CENTRAL VALUES OF DEGREE SIX L-FUNCTIONS

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Contents

1. Introduction 1
2. Rationality 2
3. The central value formula 4
References 7

1. Introduction

Let $\kappa', \kappa \geq 3$ be two odd integers. Let f (resp. g) be a normalized holomorphic modular form of weight 2κ (resp. $\kappa'+1$) and level one on the upper half plane $\mathfrak{h}$. Assume that they are Hecke eigenforms. Let $L(s, \mathrm{Sym}^2 g \times f)$ be the completed degree six L-function and we normalize so that $s = \frac{1}{2}$ is the center of symmetry. Let $\langle -, - \rangle$ be the Petersson inner product, defined using the usual measure on $\mathfrak{h}$ so that the volume of $\Gamma_0(1)\backslash\mathfrak{h}$ equals $\frac{\pi}{3}$. Let $c^+(f)$ be the fundamental period of f defined in [Shi77].

The goal of this short note is to prove the following result.

Theorem 1.1. *For any $\tau \in \mathrm{Aut}(\mathbb{C})$, we have*

$$\frac{L(\frac{1}{2}, \mathrm{Sym}^2 g^\tau \times f^\tau)}{\langle g^\tau, g^\tau \rangle^2 c^+(f^\tau)} = \left(\frac{L(\frac{1}{2}, \mathrm{Sym}^2 g \times f)}{\langle g, g \rangle^2 c^+(f)} \right)^\tau.$$

One can prove that if $\kappa' < \kappa$, then $\epsilon(\frac{1}{2}, \mathrm{Sym}^2 g \times f) = -1$. Thus $L(\frac{1}{2}, \mathrm{Sym}^2 g \times f) = 0$ and the theorem is vacuous. Thus from now on we always assume $\kappa' \geq \kappa$. Ichino [Ich05, Corollary 2.6] proved Theorem 1.1 under the assumption $\kappa' = \kappa$. Ichino deduced his result from the explicit calculation of the periods of Saito-Kurokawa liftings. We deduce the result from the periods of Jacobi forms instead. This is easier and better for generalizations. One should note that Theorem 1.1 does not follow in some easy way from the rationality of the central value of the triple product L-functions, as

$$L(\frac{1}{2}, g \times g \times f) = L(\frac{1}{2}, \mathrm{Sym}^2 g \times f) L(\frac{1}{2}, f) = 0.$$

The restrictions on the weight, the level and on the ground field are not really necessary. It is the author's ongoing work to generalize the result to the case of Hilbert modular forms. The restrictions on the weight and the level can be significantly weakened. However, the author feels that working out the current particularly simple case is still worth the effort, for its beauty and brevity.

<u>*Notation*</u> We denote by $\mathbb{A}$ the ring of adeles of $\mathbb{Q}$ and $\widehat{\mathbb{Z}} = \prod_p \mathbb{Z}_p$. We fix an additive character $\psi = \otimes \psi_v : \mathbb{Q}\backslash\mathbb{A} \to \mathbb{C}^\times$ as follows: if $v = \infty$, then $\psi_\infty(x) = e^{2\pi\sqrt{-1}x}$ for $x \in \mathbb{R}$;

if $v=p$, then $\psi_p(x)=e^{-2\pi\sqrt{-1}x}$ for $x\in\mathbb{Z}[p^{-1}]$. The measures on the unipotent groups are always the self-dual measure with respect to ψ.

We denote by R the Jacobi group, which is a subgroup of Sp_4 consisting of elements of the form

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}(\lambda,\mu,\xi)=\begin{pmatrix} a & & b & \\ & 1 & & \\ c & & d & \\ & & & 1 \end{pmatrix}\begin{pmatrix} 1 & & & \mu \\ \lambda & 1 & \mu & \xi \\ & & 1 & -\lambda \\ & & & 1 \end{pmatrix}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix}\in\mathrm{SL}_2\,.$$

We view SL_2 as a subgroup of R in this way.

Let π be a smooth representation of some Lie group. We denote by $\mathrm{d}\pi$ the infinitesimal action of the its Lie algebra.

