, we have defined a local measure dg_v on $G(F_v)$ in Subsection 14.2. By Remark 14.1, the normalized product

$$dg = (\ _G)^{-1} \prod_v \ _{G,v} dg_v \tag{19.1}$$

is a factorization of the Tamagawa measure on $G(\mathbb{A})$ defined in Subsection 3.4. For unitary groups, for every place v of F and finite set of places S of F we have the explicit values

$$\ _{U(V),v} = \prod_{i=1}^{n} L(i; \ ^i_{E_v/F_v}); \qquad ^S_{U(V)} = \prod_{i=1}^{n} L^S(i; \ ^i_{E/F}); \tag{19.2}$$

where ${}_{E_v=F_v}$ (resp. ${}_{E=F}$) is the quadratic character attached to $E_v=F_v$ (resp. $E=F$) by class eld theory, and L is the Artin L-function (resp. L^S is the partial Artin L-function).

19.2.2. Factorization of relative characters on the general linear groups. Let v be a place of F. Let W_v be in the Whittaker model $W({}_{;v};{}_{N;v})$ and ${}_v \, 2 \, S(E_{v;n})$. We have de ned in (15.4) the local Rankin{Selberg integral, in (15.5) the local Flicker{Rallis integral and in (15.6) the Whittaker inner-product respectively by

$$
\begin{aligned}
{}_v(W_v; {}_v) &= \int_{N_H(F_v)nH(F_v)} W_v(h_v)(R \; {}_1(h_v) \; {}_v)(e_n)dh_v; \\
{}_{;v}(W_v) &= \int_{N^0(F_v)nP^0_n(F_v)} W_v(p_v) \; {}_{n+1}(p_v)dp_v; \\
hW_v; W_v i_{\mathrm{Whitt} \; ;v} &= \int_{N(F_v)nP_{n;n}(F_v)} jW_v(p_v)j^2 dp_v:
\end{aligned}
$$

It was shown in Subsection 15.1 that these integrals are absolutely convergent as ${}_{;v}$ is tempered. As in (15.11), we de ne the local relative character to be

$$
I \; {}_{;v}(f_v \quad {}_v) := \sum_{W_v} \frac{{}_v(\; {}_{;v}(f_v)W_v; \; {}_v) \overline{{}_{;v}(W_v)}}{hW_v; W_v i_{\mathrm{Whitt} \; ;v}}; \; f_v \, 2 \, S(G(F_v)); \quad {}_v \, 2 \, S(E_{n;v}):
$$

where W_v runs through an orthonormal basis $W({}_{;v};{}_{N;v})$. This character extends by continuity to $S(G_+(F_v))$ by Lemma 15.6.

If $= \; {}_1 \quad {}_2$ we set

$$
\begin{aligned}
L(s; \; ; As_G) &= L(s; \; {}_1; As^{(\; 1)^{n+1}})L(s; \; {}_2; As^{(\; 1)^{n+1}}); \\
L(s; \; ; As^0_G) &= L(s; \; {}_1; As^{(\; 1)^n})L(s; \; {}_2; As^{(\; 1)^n}):
\end{aligned}
$$

where $L(s; \; {}_1; As^{(\; 1)^i})$ and $L(s; \; {}_2; As^{(\; 1)^i})$ are the Asai L-functions.

By [Zha14b, Section 3] and the equality vol$H(O_v) = {}^{\;\,1}_{H;v}$ for almost all v (see (3.1)), up to enlarging S we have for $v \, 2 \, S$, $W_v \, 2 \, W({}_{;v};{}_{N;v})^{K_v}$ with $W_v(1) = 1$ and ${}_v = 1_{O_{E_{v;n}}}$ the unrami ed computation

$$
{}_v(\; {}_{;v}(1_{G(O_v)})W_v; \; {}_v) = {}^{\;\,1}_{G;v} \; {}^{\;\,1}_{H;v} L(\frac{1}{2}; \; {}_{;v} \quad {}_v^{\;1}): \tag{19.3}
$$

For a su ciently large nite set S of places of F, set

$$
hW; W i_{\mathrm{Whitt}} = (\; {}^{S;}_{G} \;)^{\;1} L^{S;}(1; \; ; Ad) \int_{N(F_S)nP_{n;n}(F_S)} jW(p_S)j^2 dp_S; \tag{19.4}
$$

where $L(s; \; ; Ad) = L(s; \; {}_{1;} \quad {}_{\overline{1};})L(s; \; {}_{2;} \quad {}_{\overline{2};})$. Let $W = {}^0_v W_v \, 2 \, W(\; ; \; {}_N)$ and $= {}^0_v \; {}_v \, 2 \, S(A_{E;n})$ be factorizable vectors. By Theorem 10.4, the de nition of the global Flicker{Rallis period in (10.18), and the factorization of the Tamagawa measure given in (19.1), we have the factorizations for S large enough

$$Z^{RS}(W; \ ; \ ; 0) = (\ ^{S;}_{G}) \ ^{1}(\ ^{S;}_{H}) \ ^{1}L^{S}(\frac{1}{2}; \qquad ^{1}) \prod_{v2S} {}_{v}(W_v; \ {}_{v});$$

$$(W) = (\ ^{S;}_{G^0}) \ ^{1}L^{S;} (1; \ ; As_G) \prod_{v2S} {}_{;v}(W_v);$$

$$hW; W i_{Whitt} = (\ ^{S;}_{G}) \ ^{1}L^{S;} (1; \ ; Ad) \prod_{v2S} hW_v; W_v i_{Whitt \ ;v};$$

where L^S and $L^{S;}$ are the partial L functions outside of S (resp. its regularized value at $s = 1$). The following is [BPCZ22, Theorem 8.1.2.1].

Theorem 19.1. For all ' 2 and 2 ia_{M_P} we have

$$h; ' \ i_{Pet} = hW(' ; \ ); W(' ; \ )i_{Whitt} :$$

Our global relative character I can therefore be written as

$$(19.5) \qquad I \quad (f \quad) = \sum_{' \ 2B_{P;}} \frac{Z^{RS}(W(I_P(f; \ )' ; \ ); \ ; \ ; 0) \overline{(W(' ; \ ))}}{hW(' ; \ ); W(' ; \ )i_{Whitt}} :$$

It now follows from the equality of L functions $L(s; \ ; Ad) = L(s; \ ; As_G)L(s; \ ; As^0_G)$ that for a factorizable test function $f_+ = (\ ^{S;}_{H} \ ^{S;}_{G^0}) \prod_{v2S} f_{+;v} \prod_{v2S} 1_{G_+(O_v)}$ we have

$$(19.6) \qquad I \quad (f_+) = \frac{L^S(\frac{1}{2}; \qquad ^{1})}{L^{S;}(1; \ ; As^0_G)} \prod_{v2S} I \ {}_{;v}(f_{v;+}):$$

19.2.3. Local relative characters on the unitary groups. Write the decomposition $= \ _v \ _v$, and define for every v the tempered representation $_{;v} = Ind^{U_V(F_v)}_{Q_V(F_v)}(\ _v \quad bc(\ ))$. Recall that for f^V_v 2 $S(U_V(F_v))$ and $_{1;v}; \ _{2;v}$ 2 $S(L_-(F_v))$ we have defined in Subsection 15.1 a relative character

$$J \ _{;v}(f^V_v \quad _{1;v} \quad _{2;v}) = \int_{U^0_V(F_v)} Trace(\ _{;v}(h_v) \ _{;v}(f^V_v))hI_-(h_v) \ _{1;v}; \overline{\ _{2;v}}i_{L^2;v} dh_v;$$

where this integral is absolutely convergent by Lemma 15.1 and (15.2), and the character extends by continuity to $S(U_{V;+}(F_v))$.

It follows from the equality of local L functions $L(s; \ _{;v}; As^0_G) = L(s; \ _{;v}; Ad)$ and from [Xue16, Appendix D] that for v 2 S and $f^V_{+;v} = f^V_v \quad _{1;v} \quad _{2;v} = 1_{U_{V;+}(O_{F;v})}$ we have the unramified computation

$$(19.7) \qquad J \ _{;v}(f^V_{+;v}) = {}^{2}_{U(V);v} \frac{L(\frac{1}{2}; \ _{;v} \qquad ^{1})}{L(1; \ _{;v}; As^0_G)} :$$

Note that the square factor ${}^{2}_{U(V);v}$ comes from the fact that if ' is a spherical vector in $_{;v}$, we have $_{;v}(f^V_v)' = {}^{2}_{U(V);v}'$ by our choice of measure (see (3.1)).

19.2.4. End of the proof. We now follow closely [BPC, Section 7.3].

If there exists a place $v \in S$ such that $\sigma_{,v}$ doesn't have any nonzero continuous $U(V)(F_v)$-invariant functional, then both sides of (1.6) are automatically zero, so that we may assume that it is not the case. It follows from [Xue16, Appendix D] that the product of distributions $\prod_{v\in S} J_{\sigma,v}$ doesn't vanish identically. By the same argument as [Zha14b, Lemma 1.7], Theorem 1.5 is now equivalent to the following assertion: for every factorizable test function $f_+^V \in S(U_{V,+}(\mathbb{A}))$ of the form $f_+^V = (\Delta^S_{U(V)})^2 \prod_{v\in S} f^V_{+,v} \prod_{v\notin S} 1_{U_{V,+}(O_v)}$, we have

$$J_{Q_V,\sigma}(bc(\sigma), f_+^V) = |S_\sigma|^{-1} \frac{L^S(\frac{1}{2}, \sigma^{\vee})}{L^{S,*}(1, \sigma, As^0_G)} \prod_{v\in S} J_{\sigma,v}(f^V_{+,v}). \tag{19.8}$$

Indeed, note that in Theorem 1.5 we had used a factorization $dh = \prod_v dh_v$ of the Tamagawa measure of $U(V)$, while here it is of the form $dh = \Delta^{-1}_{U(V)} \prod_v \Delta_{U(V),v} d_v h_v$ with $\Delta_{U(V)} = \prod_{i=1}^n L(i, \eta^i)$ (see (19.2)).

Let f_+^V be as in (19.8). By Theorem 13.4 and because both sides of (19.8) are continuous, we may assume that for all $v \in S$ there exists $f_{+,v} \in S(G_+(F_v))$ such that

- $f_{+,v}$ matches with $f^V_{+,v}$,
- for every $V' \in H^S$ different from V, $f_{+,v}$ matches with $0 \in S(U_{V',+}(F_v))$.

