1. Introduction

1.1. Main results. Let $E/F$ be a quadratic extension of local nonarchimedean fields and $\eta : F^\times/N E^\times \to \{\pm 1\}$ the quadratic character associated to this extension. Let $D$ be a quaternion algebra over $F$ (maybe split) with a fixed embedding $E \to D$. Let $G = \text{GL}_n(D)$ and $H = \text{GL}_n(E)$, both viewed as algebraic groups over $F$. Let $\pi$ be an irreducible admissible representation of $G$. We say that $\pi$ is $H$-distinguished if

\[ \text{Hom}_H(\pi, \mathbb{C}) \neq 0. \]

Let $\pi'$ be the Jacquet–Langlands transfer of $\pi$ to $\text{GL}_{2n}(F)$ and $\pi'_E$ be its base change to $\text{GL}_{2n}(E)$. Let $H' = \text{GL}_n(F) \times \text{GL}_n(F)$, embedded in $\text{GL}_{2n}(F)$ as diagonal blocks. We say that $\pi'$ is $H'$-distinguished (resp. $(H', \eta)$-distinguished) if

\[ \text{Hom}_{H'}(\pi', \mathbb{C}) \neq 0, \quad \text{resp. } \text{Hom}_{H'}(\pi' \otimes \eta, \mathbb{C}) \neq 0. \]

These Hom spaces are all at most one dimensional [JR96, Guo97]. Let $\epsilon(\pi'_E) = \epsilon(\frac{1}{2}, \pi'_E, \psi)$ where the epsilon factor is the standard one defined by Godement and Jacquet. It equals $\pm 1$ as $\pi'_E$ is self-dual.

The goal of this paper is to prove the following theorem. When $n = 1$, this is precisely the Saito–Tunnell theorem [Tun83]. As we are going to see below, our method is even new (though...
very simple) in this case. In general this confirms a large part of a conjecture of Prasad and Takloo-
Bighash [PTB11]. Strictly speaking, the theorem is conditional on the so-called fundamental lemma
for the full Hecke algebra proposed in [Guo96], see the remark below.

**Theorem 1.1.** Let \( \pi \) be an irreducible supercuspidal representation of \( G \). Then \( \pi \) is \( H \)-distinguished if and only if the following two conditions hold:

1. \( \pi' \) is \( H' \)-distinguished and \((H',\eta)\)-distinguished.
2. \( \epsilon(\pi'_E)\eta(-1)^n = \epsilon(D)^{\alpha} \), where \( \epsilon(D) = 1 \) (resp. \(-1\)) if \( D \) splits (resp. ramifies).

This theorem has the following corollary.

**Corollary 1.2.** Let \( G_1 = GL_{2n}(F) \) (resp. \( G_2 = GL_n(D) \) with \( D \) nonsplit). Let \( \pi_1 \) and \( \pi_2 \) be irre-
ducible supercuspidal representations of \( G_1 \) and \( G_2 \) respectively and are Jacquet–Langlands transfer
of each other. Then

1. Assume \( n \) is odd and \( \pi_1 \) is \( H' \)-distinguished and \((H',\eta)\)-distinguished. Then precisely one
   of \( \pi_i \) \( (i = 1, 2) \) is \( H \)-distinguished, i.e.
   \[
   \dim \text{Hom}_H(\pi_1, \mathbb{C}) + \dim \text{Hom}_H(\pi_2, \mathbb{C}) = 1.
   \]
2. If \( n \) is even, then \( \pi_1 \) is \( H \)-distinguished if and only if \( \pi_2 \) is \( H \)-distinguished, i.e.
   \[
   \dim \text{Hom}_H(\pi_1, \mathbb{C}) = \dim \text{Hom}_H(\pi_2, \mathbb{C}).
   \]

Several remarks are in order.

**Remark 1.3.** As we have said, the theorem is conditional on the so-called “fundamental lemma”
for the full Hecke algebra, proposed in [Guo96]. The main result of [Guo96] is to confirm this
fundamental lemma for the unit of the Hecke algebra. The statement of the fundamental lemma
will be recalled in Section 6. We need this as we make use of the relative trace formulae in some
precise form (i.e. not just for nonvanishing). It appears to the author that the “fundamental lemma
for units” alone is not sufficient for our purposes. However, with the current trace formula technique
at hand, the full fundamental lemma is within reach via global methods and it is believable that it
will be established in the near future (I will do it!). Such a condition should not be considered as
a serious obstacle.

**Remark 1.4.** Beuzart-Plessis kindly provides a proof of the existence of distinguished supercuspidal
representations in the appendix. The argument is for general symmetric spaces. In the particular
situation that we consider in this paper, one can also explicitly construct distinguished depth zero
supercuspidal representations via Deligne–Lusztig theory. Thus at least Theorem 1.1 is not empty.