The coordinate on $\mathfrak{h}\times\mathbb{C}$ is written as (τ,z). We always write $\tau=x+iy$.

Let $\xi(s)=\prod_v\xi_v(s)$ be the completed Riemann zeta function.

2. Rationality

2.1. **The central value formula.** Let $h\in S^+_{\kappa+\frac12}(\Gamma_0(4))$ be the Hecke eigenform of weight $\kappa+\frac12$ associated to f by the Shimura correspondences. Let F_h be the Jacobi form of index one associated to h. The Fourier expansion of $F_h(\tau,z)$ is given by

$$\sum_{n,m\in\mathbb{Z}} c_h(4n-m^2)e^{2\pi in\tau+2\pi imz},$$

where $h=\sum c_h(n)e^{2\pi in\tau}$ is the Fourier expansion of h.

Define the following differential operator

$$\Delta=\frac{1}{2\pi i}\frac{\partial}{\partial z}+\frac{z-\overline{z}}{iy}$$

on smooth functions on $\mathfrak{h}\times\mathbb{C}$. For any function F on $\mathfrak{h}\times\mathbb{C}$, we denote its restriction to $\mathfrak{h}\times\{0\}$ simply by $F|_{\mathfrak{h}}$.

Proposition 2.1. *There is a nonzero rational number $c_{\kappa,\kappa'}$, such that*

$$|\langle g,(\Delta^{\kappa'-\kappa}F_h)|_{\mathfrak{h}}\rangle|^2=c_{\kappa,\kappa'}\frac{\langle h,h\rangle}{\langle f,f\rangle}L\left(\frac12,\mathrm{Sym}^2 g\times f\right).$$

The proof of this proposition will be given in Section 3. The constant $c_{\kappa,\kappa'}$ can be made explicit. The expression is elementary but messy. We decide not to do so.

2.2. **Nearly holomorphic forms.** We begin with an elementary lemma.

Lemma 2.2. *Suppose that $\varphi(\tau,z)$ is a holomorphic function on $\mathfrak{h}\times\mathbb{C}$. Let $l\geq 0$ be an integer. Then $(\Delta^l\varphi)|_{\mathfrak{h}}$ is of the form*

$$\sum_{\nu=0}^{[\frac{l}{2}]}\frac{c_\nu}{(2\pi y)^\nu}\left(\frac{1}{2\pi i}\frac{\partial}{\partial z}\right)^{l-2\nu}\varphi\Big|_{\mathfrak{h}}$$

where c_ν's are some rational numbers.

The proof is left for the reader.

We make use of the notion of nearly holomorphic modular forms in the sense of Shimura [Shi76]. By a nearly holomorphic modular form of level one, weight λ and order r, we mean a real analytic function f on $\mathfrak{h}$, such that $f|_\lambda\gamma = f$ for all $\gamma \in \Gamma_0(1)$ and that there are holomorphic functions $f_0, \cdots, f_r$ on $\mathfrak{h}$ so that $f = f_0 + y^{-1}f_1 + \cdots + y^{-r}f_r$. Let

$$\delta_\lambda = \frac{1}{2\pi i}\left(\frac{\partial}{\partial z} + \frac{\lambda}{2iy}\right), \quad \delta_\lambda^l = \delta_{\lambda+2l-2} \circ \cdots \circ \delta_{\lambda+2} \circ \delta_\lambda,$$

be the usual Maass–Shimura differential operators.

For brevity, from now on, we put $r = \kappa' - \kappa \geq 0$. Note that r is even. It follows from Lemma 2.2 that $(\Delta^r F_h)|_{\mathfrak{h}}$ is a nearly holomorphic form of weight $\kappa' + 1$, level one and order at most $\frac{r}{2}$. By [Shi76, Lemma 7], there are holomorphic modular forms $g_0, \cdots, g_{\frac{r}{2}}$ of level one and weight $\kappa'+1, \kappa'-1, \cdots, \kappa'-r+1$ respectively, such that

$$(\Delta^r F_h)|_{\mathfrak{h}} = g_0 + \delta^1_{\kappa'-1} g_1 + \cdots + \delta^{\frac{r}{2}}_{\kappa'-r+1} g_{\frac{r}{2}}.$$

The $g_0, \cdots, g_{\frac{r}{2}}$ are uniquely determined. We call g_0 the holomorphic projection of $(\Delta^r F_h)|_{\mathfrak{h}}$.