We now set $f_+ = (\Delta^{S,*}_H \Delta^{S,*}_{G'}) \prod_{v\in S} f_{+,v} \prod_{v\notin S} 1_{G_+(O_v)}$. Setting $f_+^{V'} = 0$ for $V' \neq V$, the function f_+ and the family $\{f_+^{V'}\}$ satisfy the hypothesis of Theorem 18.1 thanks to the fundamental lemma (Theorem 13.9) so that by Theorem 18.1

$$J^V(\chi, f_+^V) = |S_\sigma|^{-1} I_\pi(f_+). \tag{19.9}$$

Recall that

$$J^V(\chi, f_+^V) = \sum_{(M_{Q'_V}, \sigma')} J_{Q'_V,\sigma'}(bc(\sigma'), f_+^V),$$

where the sum ranges over representatives of classes in χ^V. By the local Gan–Gross–Prasad conjecture for Fourier–Jacobi models ([GI16] in the p-adic case, [Xue] in the Archimedean case) and the classification of cuspidal automorphic representations of U_V in terms of L-packets ([Mok15], [KMSW]), all the terms in this sum disappear except possibly $J_{Q_V,\sigma}(bc(\sigma), f_+^V)$, so that (19.9) now reads

$$J_{Q_V,\sigma}(bc(\sigma), f_+^V) = |S_\sigma|^{-1} I_\pi(f_+). \tag{19.10}$$

Since $\pi_v = BC(\sigma_v)$, we also have $\pi_v = BC(\sigma_v)$ by the properties of the local Langlands correspondence. By Proposition 16.7 and Theorem 16.9 there exist constants $c_v \in \mathbb{C}^\times$ such that $\prod_{v\in S} c_v = 1$ and

$$I_{\pi,v}(f_{+,v}) = c_v J_{\sigma,v}(f^V_{+,v}). \tag{19.11}$$

Therefore, (19.8) follows from the global equality (19.10), the factorization (19.6) and the local equality (19.11). This concludes the proof of Theorem 1.5.

20. Unfolding formulae for Fourier–Jacobi Periods

In this section we study Fourier–Jacobi periods in positive corank. We will use the same notations as in Section 5. Unless specified otherwise, E/F will be a quadratic extension of number fields, (V, q_V) a n-dimensional nondegenerate skew-Hermitian space over E/F and $W \subset V$ a nondegenerate subspace of dimension m such that $n = m + 2r$ and $W^{\perp}$ is split. We will always assume that $r \geq 1$ in this section.

20.1. Preliminaries.

20.1.1. Fourier–Jacobi groups. We define the following products

$$U_V := U(V) \times U(V); \quad U_W := U(V) \times U(W); \quad U_V^0 := U(V) \hookrightarrow U_V;$$

where the last embedding is the diagonal injection.

Since $W^{\perp}$ is split, there exists a basis $\{x_i, x_{-i} \mid i = 1, \dots, r\}$ of $W^{\perp}$ such that for all $1 \leq i, j \leq r$ we have

$$q_V(x_i, x_j) = 0, \quad q_V(x_{-i}, x_{-j}) = 0, \quad q_V(x_i, x_{-j}) = \delta_{i,j}. \tag{20.1}$$

Set $X = \mathrm{span}_E(x_1, \dots, x_r)$ and $X^{\vee} = \mathrm{span}_E(x_{-1}, \dots, x_{-r})$, so that $V = X \oplus W \oplus X^{\vee}$. This determines an embedding $U(W) \hookrightarrow U(V)$.

Let $P(X) = M(X)N(X)$ be the maximal parabolic subgroup of $U(V)$ stabilizing X, where $M(X)$ is the Levi subgroup stabilizing $X^{\vee}$ and $N(X)$ is the unipotent radical of $P(X)$. By restricting to $X \oplus W$, $M(X)$ is identified with $G_r \times U(W)$ where $G_r = \mathrm{Res}_{E/F}\, GL_r$. Denote by $B_V = T_V N_V$ and $B_W = T_W N_W$ Borel subgroups of $U(V)$ and $U(W)$ such that B_V is included in $P(X)$ and $B_V \cap M(X) = B_r \times B_W$ with $B_r = T_r N_r$ is the subgroup of upper triangular matrices of GL_r with respect to the basis $(x_1, \dots, x_r)$.

The subgroup of $U(V)$ of $g \in U(V)$ stabilizing X and $X^{\vee}$ and being trivial on W is identified with G_r by restricting to X. If $g \in G_r$, then g acts on $X^{\vee}$ by $(g^*)^{-1}$, where $g^* = {}^t g^c$.

Set $\mathbb{V} := \mathrm{Res}_{E/F}\, V$ and $\mathbb{W} := \mathrm{Res}_{E/F}\, W$. These spaces are equipped with the symplectic form $q_{\mathbb{V}} := \mathrm{Tr}_{E/F} \circ q_V$. We fix a polarization $\mathbb{W} = Y \oplus Y_-$. By choosing an isotropic basis $(y_1, \dots, y_m)$ of Y and the dual basis $(y_{-1}, \dots, y_{-m})$ of Y_-, we have integral models over $\mathcal{O}_F$ of Y and Y_-. For any place v, this yields a polarization $\mathbb{W}(F_v) = Y(F_v) \oplus Y_-(F_v)$. We will always assume that our local polarizations are of this form. We also obtain a polarization $\mathbb{V} = (X \oplus Y) \oplus (X^{\vee} \oplus Y_-)$, with an integral model over $\mathcal{O}_F$.

Let $S(\mathbb{W}) = \mathbb{W} \times F$ be the Heisenberg group. We have defined in (5.3) the Jacobi group

$$J(W) := S(\mathbb{W}) \rtimes U(W).$$

As $r \geq 1$ there exists an embedding $h : S(W) \to U(V)$ characterized by

$$
(20.2) \qquad \begin{cases} h(w,z)(x_i) = x_i & 1 \leq i \leq r; \\ h(w,z)(w') = w' - q_V(w', w)v_r & w' \in W; \\ h(w,z)(x_r^\vee) = (-\frac{1}{2}q_V(w,w) + z)x_r + w + x_r^\vee; & \\ h(w,z)(x_i^\vee) = x_i^\vee & 1 \leq i \leq r-1. \end{cases}
$$

This makes $J(W)$ a subgroup of $U(V)$. Let U_{r-1} and U_r be the unipotent radicals of the parabolic subgroups of $U(V)$ stabilizing the flags $\mathrm{span}_E(x_1) \subset \mathrm{span}_E(x_1, x_2) \subset \cdots \subset \mathrm{span}_E(x_1, \dots, x_{r-1})$, and the full flag $\mathrm{span}_E(x_1) \subset \cdots \subset X$ respectively. Define the Fourier–Jacobi group

$$H := U_{r-1} \rtimes J(W) \hookrightarrow U_W;$$

where the embedding is the product of inclusion $H \subset U(V)$ and the projection $H \to U(W)$. Note that $S(W) \ltimes U_{r-1} = U_r$, so that we have the alternative definition

$$H = U_r \rtimes U(W).$$

We will write H' if we consider it as a subgroup of U_V' via the inclusion.

Set $L := U_W \times G_r$. It is a Levi subgroup of the parabolic $U(V) \cap P(X)$ of U_V. Set

$$H_L := H \cap N_r \times L.$$

20.1.2. Measures. For every algebraic group G over F we have considered so far, we let dg be the Tamagawa measure on $G(\mathbb{A})$ defined in Subsection 3.4. We fix a factorization $dg = \prod_v dg_v$ on $G(\mathbb{A})$ such that for almost all v the volume of $G(O_v)$ is 1.

For unipotent groups, it is convenient to make a specific choice for the local measures dg_v. For every place v of E, we set $d_{\psi_v} x_v$ to be the unique Haar measure on E_v that is self-dual with respect to $\psi_{E,v}$. This yields a measure $dx = \prod_v d_{\psi_v} x_v$ on $\mathbb{A}_E$. More generally, for every k we equip $N_k(F_v)$ with the product measure

$$dn_v = \prod_{1 \leq i<j \leq k} d_{\psi_v} n_{i,j};$$

and set $dn = \prod_v dn_v$. Then it is well known that dn_v gives volume 1 to $N_k(O_v)$ for almost all v, and that dn is indeed the Tamagawa measure on $N_k(\mathbb{A})$.

For every k, the measure $d_{\psi_v} g_v$ chosen in Subsection 3.4 is

$$d_{\psi_v} g_v := \frac{\prod_{1 \leq i,j \leq k} d_{\psi_v} g_{i,j}}{|\det g_v|^k}.$$

We will denote by $\Delta(G_{k,v}) > 0$ the quotient of Haar measures $dg_v (d_{\psi_v} g_v)^{-1}$. This number will appear in the following Fourier inversion formula: for every place v, if $f \in C_c^\infty(G_k(F_v))$ then for every $g \in G_k(F_v)$ we have

$$
(20.3) \qquad f(g) = \Delta(G_{k-1,v})^{-1} \int_{N_{k-1}(F_v)\backslash G_{k-1}(F_v)} \int_{N_k(F_v)} f(\gamma^{-1} n \gamma g)\overline{\psi_k(n)} dn_v d\gamma_v.
$$

It follows from (3.1) that for any sufficiently large finite set of places S of F we have $(G_{k;v}) = \quad_{G_{k};v}$ for $v \notin S$, and that

$$(20.4) \qquad \prod_{v \in S} (G_{k;v}) = (\quad^{S;}_{G_k})^{\ 1}.$$

Let P_r be the mirabolic subgroup of G_r, i.e. the group of matrices whose last row is e_r. We identify it to the subgroup of $\quad \in G_r$ such that $\quad x_r = x_r$. The isomorphism $P_r(F_v)\backslash G_r(F_v) = X \ (E_v) \backslash \{0\}$ yields for every $f \in C_c^\infty(X \ (E_v))$ the formula

$$(20.5) \qquad \int_{X \ (E_v)} f(x)dx_v = \frac{(G_{r \ 1;v})}{(G_{r;v})} \int_{P_r(F_v)\backslash G_r(F_v)} |\det \ | f(\quad x_r) d \ _v.$$

20.2. Local Fourier–Jacobi periods. In this section we are only concerned with local theory, so that we will drop all v's to lighten notations. Moreover, we will also use G instead of $G(F)$ if G is linear algebraic group over F. We treat the inert and split cases in an uniform way: this means that E will either be a quadratic extension of F a local field of characteristic zero, or $E = F \quad F$. In the latter case, our groups will be general linear groups as described in Section 17.

20.2.1. Spaces of tempered functions. For every linear algebraic group G over F, we fix ς a (class of) logarithmic height function on G, as in [BP20, Section 1.2]. Note that if $G^0 \quad G$ we may take $\varsigma_{|G^0}$ as the logarithmic height function on G^0, hence the absence of reference to the group in the notation. Denote by $\ ^G$ the Harish-Chandra special spherical function on G and by $C^w(G)$ the space of tempered functions on G (see [BP21b, Section 2.4]). It contains $C_c^\infty(G)$ as a dense subset. By definition, $C^w(G) = \bigcup_{d>0} C^w_d(G)$ where for every $f \in C^w_d(G)$ we have

$$(20.6) \qquad |f(g)| \quad ^G(g)\varsigma(g)^d; \ g \in G.$$

If $\quad$ is an irreducible tempered representation of G, then for every $v \in \quad$ and $v^\vee \in \quad^\vee$ the matrix coefficient

$$g \in G \mapsto v^\vee(\ (g)v)$$

belongs to $C^w(G)$ by [CHH88].

20.2.2. Fourier–Jacobi models. We now make use of the local counterparts of the representations ω and $\quad$ defined in Subsection 5.4. By the local Stone–von Neumann theorem there exists a unique irreducible representation $\quad$ of $S(W)$ with central character $\quad$. Consider the S^1 metaplectic cover $Mp(W)$ of the symplectic group $Sp(W)$. There is a natural map $U(W) \to Sp(W)$. The data of $\quad$ determines a representation $\omega \quad$ of $Mp(W)$ ([MVW87, Section 2.II]), realised on $\quad$. Recall that $\quad$ is a character of $E \quad$ lifting $\quad_{E=F}$. Then the data of $(\ ; \quad)$ determines a splitting of $Mp(W)$ over $U(W)$ which yields the Weil representation $\omega_{\ ;}$ of $U(W)$. Set $\omega_{\overline{W}} = \omega_{\ ^1; \ ^1}$ and $\ ^\vee = \quad_1$. We will also need the Weil representation of $U(V)$ associated to $(\ ^1; \quad ^1)$, denoted by $\omega_{\overline{V}}$.

Define a morphism $: U_{r-1} \to G_a$ by

$$(20.7) \qquad (u) = \mathrm{Tr}_{E/F}\left(\sum_{i=1}^{r-1} q_V(u(x_{i+1}); x_i)\right); \quad u \in U_{r-1}.$$

Note that the action of $J(W)$ by conjugation on U_{r-1} is trivial on , so that we may extend to $H = U_{r-1} \rtimes J(W)$. Consider the character

$$(20.8) \qquad {}_U(h) := {}_E((h)); \quad h \in H.$$

By definition, for any $g \in U(W)$ the automorphism $\omega_W(g)$ is an intertwining operator of , so that we can define the representation $-$ of $H = U_{r-1} \rtimes J(W)$ by the rule

$$(20.9) \qquad -(uhg_W) = \overline{{}_U(u)} \; -(h)\omega_{\overline{W}}(g_W); \quad u \in U_{r-1}; \; g_W \in U(W); \; h \in S(W).$$

Define ${}_r$ to be the restriction of ${}_U$ to N_r. We will also consider the representation of H_L

$${}_L := - \; {}_r.$$

Consider the polarization (over the local field F) $W = Y \; Y_-$. This yields $V = (X \; Y) \bigoplus (X \; Y_-)$ a polarization of V. We can realize $\omega_{\overline{V}}$ and $-$ on $S(X \; Y_-)$ and $S(Y_-)$ the spaces of Schwartz functions on $X \; Y_-$ and Y_- respectively by the mixed model described in Subsection 5.4. For $\in \omega_{\overline{V}}$, define

$${}_{Y_-} := (x_r;) \in -.$$

We have pairings $h; i_V$ and $h; i_W$ given by

$$(20.10) \qquad h_1; {}_2 i_V = \int_X \int_{Y_-} {}_1(x; y_-)\overline{{}_2(x; y_-)}dy_- dx; \quad h_1; {}_2 i_W = \int_{Y_-} {}_1(y_-)\overline{{}_2(y_-)}dy_-.$$

where ${}_1; {}_2 \in \omega_{\overline{V}}$ and ${}_1; {}_2 \in -$.

20.2.3. Estimates. As $W^?$ is split, we have an explicit formula for the action of G_r by the mixed model (5.15):

$$(20.11) \qquad \omega_{\overline{V}}()(x; y_-) = {}^1()\,|\det|^{\frac{1}{2}}(x; y_-); \quad \in G_r; \; x \in X; \; y_- \in Y_-.$$

Lemma 20.1. We have the following assertions.

(1) There exists $" > 0$ such that

$$h\omega_{\overline{V}}(g_V)_1; {}_2 i_V \quad {}_{1;2}\, e^{"\&(g_V)}; \quad g_V \in U(V); \; {}_1; {}_2 \in \omega_{\overline{V}};$$

(2) There exists $" > 0$ such that

$$h-(ug_W)_1; {}_2 i_W \quad {}_{1;2}\, e^{"\&(g_W)}; \quad u \in U_r; \; g_W \in U(W); \; {}_1; {}_2 \in -.$$

(3) For any non degenerate skew c-Hermitian vector space V^0 and for all $" > 0$ we have

$$\int_{U(V^0)} {}_{U(V^0)}\,{}_{U(V^0)}(h)e^{"\&(h)}dh < 1.$$

(4) For all $\delta > 0$ there exists $\varepsilon > 0$ such that for every $\varphi_1, \varphi_2 \in \mathcal{S}_-$ we have

$$\int_H \Xi^{U_W}(h) e^{\varepsilon\varsigma(h)} \left| \langle h \cdot \omega_-(h) \varphi_1, \varphi_2 \rangle_W \right| (1 + |\sigma(h)|)^{\delta} dh < \infty;$$

where we recall that σ was defined in (20.7).

(5) For every φ_1 and $\varphi_2 \in \mathcal{S}_-$, the linear form

$$f \in C_c^\infty(U_W) \mapsto \int_H f(h) \langle h \cdot \omega_-(h) \varphi_1, \varphi_2 \rangle_W dh$$

extends by continuity to $\mathcal{C}^w(U_W)$.

(6) The linear form

$$f \in C_c^\infty(G_r) \mapsto \int_{N_r} f(n_r) \xi_r(n_r) dn_r$$

extends by continuity to $\mathcal{C}^w(G_r)$.

Proof. (1)–(2) As the Heisenberg representation acts by translations and multiplication by a unitary character (see [GI16, Section 7.4]) it is enough to prove the following: if V' is any nondegenerate skew-Hermitian space of dimension n' with a polarisation $V' = L_+ \oplus L_-$ then there exists $\varepsilon > 0$ such that for all $\varphi_1, \varphi_2 \in \mathcal{S}(L_-)$ we have

$$(20.12) \qquad \langle \omega_{V'}(g_{V'}) \varphi_1, \varphi_2(\cdot + l_-) \rangle_{V'} \ll e^{-\varepsilon\varsigma(g_{V'})}, \; g_{V'} \in U(V'), \; l_- \in L_-.$$

where $\langle \cdot,\cdot \rangle_{V'}$ is defined as in (20.10) and $\omega_{V'}$ is the Weil representation of $U(V')$ realized on $\mathcal{S}(L_-)$. Note that if we can prove that (20.12) holds for one polarization $V' = L_+ \oplus L_-$, then it holds for all. Indeed, change of Lagrangian induces an isomorphism of $\omega_{V'}$ which is also an isomorphism the Heisenberg representation of $\mathcal{S}(V')$. As the latter is irreducible, the inner products are equal up to multiplication by a non-zero constant.

Assume first that $E = F \oplus F$ so that $U(V') = GL_{n'}$. Identify $V' = E^{n'} = F^{n'} \oplus F^{n'}$ and take $L_- = \{0\} \oplus F^{n'}$. By the Cartan decomposition it is enough to show that (20.12) holds for $\lambda = \mathrm{diag}(\lambda_1, \dots, \lambda_{n'})$ with $|\lambda_1| \geq \dots \geq |\lambda_s| \geq 1 \geq |\lambda_{s+1}| \geq \dots \geq |\lambda_{n'}|$. But we have

$$\langle \omega_{V'}(\lambda) \varphi_1, \varphi_2(\cdot + l_-) \rangle_{V'} = \mu^{-1}(\det \lambda) |\det \lambda|^{\frac{1}{2}} \int_{F^{n'}} \varphi_1(\lambda x) \overline{\varphi_2(l_- + x)} dx.$$

Write (x_i) the coordinates of any x in the canonical basis, and (l_i) the ones of l_-. Choose $d > 0$ such that $\int_F \frac{1}{1+|y|^d} dy < \infty$ and let $P = \prod_i (1 + |x_i|^d)$ seen as a polynomial function on L_-. Then $\sup_x |\varphi_i(x) P(x)| < \infty$ and

$$\langle \omega_{V'}(\lambda) \varphi_1, \varphi_2(\cdot + l_-) \rangle_{V'} \ll \prod_{i=1}^{s} |\lambda_i|^{-\frac{1}{2}} \prod_{i=s+1}^{n'} |\lambda_i|^{\frac{1}{2}} \prod_{i=1}^{s} \int_F \frac{1}{1 + |y + l_i|^d} dy \prod_{i=s+1}^{n'} \int_F \frac{1}{1 + |y|^d} dy.$$

The product of integrals is finite and does not depend on l_-, so that $\varepsilon = \frac{1}{2}$ works.

In the case where E/F is a quadratic extension, let $(z_1, \dots, z_{r'}, z_1^*, \dots, z_{r'}^*)$ be a maximal hyperbolic family of V' (that is a family satisfying the conditions in (20.1), but that is not necessary linearly independent), and set $Z' := \mathrm{span}_E(z_1, \dots, z_{r'})$. By the Cartan decomposition, it is enough to prove that (20.12) holds for $\lambda = \mathrm{diag}(\lambda_1, \dots, \lambda_{r'}) \in GL(Z')$

$U(V^0)$, with $|\lambda_1| \geq \cdots \geq |\lambda_{r^0}| \geq 1$. We choose $L_- = Z \oplus L^0_{\varphi,-}$ where $L^0_{\varphi,-}$ is of dimension $n^0 - 2r^0$ and do the same proof as in the split case, using (20.11) and the description of the mixed model given in (5.15).

(3) This is [Wal03, Lemma II.1.5] in the p-adic case and [Var77, Proposition 31] in the Archimedean case.

(4) The proof is very close to [BP20, Lemma 6.5.1 (iii)], so we only recall the main steps. Since $\varsigma(g_W u) \ll \varsigma(g_W) + \varsigma(u)$ for $g_W \in U(W)$ and $u \in U_r$, by (2) and (3) applied to $V^0 = W$ and the equality $H = U_r \rtimes U(W)$, it is enough to prove that for all $\delta > 0$ and $\varepsilon_0 > 0$ there exists $\varepsilon > 0$ such that

$$I^0_{\varepsilon,\delta}(g_W) := \int_{U_r} \Xi^{U_W}(ug_W) e^{\varepsilon\varsigma(u)} (1 + |\psi(u)|)^{-\delta}\, du$$

is absolutely convergent for all $g_W \in U(W)$ and satisfies

$$I^0_{\varepsilon,\delta}(g_W) \ll \Xi^{U(W)}(g_W) e^{\varepsilon_0\varsigma(g_W)}.$$

The proof in [BP20, Lemma 6.5.1 (iii)] introduces two intermediate integrals, for $b > 0$:

$$I^0_{\varepsilon,\delta,\leq b}(g_W) := \int_{U_r} 1_{\varsigma(u) \leq b}\, \Xi^{U_W}(ug_W) e^{\varepsilon\varsigma(u)} (1 + |\psi(u)|)^{-\delta}\, du;$$

$$I^0_{\varepsilon,\delta,>b}(g_W) := \int_{U_r} 1_{\varsigma(u) > b}\, \Xi^{U_W}(ug_W) e^{\varepsilon\varsigma(u)} (1 + |\psi(u)|)^{-\delta}\, du.$$

On the one hand it is easy to see as in the proof of loc. cit. that there exists $d > 0$ such that

(20.13) $$I^0_{\varepsilon,\delta,\leq b}(g_W) \ll e^{\varepsilon b} b^d\, \Xi^{U(W)}(g_W);$$

for all $\varepsilon > 0$, $b > 0$ and $g_W \in U(W)$. But on the other hand, there exists $\alpha > 0$ such that $\Xi^{U_W}(g_1 g_2) \ll e^{\alpha\varsigma(g_2)}\, \Xi^{U_W}(g_1)$ for all $g_1, g_2 \in U_W$. Therefore

(20.14) $$I^0_{\varepsilon,\delta,>b}(g_W) \ll e^{\alpha\varsigma(g_W) - \sqrt{\varepsilon} b} \int_{U_r} \Xi^{U_W}(u) e^{(\varepsilon + \sqrt{\varepsilon})\varsigma(u)} (1 + |\psi(u)|)^{-\delta}\, du;$$

for all $g_W \in U(W)$ and $b > 0$. That the last integral converges for ε small is a consequence of [BP20, Lemma B.3.2] as ψ is the restriction of a non-degenerate additive character of N_{min} the unipotent radical of a minimal parabolic of $U(V)$ which stabilises a maximal isotropic flag obtained by completing $\mathrm{span}_E(x_1) \subset \mathrm{span}_E(x_1, x_2) \subset \cdots \subset X$ with isotropic lines in W. The result now follows for a small ε by the same trick as in [BP20, Lemma 6.5.1 (iii)].