Lemma 2.3. *For any $\tau \in \mathrm{Aut}(\mathbb{C})$, we have*

$$(\Delta^r F_{h^\tau})|_{\mathfrak{h}} = g_0^\tau + \delta^1_{\kappa'-1} g_1^\tau + \cdots + \delta^{\frac{r}{2}}_{\kappa'-r+1} g^\tau_{\frac{r}{2}}. \tag{2.1}$$

Proof. Let

$$F_h = \sum_{n,m\in\mathbb{Z}} c_h(4n - m^2) e^{2\pi i n\tau + 2\pi i m z}.$$

be the Fourier expansion of F_h as before. By Lemma 2.2, we have

$$(\Delta^r F_h)|_{\mathfrak{h}} = \sum_{0\leq t\leq \frac{r}{2}} a_t (2\pi y)^{-t} \sum_{m,n\in\mathbb{Z}} n^{r-2t} c_h(4n-m^2) e^{2\pi i n\tau},$$

where a_t's are some rational numbers. It is well-known (and easy to see) that $\delta^t_{\kappa-2t} g_t$ is of the form

$$\delta^t_{\kappa'-2t+1} g_t = \sum_{0\leq s\leq t} b_{s,t} (4\pi y)^{-s} \sum_{n\geq 0} n^{t-s} c_{g_t}(n) e^{2\pi i n\tau},$$

where $b_{s,t}$'s are some rational numbers and $g_t = \sum c_{g_t}(n) e^{2\pi i n\tau}$ is the Fourier expansion of g_t. Now expressing both side of (2.1) as polynomials in $(2\pi y)^{-1}$, one can easily verify the lemma. ∎

2.3. **Proof of Theorem 1.1.** The Fourier coefficients of f and g are all real. Fix a fundamental discriminant $-D < 0$ such that $c_h(D) \neq 0$. We may and will normalize h so that $c_h(D) = 1$. Then the Fourier coefficients of h are all real. It follows that the Fourier coefficient of g_0 are all real. Thus $\langle g, g_0\rangle$ is real.

By [Shi76, Lemma 4], for any $\tau \in \mathrm{Aut}(\mathbb{C})$, we have

$$\left(\frac{\langle g, g_0\rangle}{\langle g, g\rangle}\right)^\tau = \frac{\langle g^\tau, g_0^\tau\rangle}{\langle g^\tau, g^\tau\rangle}. \tag{2.2}$$

Moreover, by [Shi76, Lemma 6],

$$\langle g, (\Delta^r F_h)|_{\mathfrak{h}}\rangle = \langle g, g_0\rangle. \tag{2.3}$$

Let χ_{-D} be the quadratic character attached to $\mathbb{Q}(\sqrt{-D})$. Then Kohnen–Zagier's formulae reads

$$L\left(\frac{1}{2}, f \otimes \chi_{-D}\right) = 2^{-\kappa+1} D^{\frac{1}{2}} \frac{\langle f, f\rangle}{\langle h, h\rangle}, \tag{2.4}$$

where $L(s, f\otimes\chi_{-D})$ is the completed L-function. Moreover by the definition of $c^+(f)$ [Shi77], we have

$$\left(\frac{D^{-\frac{1}{2}} L(\frac{1}{2}, f \otimes \chi_{-D})}{c^+(f)}\right)^\tau = \frac{D^{-\frac{1}{2}} L(\frac{1}{2}, f^\tau \otimes \chi_{-D})}{c^+(f^\tau)}. \tag{2.5}$$

Now Theorem 1.1 follows from Proposition 2.1, Lemma 2.3 and the equalities (2.2) – (2.5).

3. The central value formula

3.1. **Automorphic forms and representations.** Via the usual procedure, we may define the adelization of the modular forms f (resp. g), which are automorphic forms on $\mathrm{PGL}_2(\mathbb{A})$. We denote it by $\mathbf{f}$ (resp. $\mathbf{g}$). Let $\pi = \otimes\pi_v$ (resp. $\tau = \otimes\tau_v$) be the automorphic representation of $\mathrm{PGL}_2(\mathbb{A})$ that $\mathbf{f}$ (resp. $\mathbf{g}$) generates. Since f, g are both Hecke eigenforms, $\mathbf{f}$ and $\mathbf{g}$ factorize, namely $\mathbf{f} = \otimes\mathbf{f}_v$, $\mathbf{g} = \otimes\mathbf{g}_v$. Let $\mathbf{h}$ be the adelization of h. This is an automorphic form on $\widetilde{\mathrm{SL}}_2(\mathbb{A})$. Let $\sigma = \otimes\sigma_v$ be the automorphic representation of $\widetilde{\mathrm{SL}}_2(\mathbb{A})$ that $\mathbf{h}$ generates. The automorphic form $\mathbf{h}$ factorizes as $\mathbf{h} = \otimes\mathbf{h}_v$. Let $\mathbf{F}$ be the adelization of F_h as explained in [BS98, Section 7.4]. This is an automorphic form on $R(\mathbb{A})$. It generates an automorphic representation $\rho = \otimes\rho_v$ of $R(\mathbb{A})$ [BS98, Section 7.3]. The automorphic form $\mathbf{F}$ factorizes as $\mathbf{F} = \otimes\mathbf{F}_v$.

Let ω_ψ be the Weil representation of $R(\mathbb{A})$ which is realized on the Schwartz spaces $\mathcal{S}(\mathbb{A})$. Then $\rho \simeq \sigma \otimes \omega_\psi$, c.f. [BS98, Theorem 7.3.3]. Let $\phi = \otimes\phi_v$ be the Schwartz function with $\phi_v = \mathbf{1}_{\mathbb{Z}_v}$ if $v \neq 2, \infty$, $\phi_2 = \mathbf{1}_{\frac{1}{2}\mathbb{Z}_2}$ and $\phi_\infty = e^{-2\pi x^2}$.

3.2. **Local components.** For the explicit computation, we list all the local components of the automorphic representations π, σ, τ and ρ. We refer the readers to [BS98, Section 7.5] for the explanation of the local components of ρ.

Suppose that $v = \infty$.

- π_∞ (resp. τ_∞) is a discrete series representation of $\mathrm{PGL}_2(\mathbb{R})$ of weight 2κ (resp. $\kappa' + 1$). $\mathbf{f}_\infty$ (resp. $\mathbf{g}_\infty$) is a lowest weight vector in π_∞.
- ρ_∞ is the discrete series representation of $R(\mathbb{R})$ of lowest K-type $\kappa+1$. $\mathbf{F}_\infty$ is a lowest weight vector.
- σ_∞ is a discrete series representation of $\widetilde{\mathrm{SL}}_2(\mathbb{R})$ of lowest K-type $\kappa+\frac{1}{2}$. $\mathbf{h}_\infty$ is a lowest weight vector.

Suppose that $v < \infty$.

- π_v (resp. τ_v) is an unramified principal series representation of $\mathrm{PGL}_2(\mathbb{Q}_v)$. $\mathbf{f}_v$ (resp. $\mathbf{g}_v$) is $\mathrm{PGL}_2(\mathbb{Z}_v)$-fixed.
- ρ_v is an unramified principal series representation of $R(\mathbb{Z}_v)$ and $\mathbf{F}_v$ is $R(\mathbb{Z}_v)$-fixed.
- If $v \neq 2$, then σ_v is an unramified principal series representation of $\widetilde{\mathrm{SL}}_2(\mathbb{Q}_v)$ and $\mathbf{g}_v$ is $\mathrm{SL}_2(\mathbb{Z}_v)$-fixed. If $v = 2$, then σ_2 is a principal series representation of $\widetilde{\mathrm{SL}}_2(\mathbb{Q}_2)$ and contains a distinguished vector, which is $\mathbf{h}_2$. We refer the readers to [Ich05, Section 3.2] for a description of this representation.