(5) We fix the one-parameter subgroup $a : t \in \mathbb{G}_m \mapsto \mathrm{diag}(t^{r-1}, t^{r-2}, \ldots, 1) \in GL(X) \subset U(V)$. By the description of the Heisenberg representation (5.12) and of the mixed model (5.14) we find that for all $t \in F^\times$ and $h \in H$

$$\langle h \cdot \varphi(a(t)ha(t)^{-1}) \phi_1, \phi_2 \rangle_W = \overline{\psi_U}(t\nu(h)) \langle h \cdot \varphi(h) \phi_1, \phi_2 \rangle_W.$$

The proof is now exactly the same as in [BP20, Proposition 7.1.1], using (4) instead of (iii) of [BP20, Lemma 6.5.1].

(6) This can be proved as in [BP20, Proposition 7.1.1] thanks to [BP20, Lemma B.3.2].

20.2.4. *Periods.* By Lemma 20.1 (5) and (6), for every $\phi_1, \phi_2 \in \overline{\omega}$, the linear forms

$$f \in C_c^\infty(U_W) \mapsto \int_H f(h)\langle \overline{\omega}(h)\phi_1, \phi_2\rangle_W \, dh; \text{ and } f \in C_c^\infty(G_r) \mapsto \int_{N_r} f(n_r)\psi_r(n_r)dn_r$$

extend by continuity to $\mathcal{C}(U_W)$ and $\mathcal{C}(G_r)$ respectively. We denote them by $P_H(f \otimes \phi_1 \otimes \phi_2)$ and $P_{N_r}(f)$. For $\phi_1, \phi_2 \in \omega_{\overline{\nabla}}$, we also introduce

$$P_{U_V^0}(f \otimes \phi_1 \otimes \phi_2) := \int_{U_V^0} f(g)\langle \omega_{\nabla}(g)\phi_1, \phi_2\rangle_V dg; \; f \in \mathcal{C}^w(U_V):$$

This integral converges absolutely by Lemma 20.1 (1) and (3), and the estimate (20.6).

For $(G, H) \in \{(U_V, U_V^0), (U_W, H), (L, H_L)\}$, if π is a tempered representation of G equipped with an invariant inner product $\langle\cdot,\cdot\rangle$, then for every $\varphi_1, \varphi_2 \in \pi$ the map $c_{\varphi_1,\varphi_2} : g \mapsto \langle \pi(g)\varphi_1, \varphi_2\rangle$ belongs to $\mathcal{C}^w(G)$. We set for $\phi_i \in \omega_{\nabla}$, $\overline{\omega}$ or ν_Γ

$$P_H(\varphi_1, \phi_1, \varphi_2, \phi_2) := P_H(c_{\varphi_1,\varphi_2}, \phi_1, \phi_2): \tag{20.15}$$

If $\varphi_1 = \varphi_2$ and $\phi_1 = \phi_2$, we will simply write $P_H(\varphi, \phi)$ for $P_H(\varphi_1, \phi_1, \varphi_2, \phi_2)$. We also define, for σ a tempered representation of G_r and $\varphi \in \sigma$, the period

$$P_{N_r}(\varphi) := P_{N_r}(c_{\varphi,\varphi}):$$

20.2.5. *Representations.* Let σ_V, σ_W and τ be smooth irreducible representations of $U(V)$, $U(W)$ and G_r respectively, equipped with invariant inner products $\langle\cdot,\cdot\rangle$. Set $\sigma = \sigma_V \boxtimes \sigma_W$, an irreducible representation of U_W. For $s \in \mathbb{C}$, set $\tau_s = \tau|\det|^s$ and $\pi_s = \operatorname{Ind}_{P(X)}^{U(V)} \sigma_W \boxtimes \tau_s$, equipped with its canonical inner product $\langle\cdot,\cdot\rangle$. These representations are fixed for the rest of this section.

20.2.6. *Reduction to the corank zero.* Let

$$\mathcal{L}_{H_L} \in \operatorname{Hom}_{H_L}((\sigma \boxtimes \tau) \otimes \nu_\Gamma; \mathbb{C}):$$

By multiplicity one results of [AGRS10, GGP12, LS13], we know that $\mathcal{L}_{H_L}$ factors as $\mathcal{L}_{H_L} = \mathcal{L}_{N_r} \otimes \mathcal{L}_H$ with $\mathcal{L}_{N_r} \in \operatorname{Hom}_{N_r}(\tau; \overline{\psi_r})$ and $\mathcal{L}_H \in \operatorname{Hom}_H(\sigma \otimes \overline{\omega}; \mathbb{C})$.

Proposition 20.2. *There exists $c > 0$ such that for $\Re(s) > c$ the functional*

$$\mathcal{L}_{U_V^0,s} : \varphi_V \otimes \varphi_s \in \sigma_V \otimes \pi_s \otimes \omega_{\nabla} \mapsto \int_{H'\backslash U_V^0} \mathcal{L}_{H_L}(\sigma_V(g)\varphi_V \otimes \varphi_s(g); (\omega_{\nabla}(g)\phi)_{Y_-})dg \tag{20.16}$$

converges absolutely. If all representations are tempered, we may take $c = \frac{1}{2}$. Furthermore, for $\Re(s) > c$ the following assertions are equivalent.

(1) *There exist $\varphi_V, \varphi_W, \varphi_\tau$ and ϕ such that $\mathcal{L}_{H_L}(\varphi_V \otimes \varphi_W \otimes \varphi_\tau, \phi) \neq 0$.*
(2) *There exist φ_V, φ_s and ϕ such that $\mathcal{L}_{U_V^0,s}(\varphi_V \otimes \varphi_s, \phi) \neq 0$.*

Note that $\mathcal{L}_{U_V^0,s}$ is a Fourier–Jacobi functional in corank zero.

Proof. The convergence can be proved as in [BPC, Proposition 8.6.1.1] by the Iwasawa decomposition $U(V) = T_r H K_V$ where K_V is a maximal compact subgroup, using (20.11), and Lemma 20.1 (1) and (2). The implication (2) $\Rightarrow$ (1) is automatic. For (1) $\Rightarrow$ (2), we see that for $\Re(s) > c$ we have by (20.11),

$$L_{U_V^0,s}(\varphi_V \otimes \varphi_s;) = \int_{P(X)\backslash U(V)} \int_{N_r\backslash G_r} L_{H_L}(\;_V(gh) \quad (g))(\varphi_V \otimes \varphi_s(h)); (\omega_\nabla(h))(\; g\, x_r; \;) \; (|\;|^{s+\frac{1}{2}} - \;_{P(X)}^{\frac{1}{2}})(g)\,dg\,dh.$$

As $_s$ is stable by multiplication by $C^1(P(X)\backslash U(V))$, it is enough to show that there exists $\varphi_V \in \;_V$, $\varphi_W \in \;_W$, $\varphi \in$, $\in S(X)$ and $\in \;_-$ such that

$$\int_{N_r\backslash G_r} (\; g\, x_r) L_{H_L}(\;_V(g)\varphi_V \otimes \varphi_W \quad (g)\varphi\;;\;)(|\;|^{s+\frac{1}{2}} - \;_{P(X)}^{\frac{1}{2}})(g)\,dg \neq 0.$$

This condition can be rewritten as

$$\int_{P_r\backslash G_r} (\; g\, x_r) \int_{N_r\backslash P_r} L_{H_L}(\;_V(pg)\varphi_V \otimes \varphi_W \quad (pg)\varphi\;;\;)(|\;|^{s+\frac{1}{2}} - \;_{P(X)}^{\frac{1}{2}})(pg)\,|p|^{-1}\,dp\,dg \neq 0.$$

The map $g \in P_r\backslash G_r \mapsto g\, x_r \in X$ induces an embedding $S(P_r\backslash G_r) \hookrightarrow S(X)$, so it is enough to prove that we have elements such that

$$\int_{N_{r-1}\backslash G_{r-1}} L_H(\varphi_W \quad _V(g)\varphi_V;\;) L_{N_r}(\;(g)\varphi\;)(|\;|^{s-\frac{1}{2}} - \;_{P(X)}^{\frac{1}{2}})(g)\,dg \neq 0.$$

By [GK75, Theorem 6] and [Kem15, Theorem 1], for every $f \in C_c^1(N_{r-1}\backslash G_{r-1}; \overline{\;_r})$ there exists $\varphi \in$ such that $L_{N_r}(\;(g)\varphi\;) = f(g)$ for every $g \in N_{r-1}\backslash G_{r-1}$. The claim now follows from the non-vanishing of L_H.

20.2.7. Tempered unfolding. Assume that $_V$, $_W$ and are tempered. The following is [Boi, Proposition 6.3.2].

Proposition 20.3. For every $\varphi_V^1;\varphi_V^2 \in \;_V$, $\varphi^1;\varphi^2 \in$, and $_1;\;_2 \in \omega_\nabla$ we have

$$P_{U_V^0}(\varphi_V^1 \otimes \varphi^1;\;_1;\varphi_V^2 \otimes \varphi^2;\;_2)$$
$$= (G_r)^{-1}\int_{(H^0\backslash U_V^0)^2} P_{H_L}(\;_V(h_i)\varphi_V^i \otimes \varphi^i(h_i);(\omega_\nabla(h_i)\;_i)_{Y_-})_{i=1,2}\,dh_1 dh_2.$$

Corollary 20.4. For every $\varphi_V \in \;_V$, $\varphi_W \in \;_W$ and $\in \;_-$ we have

$$P_H(\varphi_V \otimes \varphi_W;\;) \geq 0.$$

Moreover, if $\mathrm{Hom}_H(\;_V \otimes \;_W \otimes \;_-;\mathbb{C}) \neq \{0\}$, then P_H is not identically zero.