3.3. **The coarse form of the central value formula.** If $v \neq \infty$, let $\mathrm{d}g_v$ be the measure on $\mathrm{SL}_2(\mathbb{Q}_v)$ so that the volume of $\mathrm{SL}_2(\mathbb{Z}_v)$ equals one. On $\mathrm{SL}_2(\mathbb{R})$, let $\mathrm{d}g_\infty = y^{-2}\mathrm{d}x\mathrm{d}y\mathrm{d}k_\infty$, where $g_\infty = \left(\begin{smallmatrix} y & \\ & y^{-1}\end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & x \\ & 1\end{smallmatrix}\right)k_\infty$ is the Iwasawa decomposition and $\mathrm{d}k_\infty$ is the measure on $\mathrm{SO}_2(\mathbb{R})$ so that the volume of it equals one. Then $\xi(2)^{-1}\prod \mathrm{d}g_v$ is the Tamagawa measure on $\mathrm{SL}_2(\mathbb{A}_\mathbb{Q})$. This also gives a measure on $R(\mathbb{A})$. We define the Petersson inner product using these measures. Then the isomorphism $\rho \simeq \sigma \otimes \omega_\psi$ is an isometry.

Let $\mathfrak{r}$ be the Lie algebra of $R(\mathbb{R})$ and $\mathfrak{r}_\mathbb{C}$ its complexification. Define the following elements in $\mathfrak{r}_\mathbb{C}$

$$\mathsf{X}_\pm = \frac{1}{2}\begin{pmatrix} 1 & & \pm i & \\ & 0 & & \\ \pm i & & -1 & \\ & & & 0 \end{pmatrix}, \quad \mathsf{Y}_\pm = \frac{1}{2}\begin{pmatrix} 0 & & & \pm i \\ 1 & 0 & \pm i & \\ & & 0 & -1 \\ & & & 0 \end{pmatrix}.$$

Then $\mathrm{d}\rho_\infty \mathsf{X}_- \mathbf{F}_\infty = \mathrm{d}\rho_\infty \mathsf{Y}_- \mathbf{F}_\infty = 0$. Note that $\mathfrak{sl}_2$ is a Lie subalgebra of $\mathfrak{r}$ and $\mathsf{X}_\pm \in \mathfrak{sl}_{2,\mathbb{C}}$. Define $Y_+ = (-2\pi)^{-1}\mathrm{d}\rho_\infty \mathsf{Y}_+$. Then by [BS98, Remark 3.5.1], the adelization of $\Delta^r F_h$ is $Y_+^r \mathbf{F}$.

For each place v, pick some inner product on τ_v and ρ_v. Define

$$\alpha_v(\mathbf{g}_v, \mathbf{F}_v) = \int_{\mathrm{SL}_2(\mathbb{Q}_v)} \langle \tau_v(g)\mathbf{g}_v, \mathbf{g}_v\rangle \overline{\langle \rho_v(g)\mathbf{F}_v, \mathbf{F}_v\rangle} \mathrm{d}g$$

for any $\mathbf{g}_v \in \tau_v$ and $\mathbf{F}_v \in \rho_v$. Define

$$\alpha_v^\natural(\mathbf{g}_v, \mathbf{F}_v) = \left(\frac{\xi_v(2) L(\frac{1}{2}, \mathrm{Sym}^2 \tau_v \times \pi_v)}{L(1, \pi_v, \mathrm{Ad}) L(1, \tau_v, \mathrm{Ad})}\right)^{-1} \frac{\alpha_v(\mathbf{g}_v, \mathbf{F}_v)}{\langle \mathbf{g}_v, \mathbf{g}_v\rangle \langle \mathbf{F}_v, \mathbf{F}_v\rangle}.$$

Then it is proved in [Qiu14, Lemma 4.4] that if $v \neq 2, \infty$, then $\alpha_v^\natural(\mathbf{g}_v, \mathbf{F}_v) = 1$ if $\mathbf{g}_v$ (resp. $\mathbf{F}_v$) is $\mathrm{SL}_2(\mathbb{Z}_v)$ (resp. $R(\mathbb{Z}_v)$) fixed. It is proved in [Xue, Section 6] that this also holds if $v = 2$. The following proposition follows from this and [Qiu14, Theorem 4.5].

Proposition 3.1.

$$\frac{|\langle \mathbf{g}, Y_+^r \mathbf{F}|_{\mathrm{SL}_2(\mathbb{A})}\rangle|^2}{\langle \mathbf{g}, \mathbf{g}\rangle \langle Y_+^r \mathbf{F}, Y_+^r \mathbf{F}\rangle} = \frac{1}{4} \times \frac{L(\frac{1}{2}, \mathrm{Sym}^2 \tau \times \pi)}{L(1, \pi, \mathrm{Ad}) L(1, \tau, \mathrm{Ad})} \times \alpha_\infty^\natural(\mathbf{g}_\infty, Y_+^r \mathbf{F}_\infty).$$

Remark 3.2. The appearance of $\xi(2)$ in [Qiu14, Theorem 4.5] is due to the different choices of the measures.

3.4. **Computation at the Archimedean place.** To compute $\alpha_\infty^\natural(\mathbf{g}_\infty, Y_+^r \mathbf{F}_\infty)$, we make use of an explicit model for the representation ρ_∞. For the full description of this model, we refer the readers to [BS98, Proposition 3.1.7]. We only list here the action of $\mathsf{X}_\pm$ and $\mathsf{Y}_\pm$. We denote this model by $D(\kappa+1)$. As a vector space,

$$D(\kappa+1) = \bigoplus_{k,l \geq 0,\ l \text{ even}} \mathbb{C} v_{k,l},$$

and elements of $\mathrm{SO}_2(\mathbb{R})$ acts on $v_{k,l}$ via the character $u \mapsto u^{\kappa+1+k+l}$. The action of $\mathsf{X}_\pm$ and $\mathsf{Y}_\pm$ is given by

$$\begin{aligned} \mathrm{d}\rho_\infty \mathsf{Y}_+ v_{k,l} &= v_{k+1,l}, & \mathrm{d}\rho_\infty \mathsf{X}_+ v_{k,l} &= -\frac{1}{2\pi} v_{k+2,l} + v_{k,l+2}, \\ \mathrm{d}\rho_\infty \mathsf{Y}_- v_{k,l} &= -2\pi k v_{k-1,l}, & \mathrm{d}\rho_\infty \mathsf{X}_- v_{k,l} &= \pi k(k-1) v_{k-2,l} - \frac{l}{2}(\kappa - \frac{1}{2} + \frac{l}{2}) v_{k,l-2}. \end{aligned}$$

There is an inner product on $D(\kappa+1)$ such that $v_{k,l}$'s form an orthogonal basis. Denote this inner product by $\langle -,-\rangle$ and $\|v\|^2 = \langle v, v\rangle$. Then by [BS98, p. 46–47], we have

$$\|v_{k,l+2}\|^2 = \frac{l+2}{2}(\kappa + \frac{1+1}{2})\|v_{k,l}\|^2, \quad \|v_{k+1,l}\|^2 = 2\pi(k+1)\|v_{k,l}\|^2.$$

We may normalize the inner product so that $\|v_{r,0}\| = 1$. Then for any $2 \le l \le r$, l even, we have

$$\|v_{r-l,l}\|^2 = (4\pi)^{-l} \prod_{0\le j\le l-2,\ j \text{ even}} \frac{(j+2)(2\kappa+j+1)}{(r-j)(r-j-1)}. \tag{3.1}$$

The space

$$D(\kappa+1, r) = \bigoplus_{k+l=r,\ l \text{ even}} \mathbb{C}v_{k,l}$$

is the largest subspace of $D(\kappa+1)$ on which $\mathrm{SO}_2(\mathbb{R})$ acts via the character $u \mapsto u^{\kappa'+1}$.