Proof. For all $f \in C_c^1(U_W)$ satisfying $f(g) = \overline{f(g^{-1})}$, we have $P_H(f \quad) \in \mathbb{R}$. Therefore it follows by continuity that $P_H(\varphi_V \otimes \varphi_W \quad)$ is real. By [LM15] and [BP21b, Proposition 2.14.3], there exists a smooth irreducible tempered representation of G_r such that

$$P_{N_r} = L_{N_r} \otimes \overline{L_{N_r}}$$

with $L_{N_r} \in \mathrm{Hom}_{N_r}(; \overline{}_r)$ non zero. Using Proposition 20.3 and following the same steps as in the proof of Proposition 20.2 in reverse order, one shows that the existence of $_V ; '_W$ and such that $P_H(' _V \; '_W \;) < 0$ implies the existence of $\in := \mathrm{Ind}\,^{U(V)}_{P(X)} \; _W$ and $\in\; ! \; _-$ satisfying

$$P_{U_V^0}(' _V \; ' \; ; \;) < 0.$$

This is known to be a contradiction by [Xue16, Proposition 1.1.1 (2)], and the first point follows.

For the second point, assume that $P_H = 0$. By Proposition 20.3, this implies that $P_{U_V^0} = 0$, which in turn implies that $\mathrm{Hom}_{U_V^0}(_V \; ! \; _-; \mathbb{C}) = \{0\}$ by [Xue16, Proposition 1.1.1 (2)]. It now follows from Proposition 20.2 that $\mathrm{Hom}_H(_V \; _W \; _-; \mathbb{C}) = \{0\}$.

20.3. Unramified computations. We now go back to the global situation.

20.3.1. Unramified places. Recall that we have fixed in Section 20.1.1 a polarization $W = Y \; Y_-$ (over the global field F), and an isotropic basis $(y_1; \dots; y_m)$ of Y which yields a dual basis $(y_{\bar 1}; \dots; y_{\bar m})$ of Y_-. We therefore have integral models of Y and Y_- over O_F. Recall that $_{E/F}$ is the nonzero element in E chosen in Subsection 3.2. Let S be a finite set of places of F containing all the Archimedean places and such that $v \notin S$ implies that the following conditions hold.

- The extension E_v/F_v is unramified (this includes the case $E_v = F_v \; F_v$), the residual characteristic of F_v is not 2.
- $q_V(y_i; y_j)$, $q_V(y_i; y_{\bar j})$, $q_V(y_{\bar i}; y_{\bar j})$ are all in O_{E_v} for $1 \; i; j \; m$.
- $_{E/F} \in O_{E_v}$.
- The characters $_v$ and $_v$ are unramified.

Let $v \notin S$. The above hypotheses ensure that the set $_{W;v} = Y(O_v) \; Y_-(O_v)$ is an O_v-autodual lattice in $W(F_v)$, and also an $O_{E;v}$-autodual lattice in $W(E_v)$. Moreover, the set $_V :=$ $\mathrm{span}_{O_{E;v}}(x_1; \dots; x_r) \bigoplus \; _{W;v} \bigoplus \mathrm{span}_{O_{E;v}}(x_1; \dots; x_r)$ is an autodual lattice of $V(E_v)$ with respect to $q_V(;)$. Define $K_{W;v}$ to be the stabilizer of $_{W;v}$ in $U(W)(F_v)$, and $K_{V;v}$ to be the stabilizer of $_{V;v}$ in $U(V)(F_v)$. These are special maximal open compact subgroups of $U(W)(F_v)$ and $U(V)(F_v)$ respectively. Write $K_{r;v} := G_r(O_v)$. Up to enlarging S, we also assume that our local measures give volume 1 to $K_{V;v}$, $K_{W;v}$ and $K_{r;v}$ when $v \notin S$.

20.3.2. Unramified vectors of the Weil representation. Let $v \notin S$. The local representation $_{\bar v}$ admits a lattice model with respect to $_{W;v}$ (see [GKT23, Section 1.2.3]). It is realised on the vector space $C_c^1(W(F_v); \; ^1)$ of $f \in C_c^1(W(F_v))$ which satisfy

$$f(l + w) = \; ^1 \; \frac{1}{2} q_W(l; w) \;\; f(w); \; l \in \; _{W;v}; \; w \in W(F_v).$$

There is a unitary isomorphism between the models $S(Y_-(F_v))$ and $C_c^1(W(F_v); \; ^1)$ described in [GKT23, Proposition 1.23], and it sends $1_{Y_-(O_v)}$ to $1 \; _{W;v}$.

In the lattice model, the Heisenberg representation is given by

$$_1(w; z) f(w^0) = \; ^1 \; \frac{1}{2} q_W(w^0; w) + z \;\; f(w + w^0); \; (w; z) \in S(W)(F_v); \; w^0 \in W(F_v);$$

and $K_{W,v}$ acts through the Weil representation $!_{\overline{W},v}$ by left-translations ([GKT23, Section 1.4.6]). It follows that $1_{\;W,v} \;2\; (\;_{\overline{V}})^{K_{V,v} \cap H\;(F_v)}$, and therefore that in our Schrödinger model $1_{Y_-(O_v)}$ 2 $(\;_{\overline{V}})^{K_{V,v} \cap H\;(F_v)}$. Moreover, it is easily seen using the explicit description of the Heisenberg representation given in (5.12) that the space $(\;_{\overline{V}})^{K_{V,v} \cap H\;(F_v)}$ has dimension one. We show similarly that $1_{X\;(O_{E,v})\;\;Y_-(O_v)}$ 2 $(!_{\overline{V},v})^{K_{V,v}}$ (but this space is not of dimension one).

20.3.3. Unramified unfolding. Let $v \not\in S$. Let $\;_{V,v}$, $\;_{W,v}$ and $\;_v$ be smooth irreducible representations of $U(V)(F_v)$, $U(W)(F_v)$ and $G_r(F_v)$ respectively. Let $\;_v := \mathrm{Ind}_{P(X)(F_v)}^{U(V)(F_v)}\;_{W,v}\;\;\;_v$ be the normalized induction, and set $\;_{v,s} = \mathrm{Ind}_{P(X)(F_v)}^{U(V)(F_v)}\;_{W,v}\;\;\;_{v,s}$. We assume that our representations are unramified, i.e. $\;_{V,v}^{K_{V,v}} \neq \{0\}$, $\;_{W,v}^{K_{W,v}} \neq \{0\}$ and $\;_v^{K_{r,v}} \neq \{0\}$. Note that this implies that $\;_{v,s}^{K_{V,v}} \neq \{0\}$ for all $s \in \mathbb{C}$. Recall that the linear form $L_{U_V^0,v,s}$ was defined in (20.16).

Theorem 20.5. Let $'\;_{V,v} \in \;_{V,v}^{K_{V,v}}$, $'\;_{v,s} \in \;_{v,s}^{K_{V,v}}$ and $\;_v \in (\;_{\overline{V}})^{K_{V,v} \cap H\;(F_v)}$. Set $\;_v := 1_{X\;(O_{E,v})}\;\;\;_v \in !_{\overline{V},v}$. For $\Re(s)$ sufficiently large we have

(20.17)

$$L_{U_V^0,v,s}('\;_{V,v}\;\;\;'\;_{v,s};\;\;_v) = \frac{L(\frac{1}{2}+s;\;_v\;\;\;_{V,v}\;\;\;_v{}^{1})}{L(\frac{1}{2}+s;\;_v^c\;\;\;_{W,v})L(1+2s;\;_v;As^{(\;1)^m})}L_{H_L,v}('\;_{V,v}\;\;\;'\;_{v,s}(1);\;\;_v):$$

where c is the non trivial element in $\mathrm{Gal}(E_v/F_v)$, $\;_v^c = \;_v\;\;c$ and m is the dimension of W. Moreover, if $\;_{V,v}$, $\;_{W,v}$ and $\;_v$ are tempered, (20.17) holds for $\Re(s) > \;\frac{1}{2}$.

Proof. By [Boi, Proposition 7.3.1], we have the unramified unfolding equality for $\Re(s)$ large enough

$$\sum_{r \in \;_{r,v}} L_{H_L,v}\;\;_{V,v}(\;r)'\;_{V,v}\;\;\;'\;_{v,s}(\;r);(!_{\overline{V},v}(\;r)\;\;_v)_{Y_-}\;\;(\;_{P(X)}^{\frac{1}{2}}\;P^{1})(\;r)$$

(20.18)
$$= \frac{L(\frac{1}{2}+s;\;_v\;\;\;_{V,v}\;\;\;_v{}^{1})}{L(\frac{1}{2}+s;\;_v^c\;\;\;_{W,v})L(1+2s;\;_v;As^{(\;1)^m})}L_{H_L,v}('\;_{V,v}\;\;\;'\;_{v,s}(1);\;\;_v);$$

where P is the parabolic subgroup of $U(V)$ stabilizing the flag $0 \;\; Ex_1 \;\; \dots \;\; X$ and we have set $\;_{r,v} = T_r(F_v)/T_r(O_v)$. Note that with our choices of measures, the volumes $\mathrm{vol}(K_{V,v})$ and $\mathrm{vol}(K_{V,v} \cap H^0(F_v))$ are 1. The result now follows from the definition of $L_{U_V^0,v,s}$ given in (20.16) and the Iwasawa decomposition $U(V)(F_v) = T_r(F_v)H(F_v)K_{V,v}$.

20.4. Global Fourier–Jacobi periods. In this section we stay in the global setting.

20.4.1. Global Weil representations. Note that the restricted tensor product $S(X\;(\mathbb{A})\;\;Y_-(\mathbb{A}))$ and $S(Y_-(\mathbb{A}))$ were taken with respect to the functions $1_{X\;(O_{E,v})\;\;Y_-(O_v)}$ and $1_{Y_-(O_v)}$. According to the discussion in Subsection 20.3.2, we therefore write $\;_{\overline{V}} = \;{}_v^0 !_{\overline{V},v}$, $\;_- = \;{}_v^0\;_{\overline{V}}$ and $\;_{\overline{L}} = \;{}_v^0\;_{\overline{L},v}$ for the restricted tensor products over the places of F. These are representations of $U(V)(\mathbb{A})$, $H(\mathbb{A})$ and $H_L(\mathbb{A})$ respectively, realized on $S(X\;(\mathbb{A})\;\;Y_-(\mathbb{A}))$ and $S(Y_-(\mathbb{A}))$ respectively.