Lemma 3.3. *There is a unique (up to a scalar) vector v_r^{hol} in $D(\kappa+1,r)$ with the property that $\mathrm{d}\rho_\infty \mathsf{X}_- v_r^{\mathrm{hol}} = 0$. It is given by*

$$\sum_{0\le l\le r,\ l \text{ even}} c_l v_{r-l,l}, \quad c_0 = 1, \quad c_l = (2\pi)^{-\frac{l}{2}} \prod_{0\le j\le l-2,\ j \text{ even}} \frac{(j+2)(j+2\kappa+1)}{(r-j)(r-j-1)}, \ (l \ge 2).$$

Proof. Suppose that

$$v_r^{\mathrm{hol}} = \sum_{0\le l\le r,\ l \text{ even}} c_l v_{r-l,l} \in D(\kappa+1, r)$$

and $\mathrm{d}\rho_\infty \mathsf{X}_- v_r^{\mathrm{hol}} = 0$. Then by the formula for the action of X_-, we conclude that for any $0 \le l \le r-2$ and l even, we have

$$c_{l+2} \cdot \frac{l+2}{2}(\kappa - \frac{1}{2} + \frac{l+2}{2}) = c_l \cdot \pi \cdot (r-l)(r-l-1).$$

Let $c_0 = 1$. Then we may recursively solve for c_l's. □

With this choice of the model of ρ_∞, we realize τ_∞ as a subrepresentation of $\rho_\infty|_{\mathrm{SL}_2(\mathbb{R})}$ generated by v_r^{hol}. We may assume that the inner product on τ_∞ is given by the restriction of that of ρ_∞. Since $\alpha_\infty^\natural(\mathbf{g}_\infty, Y_+^r \mathbf{F}_\infty)$ does not change if we replace $\mathbf{g}_\infty$ or $\mathbf{F}_\infty$ by a scalar multiple of them, we may assume that $\mathbf{g}_\infty = v_r^{\mathrm{hol}}$ and $Y_+^r \mathbf{F}_\infty = v_{r,0}$.

Proposition 3.4. $\alpha_\infty^\natural(\mathbf{g}_\infty, Y_+^r \mathbf{F}_\infty) \in \mathbb{Q}^\times \pi^r$.

Proof. The orthogonal projection of $v_{r,0}$ to the line generated by v_r^{hol} is $\|v_r^{\mathrm{hol}}\|^{-2} v_r^{\mathrm{hol}}$. It follows that

$$\alpha_\infty(\mathbf{g}_\infty, Y_+^r \mathbf{F}_\infty) = \frac{1}{\|v_r^{\mathrm{hol}}\|^4} \int_{\mathrm{SL}_2(\mathbb{R})} |\langle \tau_\infty(g) v_r^{\mathrm{hol}}, v_r^{\mathrm{hol}}\rangle|^2 \mathrm{d}g.$$

As τ_∞ is the discrete series representation of $\mathrm{SL}_2(\mathbb{R})$ with lowest K-type $\kappa'+1$, it is well-known that

$$\langle \tau_\infty(\mathrm{diag}[e^t, e^{-t}]) v_r^{\mathrm{hol}}, v_r^{\mathrm{hol}}\rangle = \|v_r^{\mathrm{hol}}\|^2 \times (\cosh t)^{-(\kappa'+1)}, \quad t \ge 0.$$

Let $g = k_1 \operatorname{diag}[e^t, e^{-t}] k_2$ be the Cartan decomposition. Then $\mathrm{d}g = 2\pi \sinh 2t \mathrm{d}t \mathrm{d}k_1 \mathrm{d}k_2$ where $\mathrm{d}k_1, \mathrm{d}k_2$ are the measure on $\mathrm{SO}_2(\mathbb{R})$ so that the volume is one and $\mathrm{d}t$ is the usual Lebesgue measure on $\mathbb{R}$. Therefore

$$\alpha_\infty^\natural(\mathbf{g}_\infty, Y_+^r \mathbf{F}_\infty) = \left(\frac{\xi_\infty(2) L(\frac{1}{2}, \operatorname{Sym}^2 \tau_\infty \times \pi_\infty)}{L(1, \pi_\infty, \mathrm{Ad}) L(1, \tau_\infty, \mathrm{Ad})} \right)^{-1} \frac{1}{\|v_r^{\mathrm{hol}}\|^2} \int_0^\infty (\cosh t)^{-(\kappa'+1)} 2\pi \sinh 2t \mathrm{d}t.$$