20.4.2. Global periods. For $(H, \omega_\psi, L_\psi) \in \{(U(V), \omega_{\overline{\psi}}, X \otimes Y_\psi), (H, \omega_\psi, Y_\psi), (H_L, \omega_{\Gamma}, Y_\psi)\}$, write the theta series

$$\theta_H(h, \phi) = \sum_{l \in L_\psi(F)} (\omega_\psi(h)\phi)(l), \quad h \in H(\mathbb{A}), \quad \phi \in \omega_\psi.$$

We will also make use of

$$\theta^{W,\psi}_V(g, \phi) := \sum_{y_\psi \in Y_\psi(F)} (\omega_{\overline{\psi}}(g)\phi)_{Y_\psi}(y_\psi), \quad g \in U(V)(\mathbb{A}), \quad \phi \in \omega_{\overline{\psi}}.$$

By (20.9) and the description of the mixed model (5.15) and (5.16), we have for $u \in U_{r-1}(\mathbb{A})$, $n \in N(X)(\mathbb{A})$, $\gamma \in P_r(F)$ (the mirabolic group of G_r), $g_W \in U(W)(\mathbb{A})$ and $g \in U(V)(\mathbb{A})$:

$$\overline{\psi}_{[U]}(u)\, \theta^{W,\psi}_V(n\, g_W\, g, \phi) = \theta_H(un, (\omega_{\overline{\psi}}(\gamma g_W g)\phi)_{Y_\psi}). \tag{20.19}$$

These theta series are of moderate growth. For triples of groups and representations of the form $(G, H, \omega_\psi) \in \{(U_V, U^0_V, \omega_{\overline{\psi}}), (U_W, H, \omega_\psi), (L, H_L, \omega_{\Gamma})\}$, consider the absolutely convergent integral

$$\mathcal{P}_{[H]}(\varphi, \phi) := \int_{[H]} \varphi(h)\, \theta_H(h, \phi)\, dh, \quad \varphi \in \mathcal{F}_{rd,H}([G]), \quad \phi \in \omega_\psi,$$

where $\mathcal{F}_{rd,H}([G])$ is the space of functions on $[G]$ rapidly decreasing when restricted to $[H]$. In particular, $\varphi \in \mathcal{A}_{cusp}([U(V)]) \otimes \mathcal{A}([U(V)])$, $\mathcal{A}_{cusp}([U(V)]) \otimes \mathcal{A}([U(W)])$ and $\mathcal{A}_{cusp}([U(V)]) \otimes \mathcal{A}([U(W) \times G_r])$ are examples of functions in these respective spaces.

For $\varphi \in \mathcal{A}([G_r])$ set

$$\mathcal{P}_{[N_r]}(\varphi) := \int_{[N_r]} \varphi(h)\, \psi_r(n)\, dn.$$

Then for $\varphi = \varphi_{W,V} \otimes \varphi_r \in \mathcal{F}_{rd,H_L}([L])$ we have

$$\mathcal{P}_{[H_L]}(\varphi, \phi) = \mathcal{P}_{[N_r]}(\varphi_r)\, \mathcal{P}_{[H]}(\varphi_{W,V}, \phi).$$

20.4.3. Relations between global periods. For $\varphi \in \mathcal{A}_{P(X)}([U(V)])$ and $s \in \mathbb{C}$, we consider the Eisenstein series $E^{U(V)}_{P(X)}(\varphi, s)$ which is defined for $\Re(s)$ large enough by

$$E^{U(V)}_{P(X)}(g, \varphi, s) = \sum_{\gamma \in P(X)(F) \backslash U(V)(F)} \varphi_s(\gamma g), \quad g \in [U(V)].$$

Proposition 20.6. Let $\varphi_V \in \mathcal{A}_{cusp}([U(V)])$, $\varphi \in \mathcal{A}_{P(X)}([U(V)])$ and $\phi \in \omega_{\overline{\psi}}$. There exists $c > 0$ such that for all s with $\Re(s) > c$ we have the equality

$$\mathcal{P}_{[U^0_V]}\left(\varphi_V \otimes E^{U(V)}_{P(X)}(\varphi, s), \phi\right) = \int_{H^0(\mathbb{A}) \backslash U^0_V(\mathbb{A})} \mathcal{P}_{[H_L]}\left(R(g)(\varphi_V \otimes \varphi_s), (\omega_{\overline{\psi}}(g)\phi)_{Y_\psi}\right) dg,$$

where the right-hand side is absolutely convergent.

Proof. By cuspidality of φ_V, $P_{[U_V^0]}$ is defined by an absolutely convergent integral so that for $\Re(s)$ large enough we have

$$P_{[U_V^0]}\left(\varphi_V \otimes E_{P(X)}^{U(V)}(\varphi; s)\right) = \int_{[U(V)]} \sum_{\gamma \in P(X)(F)\backslash U(V)(F)} \varphi_s(\gamma g)\varphi_V(g)\overline{\theta_{U_V^0}(g;\phi)}\,dg$$

$$= \int_{P(X)(F)\backslash U(V)(\mathbb{A})} \varphi_s(g)\varphi_V(g)\overline{\theta_{U_V^0}(g;\phi)}\,dg$$

$$(20.20)\qquad = \int_{M(X)(F)N(X)(\mathbb{A})\backslash U(V)(\mathbb{A})} \varphi_s(g)\int_{[N(X)]}\varphi_V(ng)\overline{\theta_{U_V^0}(ng;\phi)}\,dn\,dg.$$

For a fixed g the inner integral over $[N(X)]$ is given by

$$\int_{[N(X)]}\varphi_V(ng)\overline{\sum_{x\in X^\vee(F)}\sum_{y_-\in Y_-(F)}\omega_\psi(ng)\phi(x;y_-)}\,dn.$$

Note that this triple integral is absolutely convergent. It follows from the description of the mixed model given in (5.15) and (5.16) that $N(X)$ only acts on the y_- coordinate, so that for every x the map $n \mapsto \sum_{y_-\in Y_-(F)}\omega_\psi(ng)\phi(x;y_-)$ is $N(X)(F)$ left-invariant, and $n\mapsto \sum_{y_-\in Y_-(F)}\omega_\psi(ng)\phi(0;y_-)$ is constant on $[N(X)]$. Then by cuspidality of φ_V and (5.15) we have

$$\int_{[N(X)]}\varphi_V(ng)\overline{\theta_{U_V^0}(ng;\phi)}\,dn = \sum_{x\in X^\vee\setminus\{0\}}\int_{[N(X)]}\varphi_V(ng)\overline{\sum_{y_-\in Y_-(F)}\omega_\psi(ng)\phi(x;y_-)}\,dn$$

$$= \sum_{\gamma\in P_r(F)\backslash G_r(F)}\int_{[N(X)]}\varphi_V(ng)\overline{\theta_V^{W,\psi_-}(\gamma ng;\phi)}\,dn$$

$$= \sum_{\gamma\in P_r(F)\backslash G_r(F)}\int_{[N(X)]}\varphi_V(n\gamma g)\overline{\theta_V^{W,\psi_-}(n\gamma g;\phi)}\,dn,$$

where the last equality is obtained thanks to the change of variables $n\mapsto \gamma n\gamma^{-1}$. Now x . Set $X_{r-1}=\mathrm{span}_E(x_1,\dots,x_{r-1}) \subset X$. Let $N(X_{r-1})$ be the unipotent radical of the parabolic of $U(V)$ stabilizing X_{r-1}. Recall that we have an embedding of $S(W)$ the Heisenberg group of W in $N(X)$ described in (20.2), and we see that $N(X)=S(W)\ltimes(N(X)\cap N(X_{r-1}))$. Then

$$(20.21)\qquad \int_{[N(X)]}\varphi_V(n\gamma g)\overline{\theta_V^{W,\psi_-}(n\gamma g;\phi)}\,dn = \int_{[S(W)]}\int_{[N(X)\cap N(X_{r-1})]}\varphi_V(nh\gamma g)\overline{\theta_V^{W,\psi_-}(nh\gamma g;\phi)}\,dn\,dh.$$

Fox h fixed, consider $p\in[P_r]\mapsto \int_{[N(X)\cap N(X_{r-1})]}\varphi_V(nph\gamma g)dn$. This is well defined as $P_r(F)$ normalizes $(N(X)\cap N(X_{r-1}))(\mathbb{A})$. But $N_r\ltimes(N(X)\cap N(X_{r-1}))=U_{r-1}$ and $S(W)$ commutes with P_r so that by cuspidality of φ_V we have the Fourier expansion along $[N_r]$

$$(20.22)\qquad \int_{[N(X)\cap N(X_{r-1})]}\varphi_V(nh\gamma g)\,dn = \sum_{\gamma'\in N_r(F)\backslash P_r(F)}\int_{[U_{r-1}]}\varphi_V(uh\gamma'\gamma g)\overline{\psi_{[U]}(u)}\,du.$$

As $U_{r-1}\rtimes S(W)=U_r$, using (20.19) and (20.22) we see that (20.21) reduces to

$$\int_{[N(X)]}\varphi_V(n\gamma g)\overline{\theta_V^{W,\psi_-}(n\gamma g;\phi)}\,dn = \sum_{\gamma'\in N_r(F)\backslash P_r(F)}\int_{[U_r]}\varphi_V(u\gamma'\gamma g)\overline{\psi_H(u;(\omega_\psi(\gamma'\gamma g)\phi)_{Y_-})}\,du.$$

Hence by going back to (20.20) we obtain

$$P_{[U_V^0]}\left(E_{P(X)}^{U(V)}(\varphi,s)\cdot\varphi_V\right)$$

$$(20.23)\quad = \int_{M(X)(F)N(X)(A)\backslash U(V)(A)} \sum_{\gamma\in N_r(F)\backslash G_r(F)} \varphi_s(\gamma g)\int_{[U_r]} \varphi_V(u\gamma g)\,\psi_H(u;(\omega_\nabla(\gamma g)\Phi)_{Y_-})\,du\,dg.$$

Assume for the moment that the double integral $\int_{M(X)(F)N(X)(A)\backslash U(V)(A)}\sum_{\gamma\in N_r(F)\backslash G_r(F)}$ is absolutely convergent. Then we see that

$$\begin{aligned}
&P_{[U_V^0]}\left(\varphi_V\cdot E_{P(X)}^{U(V)}(\varphi,s)\right)\\
&= \int_{N_r(F)U(W)(F)N(X)(A)\backslash U(V)(A)} \varphi_s(g)\int_{[U_r]} \varphi_V(ug)\,\psi_H(u;(\omega_\nabla(g)\Phi)_{Y_-})\,du\,dg\\
&= \int_{H(A)\backslash U(V)(A)}\int_{[N_r]}\int_{[U(W)]}\int_{[U_r]} \varphi_s(n_r g_W g)\varphi_V(u n_r g_W g)\,\psi_H(u;(\omega_\nabla(n_r g_W g)\Phi)_{Y_-})\,du\,dg_W\,dn_r\,dg\\
&= \int_{H(A)\backslash U(V)(A)}\int_{[N_r]}\int_{[U(W)]}\int_{[U_r]} \varphi_s(n_r g_W g)\varphi_V(u g_W g)\,\psi_r(n_r)\,\psi_H(u g_W;(\omega_\nabla(g)\Phi)_{Y_-})\,du\,dg_W\,dn_r\,dg\\
&= \int_{H'(A)\backslash U_V^0(A)}\int_{[H_L]} \varphi_s(hg)\varphi_V(hg)\,\psi_{H_L}(h;(\omega_\nabla(g)\Phi)_{Y_-})\,dh\,dg\\
&= \int_{H'(A)\backslash U_V^0(A)} P_{[H_L]}\left(R(g)(\varphi_V\otimes\varphi_s);(\omega_-(g)\Phi)_{Y_-}\right)dg;
\end{aligned}$$

where we have used a change of variable $u = u n_r$ and (20.19) to go from the third to the fourth line.

So it remains to show that (20.23) is absolutely convergent, i.e.

$$(20.24)\quad \int_{N_r(F)U(W)(F)N(X)(A)\backslash U(V)(A)} |\varphi_s(g)|\left|\int_{[U_r]} \varphi_V(ug)\,\psi_H(u;(\omega_\nabla(g)\Phi)_{Y_-})\,du\right| dg < \infty$$

for $\Re(s)$ large enough. Let $P = (T_r\times U(W))\ltimes(N_r N(X))$ be the parabolic subgroup of $U(V)$ stabilizing the flag $\mathrm{span}_E(x_1)\subset\cdots\subset X$. As $P(A)\backslash U(V)(A)$ is compact, it is sufficient to show the convergence of

$$\int_{[U(W)]}\int_{T_r(A)}\int_{[N_r]} |\varphi(n_r a h g_W)|\left|\int_{[U_r]} \varphi_V(u a g_W)\,\psi_H(u;(\omega_\nabla(a g_W)\Phi)_{Y_-})\,du\right|\delta_P(a)^{-1}|\det a|^s\,dn_r\,da\,dg_W$$

By (20.19), the inner integral can be bounded above by

$$\int_{[N(X)]}\left|\psi_H(n;(\omega_\nabla(a g_W)\Phi)_{Y_-})\int_{[N_r]} \varphi_V(n_r n a g_W)\overline{\psi_r(n_r)}\,dn_r\right| dn.$$

By [BPCZ22, Lemma 2.6.1.1] and [BP21a, Proposition A.1.1 (i)], since $g_r\in[G_r]\mapsto\varphi_V(g_r g)$ is rapidly decreasing for any g and that $N(X)$ commutes with T_r, for every $R>0$ we have for any $a\in T_r(A)$, $g_W\in[U(W)]$, $n\in[N(X)]$

$$\left|\int_{[N_r]} \varphi_V(n_r n a g_W)\overline{\psi_r(n_r)}\,dn_r\right| \ll \prod_{i=1}^{r}\|a_i a_{i+1}^{-1}\|_{A_E}^{-R}\|g_W n\|_{U(V)}^{R'} \ll \prod_{i=1}^{r}\|a_i a_{i+1}^{-1}\|_{A_E}^{-R}\|g_W\|_{U(W)}^{R};$$

where R^0 is chosen appropriately. But by (20.11) and (20.19), for all $R > 0$ there exists $D > 0$ such that

$$_{R}(n; (!_{\nabla}(ag_W))_{Y_-}) = {}^{W;-}_{V}(ang_W;) \quad \| \det a|^{\frac{1}{2}} \|a_r\|_{A_E}^{R} \|g_W\|^{D}_{U(W)}:$$

It follows that for all $R > 0$

$$\int_{[N_r]} '_V(nag_W)\, _{R}(n; (!_{\nabla}(ag_W))_{Y_-})dn \quad | \det a|^{\frac{1}{2}} \|a\|_{A_E^r}^{R} \|g_W\|^{R}_{U(W)}:$$

But we also know thanks to the moderate growth of ' that there exists $D > 0$ such that

$$|'(nag_W)| \quad \|a\|^D \|g_W\|^D_{U(W)};\ n \in [N_r];\ a \in T_r(A);\ g_W \in [U(W)];$$

Hence the absolute convergence of (20.24) reduces to the following easy fact: for every s with $\Re(s)$ large enough there exists $R > 0$ such that

$$\int_{T_r(A)} \|a\|^D \|a\|_{A_E^r}^{R}\ _P(a)^{-1} |\det a|^{s+\frac{1}{2}} da < \infty:$$

This completes the proof of the proposition.

21. Proof of the Gan–Gross–Prasad conjecture for Fourier–Jacobi periods

In this section we prove the Gan–Gross–Prasad conjecture for Fourier–Jacobi periods stated in Theorem 1.10, and its refinement Theorem 1.11.

21.1. Proof of Theorem 1.10.

21.1.1. Setup. We work in the global setting. In this section, we let

- E/F be a quadratic extension of global fields,
- W be a nondegenerate skew c-Hermitian space of dimension m, and V a nondegenerate skew c-Hermitian space of dimension n with $V = X \quad W \quad X$ as in section 20.1.1,
- $P(X)$ be the parabolic subgroup of $U(V)$ stabilizing X,
- $_V$ and $_W$ be irreducible cuspidal automorphic representations of $U(V)(A)$ and $U(W)(A)$ respectively,
- be an irreducible automorphic representations of $G_r(A)$ that is induced from a unitary cuspidal representation, i.e. there exists a parabolic subgroup $R = M_R N_R \quad G_r$ and a unitary cuspidal representation of $M_R(A)$ such that

$$= \{E_R^{G_r}(\ ; ';\ 0) \mid ' \in A_{R;}\ (G_r)\}:$$

- S be a set of ramified places as in Subsection 20.3.

We will also make use of the representations $_-$ and $_-$ introduced in section 20.4.2. For $s \in \mathbb{C}$ we set $_s := \quad |\det(\ )|^s_E$ and $_s := \mathrm{Ind}^{U(V)}_{P(X)}(\ _s \quad _W)$. Write tensor product decompositions $_V = {}^0_v\ _{V;v}$, $_W = {}^0_v\ _{W;v}$ and $= {}^0_v\ _v$. Then for every v the representations $_{V;v}$, $_{W;v}$ and $_v$ are of the form prescribed in Subsection 20.2.5.

21.1.2. Non-vanishing of periods. Let $\varphi_V \in \ _V$, $\varphi \in$ and $\in S(X(\mathbb{A}) \ Y_-(\mathbb{A}))$. By Proposition 20.6 and Theorem 20.5, up to enlarging S we have

$$(21.1)\quad P_{[U_V^0]}\ \varphi_V\ E_{P(X)}^{U(V)}(\varphi\ ;s);\quad = \frac{L^S(\frac{1}{2}+s;\ _V\ \ ^1)}{L^S(\frac{1}{2}+s;\ ^c\ _W)L^S(1+2s;\ ;As^{(\ 1)^m})} \int_{H^0(F_S)\backslash U_V^0(F_S)} P_{[H_L]}\ R(g)(\varphi_V\ \varphi\ _{;s});(\omega_-(g)\ )\ _{Y_-}\ dg;$$

provided that $\Re(s) \ 0$, where L^S is the partial L-function outside of S.

Proposition 21.1. The following are equivalent.

(1) There exist $\varphi_V \in \ _V$, $\varphi_W \in \ _W$ and $\in \ _-$ such that $P_{[H]}(\varphi_V\ \varphi_W;\ ) \neq 0$.

(2) There exist $\varphi_V \in \ _V$, $\varphi \in$, $\in \omega_{\overline{\psi}}$ and $s \in \mathbb{C}$ such that $E_{P(X)}^{U(V)}(\varphi;:)$ has no pole at s and $P_{[U_V^0]}\ \varphi_V\ E_{P(X)}^{U(V)}(\varphi\ ;s);\quad \neq 0$.

Proof. It follows from the hypothesis on that the Whittaker period $P_{[N_r]}$ is non zero on . By the factorization $P_{[H_L]} = P_{[H]}\ P\ _{[N_r]}$, (1) is equivalent to the non vanishing of $P_{[H_L]}$ on $_V\ \ _W\ \ _-$, which is itself equivalent to (2) by (21.1) and Proposition 20.2.

21.1.3. End of the proof. We follow the strategy described in [BPC, Section 1.3.4]. Let $= \ _n\ \ _m$ be a discrete Hermitian Arthur parameter of $G_n\ G_m$ (see section 1.1 for the definition). Let $_1;:::;\ _r$ be automorphic characters of $\mathbb{A}_E$ such that the $_1;:::;\ _r;\ _1;:::;\ _r$ are two by two distinct. Let Q_n be the standard parabolic of G_n with Levi $G_1^r\ G_m\ G_1^r$, and define for all $s \in i\mathbb{R}$

$$e_s := \ _n\quad \mathrm{Ind}_{Q_n}^{G_n}(\ _1 j{:}j_E^s\ \ :::\ \ _r j{:}j_E^s\ \ _m\ \ _1 j{:}j_E{}^s\ \ :::\ \ _r j{:}j_E{}^s):$$

These are G-regular semi-discrete Hermitian Arthur parameter of $G = G_n\ G_n$. By remark 1.1, they are $(G;H;\ ^1)$-regular. By elementary properties of the Rankin–Selberg L-function, assertion (1) of Theorem 1.10 is equivalent to the following assertion.

(1') There exists an $s \in i\mathbb{R}$ such that $L(\frac{1}{2};e_s) \neq 0$.

By Theorem 1.2 and by the definition of weak base change, cf. Subsection 1.1.2, this implies (2).

Conversely, let $= \ _V\ \ _W$ be a cuspidal automorphic representation of U_W whose base change to $G_n\ G_m$ is . Let P be the parabolic subgroup of $U(V)$ stabilizing the flag $\mathrm{span}_E(x_1)\ :::\ X$, with Levi subgroup $T_r\ U(W)$. Define

$$:= \ _1\ :::\ \ _r;$$

which is a cuspidal automorphic representation of $T_r(\mathbb{A})$. Then for all $s \in i\mathbb{R}$ the representation e_s is the weak base change of $(U(V)\ P;\ _V\ \ _s\ \ _W)$.

Define $= \mathrm{Ind}_{B_r}^{G_r}(\ )$ so that $\mathrm{Ind}_P^{U(V)}(\ _s\ \ _W) = \mathrm{Ind}_{P(X)}^{U(V)}(\ _s\ \ _W)$. Then by Proposition 21.1, (2) is equivalent to the following assertion.

(2') There exists an $s \in i\mathbb{R}$ such that the bilinear form $P_{U(V)}$ is non zero on $_V\ (\mathrm{Ind}_{P(X)}^{U(V)}\ _s\ \ _W)\ \ \omega_{\overline{\psi}}$.

But (2') implies (1') by Theorem 1.2. This concludes the proof of Theorem 1.10.

21.2. Proof of Theorem 1.11.

21.2.1. Factorization in the tempered case. We keep the notations of Subsection 21.1.1 and assume that $_V$, $_W$ and are tempered: this means that we have factorizations $_V = {}^0 {}_{V;v}$, $_W = {}^0 {}_{W;v}$ and $= {}^0 {}_v$ such that all the respective local components are tempered. Let $'_V\ 2\ _V$, $'_W\ 2\ _W$, $'\ 2$ and $'\ 2$. We also pick some Schwartz functions $2\ !_{\nabla}$ and $2\ _{-}$. We assume that all these vectors are factorizable, i.e.

$$'_V = {}^0_v{}' {}_{V;v};\ '_W = {}^0_v{}' {}_{W;v};\ ' = {}^0_v{}' {}_{;v};\ ' = {}^0_v{}' {}_{;v};\ = {}^0_v\ {}_v;\ = {}^0_v\ {}_v:$$

We now equip our representations with the following invariant inner products (where the measures are all as prescribed in Subsection 3.4).

We endow $_V$ and $_W$ with the Petersson inner product $h;\ i_{Pet}$.