By definition

$$\frac{\xi_\infty(2) L(\frac{1}{2}, \operatorname{Sym}^2 \tau_\infty \times \pi_\infty)}{L(1, \pi_\infty, \mathrm{Ad}) L(1, \tau_\infty, \mathrm{Ad})} = \frac{\pi^{-1}\Gamma(1) \cdot 2^2 (2\pi)^{-2\kappa'-1} \Gamma(\kappa'+\kappa)\Gamma(\kappa'-\kappa+1) \cdot 2(2\pi)^{-\kappa}\Gamma(\kappa)}{2(2\pi)^{-\kappa'-1}\Gamma(\kappa'+1)\pi^{-1}\Gamma(1) \cdot 2(2\pi)^{-2\kappa}\Gamma(2\kappa)\pi^{-1}\Gamma(1)}.$$

Thus it lies in $\mathbb{Q}^\times \pi^{-r+1}$. It is not hard to see, from Lemma 3.3 and the expression (3.1), that $\|v_r^{\mathrm{hol}}\| \in \mathbb{Q}^\times$. Moreover

$$\int_0^\infty (\cosh t)^{-(\kappa'+1)} 2\pi \sinh 2t \mathrm{d}t = 4\pi(\kappa'-1)^{-1}.$$

Therefore $\alpha_\infty^\natural(\mathbf{g}_\infty, Y_+^r \mathbf{F}_\infty) \in \mathbb{Q}^\times \pi^r$. □

3.5. **Proof of Proposition 2.1.** Suppose that $a, b \in \mathbb{C}^\times$. The notation $a \sim b$ means $ab^{-1} \in \mathbb{Q}^\times$. Since the volume of $\Gamma_0(1)\backslash\mathfrak{h}$ equals $\frac{\pi}{3}$, we have

$$\langle \mathbf{g}, Y_+^r \mathbf{F} \rangle \sim \pi^{-1} \langle g, (\Delta^r F_h)|_{\mathfrak{h}} \rangle, \quad \langle \mathbf{g}, \mathbf{g} \rangle \sim \pi^{-1} \langle g, g \rangle.$$

By [BS98, Theorem 7.3.3], we have $\langle Y_+^r \mathbf{F}, Y_+^r \mathbf{F} \rangle = \langle \mathbf{h}, \mathbf{h} \rangle \langle \mathrm{d}\omega_\psi \mathsf{Y}_+^r \phi, \mathrm{d}\omega_\psi \mathsf{Y}_+^r \phi \rangle$. By [BS98, Lemma 3.2.1]

$$\mathrm{d}\omega_\psi \mathsf{Y}_+^r \phi_\infty = \left(\frac{1}{4\pi} \frac{\mathrm{d}}{\mathrm{d}x} - x \right)^r \phi_\infty = (4\pi)^{-\frac{r}{2}} H_r(2\pi^{\frac{1}{2}} x) e^{-2\pi x^2}.$$

where H_r is the r-th Hermite polynomial. It has integer coefficients and contains only even powers of x since r is even. Therefore $\langle \phi, \phi \rangle \sim \pi^{-r}$ and

$$\langle Y_+^r \mathbf{F}, Y_+^r \mathbf{F} \rangle = \langle \mathbf{h}, \mathbf{h} \rangle \langle \mathrm{d}\omega_\psi \mathsf{Y}_+^r \phi, \mathrm{d}\omega_\psi \mathsf{Y}_+^r \phi \rangle \sim \pi^{-r} \langle \mathbf{h}, \mathbf{h} \rangle \sim \pi^{-r-1} \langle h, h \rangle.$$

It is also well-known that $\langle f, f \rangle = 2^{-2\kappa} L(1, \pi, \mathrm{Ad})$ and $\langle g, g \rangle = 2^{-\kappa'-1} L(1, \tau, \mathrm{Ad})$. Proposition 2.1 then follows from Proposition 3.1 and Proposition 3.4.

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