We endow $!_{\nabla}$ with

$$h\ _1;\ _2 i_V := \int_{[X\ \ Y_{-}]} {}_1(x)\overline{{}_2(x)}dx;\quad {}_1;\ {}_2\ 2\ !_{\nabla};$$

and $_{-}$ with

$$h\ _1;\ _2 i_W := \int_{[Y_{-}]} {}_1(x)\overline{{}_2(x)}dx;\quad {}_1;\ {}_2\ 2\ _{-}:$$

On we put

$$hE_R^{G_r}(';\ 0); E_R^{G_r}('\ ^0; 0)i = \int_{R(A)nG_r(A)}\int_{[M_R]^1} '(mg)\overline{'^0(mg)}dmdg;\ ';\ '^0\ 2\ \mathrm{Ind}_{R(A)}^{G_r(A)}:$$

We equip with

$$h';\ '^0 i = \int_{P(X)(A)nU(V)(A)} h'(g); '^0(g)i\ _w dg;$$

where $h;\ i\ _w$ is the inner product on $_w$.

We also x factorizations of these inner products: they give rise to the integral of matrix coe cients $P_{U_V^0;v}$, $P_{H;v}$ and $P_{H_L;v}$ described in (20.15). Up to enlarging S, we assume that for $v\ 2\ S$ we have $h'_{V;v}; '_{V;v}i_v = h'_{W;v}; '_{W;v}i_v = h'_{;v}; '_{;v}i_v = h'_{;v}; '_{;v}i_v = h\ _v;\ _v i_v = h\ _v;\ _v i_v = 1$.

Proposition 21.2. For $v\ 2\ S$ we have

$$P_{U_V^0;v}('_{V;v}\ \ '_{;v};\ _v) = {}_{U(V);v}\frac{L(\frac{1}{2};\ _{V;v}\ \ _{V;v}\ \ _v{}^1)}{L(1;\ _{V;v}; Ad)\,L(1;\ _v; Ad)}; \tag{21.2}$$

$$P_{H;v}('_{V;v}\ \ '_{W;v};\ _v) = {}_{U(V);v}\frac{L(\frac{1}{2};\ _{V;v}\ \ _{W;v}\ \ _v{}^1)}{L(1;\ _{V;v}; Ad)\,L(1;\ _{W;v}; Ad)}; \tag{21.3}$$

$$P_{N_r;v}('_{;v}) = \frac{{}_{G_r;v}}{L(1;\ _v; Ad)}: \tag{21.4}$$

Note that this implies for $v \not\in S$

$$(21.5)\qquad \mathcal{P}_{H_L,v}(\varphi_{V,v} \otimes \varphi_{,v} \otimes \varphi_{W,v};\ _v) = \ _{U(V),v}\ _{G_r,v}\frac{L(\frac{1}{2};\ _{V,v} \ _{W,v}\ _v{}^{1})}{L(1;\ _{V,v};\mathrm{Ad})\,L(1;\ _{W,v};\mathrm{Ad})\,L(1;\ _v;\mathrm{Ad})}:$$

Proof. Equation (21.2) is [Xue16, Proposition 1.1.1 (3)], (21.3) is [Boi, Theorem 1.0.3] and (21.4) is a consequence of [CS80], where we use the explicit values

$$_{U(V),v} = \prod_{i=1}^{n} L(i;\ ^{i}_{E_v=F_v});\qquad _{G_r,v} = \prod_{i=1}^{n}\ _{E_v}(i);$$

where $_{E_v}$ is the local Eulerian factor of $_E$ at v (if v splits, it is a product of two local factors).

Set

$$\prod_v{}' \mathcal{P}_{U_V^0,v}(\varphi_{V,v} \otimes \varphi_{,v};\ _v) := \ ^{S}_{U(V)}\frac{L^S(\frac{1}{2};\ _V \ {}^{1})}{L^S(1;\ _V;\mathrm{Ad})\,L^{S,}(1;\ ;\mathrm{Ad})}\prod_{v\in S}\mathcal{P}_{U_V^0,v}(\varphi_{V,v} \otimes \varphi_{,v};\ _v);$$

$$\prod_v{}' \mathcal{P}_{H,v}(\varphi_{V,v} \otimes \varphi_{W,v};\ _v) := \ ^{S}_{U(V)}\frac{L^S(\frac{1}{2};\ _V \ _W{}^{1})}{L^S(1;\ _V;\mathrm{Ad})\,L^S(1;\ _W;\mathrm{Ad})}\prod_{v\in S}\mathcal{P}_{H,v}(\varphi_{V,v} \otimes \varphi_{W,v};\ _v);$$

$$\prod_v{}' \mathcal{P}_{N_r,v}(\varphi_{,v};\varphi_{,v}) := \frac{^{S,}_{G_r}}{L^{S,}(1;\ ;\mathrm{Ad})}\prod_{v\in S}\mathcal{P}_{N_r,v}(\varphi_{,v}):$$

where $L^{S,}(1;\ ;\mathrm{Ad})$ and $L^{S,}(1;\ ;\mathrm{Ad})$ are the regularized values

$$L^{S,}(1;\ ;\mathrm{Ad}) = (s-1)^d L^S(s;\ ;\mathrm{Ad})\Big|_{s=1};\quad L^{S,}(1;\ ;\mathrm{Ad}) = (s-1)^d L^S(s;\ ;\mathrm{Ad})\Big|_{s=1};$$

with $d = \dim(A_{M_R})$, and

$$^{S,}_{G_r} = \ ^{S,}_{E}(1)\prod_{i=2}^{n}\ ^{S}_{E}(i);$$

with $^{S,}_{E}(1)$ the residue of the partial zeta function $^{S}_{E}$ at $s = 1$. We de ne similarly

$$\prod_v{}' \mathcal{P}_{H_L,v}(\varphi_{V,v} \otimes \varphi_{,v} \otimes \varphi_{W,v};\ _v);$$

as well as the corresponding twisted versions for $s \in i\mathbb{R}$.

21.2.2. End of the proof. Let us use the same notations as in section 21.1.3, so that $_s$ is a regular semi-discrete Hermitian Arthur parameter for all $s \in i\mathbb{R}$, which is the weak base change of $(U(V)\ P;\ _V\ _s\ _W)$. De ne $_s := \mathrm{Ind}^{U(V)}_{P(X)}(\ _s\ _W)$. Equation (21.1) reads

(21.6)

$$\left|\mathcal{P}_{[U_V^0]}\ \varphi_V\ E^{U(V)}_{P(X)}(\varphi;s);\right|^2 = \left|\frac{L^S(\frac{1}{2}+s;\ _V \ {}^{1})}{L^S(\frac{1}{2}+s;\ ^{c}\ _W)\,L^S(1+2s;\ ;\mathrm{As}^{(-1)^m})}\right|^2$$

$$\int_{(H^0(F_S)\backslash U_V^0(F_S))^2} \mathcal{P}_{[H_L]}\ R(g_1)(\varphi_V \otimes \varphi_{,s});(\omega^{-}(g_1)\ _{Y_-}\ \overline{\mathcal{P}_{[H_L]}(R(g_2)(\varphi_V \otimes \varphi_{,s});(\omega^{-}(g_2)\ _{Y_-})}dg_1 dg_2;$$

for s 2 iR. But by Theorem 1.5 we have

$$P_{[U^0_V]} \; ' _V \quad E^{U(V)}_{P(X)}(' \quad ; s); \quad ^2 = \frac{^S_{U(V)}}{jS\ j} \frac{L^S(\frac{1}{2};\ _V \quad _s \quad ^1)}{L^{S;}(1;\ _s; Ad)\, L^S(1;\ _V; Ad)} \prod_{v 2 S} P_{U^0_V;v}(' _{V;v} \quad ' _{;s;v};\ _v) \; ^2 :$$

By Proposition 20.3, this is

$$(21.7) \quad \frac{^S_{U(V)} \;\; ^{S;}_{G_r}}{jS\ j} \frac{L^S(\frac{1}{2};\ _V \quad _s \quad ^1)}{L^{S;}(1;\ _s; Ad)\, L^T(1;\ _V; Ad)} \prod_{v 2 S} \int_{(H^0_V n(U^0_{V;v})^2} P_{H_{L;v}} \; (\; _{V;v}(g_i)' \; ^i_{V;v} \quad ' \; ^i_{;s;v}(g_i); (! \; _{\nabla;v}(g_i) \quad _{i;v})_{Y_{V^-}})_{i=1;2} \; dg_1 dg_2;$$

where we have used (20.4) to make $^{S;}_{G_r}$ appear. From (21.6), (21.7), Proposition 20.2 and the equality of partial L-functions:

$$\frac{L^S(\frac{1}{2};\ _V \quad _s \quad ^1)}{L^{S;}(1;\ _s; Ad)} = \frac{L^S(\frac{1}{2};\ _V \quad _W \quad ^1)}{L^{S;}(1;\ _W; Ad)\, L^S(1;\ ; Ad)} \; \frac{L^S(\frac{1}{2} + s;\ _V \quad ^1)}{L^S(1 + s;\ ^c \quad _W)\, L(1 + 2s;\ ; As^{(\ 1)^m})} \; ^2 :$$

we see that for every factorizable' 2 , ' $_W$ 2 $_W$ and 2 $-$ we have

$$(21.8) \quad P_{[H_L]}(' _V \quad ' \quad ' _W;\ ) \; ^2 = \frac{^S_{U(V)} \;\; ^{S;}_{G_r}}{jS\ j} \frac{L^S(\frac{1}{2};\ _V \quad _W \quad ^1)}{L^S(1;\ _V; Ad)\, L^{S;}(1;\ ; Ad)\, L^S(1;\ _W; Ad)} \prod_{v 2 S} P_{H_L;v}(' _{V;v} \quad ' _{;v} \quad ' _{W;v};\ _v)$$

$$= \frac{1}{jS\ j} \prod_v{}^0 P_{H_L;v}(' _{V;v} \quad ' _{;v} \quad ' _{W;v};\ _v):$$

Recall that

$$P_{[H_L]}(' _V \quad ' \quad ' _W;\ ) = P_{[H]}(' _V \quad ' _W;\ ) P_{[N_r]}(' \quad);$$

and

$$P_{H_L;v}(' _{V;v} \quad ' _{;v} \quad ' _{W;v};\ _v) = P_{H;v}(' _{V;v} \quad ' _{W;v};\ _v) P_{N_r;v}(' _{;v})$$

By [FLO12], it is known that

$$(21.9) \quad jP_{N_r}(' \quad)j^2 = \prod_v{}^0 P_{N_r;v}(' _{;v});$$

so that Theorem 1.11 follows by dividing (21.8) by (21.9), which is non zero for some 2 , taking into account the equality of L functions

$$L^S(1;\ _V; Ad)\, L^S(1;\ _W; Ad) = L^S(1;\ ; As^0_{n;m});$$

where $As^0_{n;m} = As^{(\ 1)^n} \quad As^{(\ 1)^m}$, and the explicit formulae for every place v

$$_{U(V);v} = \prod_{i=1}^n L(i;\ ^i_{E_v = F_v}) \quad \text{and} \quad _{U(V)} = \prod_{i=1}^n L(i;\ ^i_{E=F}):$$

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