We also prove a slight variant of the above result involving the case of central division algebras.
This setup is again covered by the conjecture of Prasad and Takloo-Bighash. Assume that \( A \) is
a central division algebra over \( F \) of dimension \( 4n^2 \) with an embedding \( E \to A \). Let \( B \) be the
centralizer of $E$ in $A$. Put $G = A^\times$ and $H = B^\times$, both viewed as algebraic groups over $F$. We say that an irreducible representation of $G$ (necessarily finite dimensional) is $H$-distinguished if $\text{Hom}_H(\pi, \mathbb{C}) \neq 0$. Let $\pi'$ be the Jacquet–Langlands transfer of $\pi$ to $\text{GL}_{2n}(F)$.

**Theorem 1.5.** We have that $\pi$ is $H$-distinguished if and only if the following two conditions hold:

1. $\pi'$ is $H'$-distinguished and $(H', \eta)$-distinguished.
2. $\epsilon(\pi'_E)\eta(-1) = -1$.

**Corollary 1.6.** Let $G_1 = \text{GL}_{2n}(F)$ (resp. $G_2 = A^\times$). Let $\pi_1$ and $\pi_2$ be irreducible supercuspidal representations of $G_1$ and $G_2$ respectively and are Jacquet–Langlands transfer of each other. Assume that $\pi_1$ is $H'$-distinguished and $(H', \eta)$-distinguished. Then precisely one of $\pi_i$ ($i = 1, 2$) is $H$-distinguished, i.e.

$$\dim \text{Hom}_H(\pi_1, \mathbb{C}) + \dim \text{Hom}_H(\pi_2, \mathbb{C}) = 1.$$ 

The same remarks to Theorem 1.1 also apply to these results. In particular they are conditional on the (same) “fundamental lemma” for the full Hecke algebra.

1.2. **Motivation.** To explain the motivation behind our work, let us switch to the global situation. Let $E/F$ be a quadratic extension of number fields. Let $D$ be a quaternion algebra (split or not) over $F$ with a fixed embedding $E \rightarrow D$. Let $G = D^\times$, $Z = F^\times$ be its center and let $H = E^\times$, all considered as algebraic groups over $F$. We then have an embedding $H \rightarrow G$. In 1980s, Waldspurger proved the following celebrated result. Let $\pi$ be an irreducible cuspidal automorphic representation of $G(\mathbb{A}_F)$ with trivial central character. Then the period integral

$$\int_{Z(\mathbb{A}_F)H(\mathbb{A}_F)\backslash H(\mathbb{A}_F)} \varphi(h)dh$$

is not identically zero if and only if $\text{Hom}_H(\mathbb{A}_F)(\pi, \mathbb{C}) \neq 0$ and $L(\frac{1}{2}, \pi'_E) \neq 0$ where $\pi'$ is the Jacquet–Langlands transfer of $\pi$ to $\text{GL}_{2n}(\mathbb{A}_F)$ and $\pi'_E$ base change to $\text{GL}_{2n}(\mathbb{A}_E)$. This result and its subsequent refinements played a definitive role in the study of the BSD conjecture for the past three decades.

There are many directions to generalize this result to groups of higher rank. One of the most successful generalizations is the Gan–Gross–Prasad conjecture. By incidental isogenies, the group $G$ is closely related to $U(2)$ and $H$ is closely related to $U(1)$ with a natural embedding $U(1) \rightarrow U(2)$. Thus the integral (1.1) is interpreted as integrating an automorphic form on $U(2)$ on $U(1)$. From this point of view, a natural generalization is to replace $U(2)$ by $U(n + 1)$ and $U(1)$ by $U(n)$, and then integrate an automorphic form on $U(n + 1)$ against another automorphic form on $U(n)$ along $U(n)$. This is a very active area of research and has witnessed tremendous advances in recent years, e.g. [BPa, Xue, Zha14a, Zha14b].

Based on his study of period integrals along the symmetric subgroups, especially the study of linear periods, Jacquet proposed a generalization of the period (1.1) in a different direction. Instead of using incidental isogenies to relate the integral (1.1) to period integrals on classical groups,
Jacquet stayed within the realm of general linear groups and the inner forms. More specifically, let \( n \) be a positive integer, and consider

\[
G = \text{GL}_n, \quad H = \text{Res}_{E/F} \text{GL}_n,
\]
as algebraic groups over \( F \), with an embedding \( H \to G \) induced by \( E \to D \). Let \( \pi \) be a cuspidal automorphic representation of \( G(\mathbb{A}_F) \). Put

\[
(1.2) \quad P_H(\varphi) = \int_{Z_H(\mathbb{A}_F)H(F)\backslash H(\mathbb{A}_F)} \varphi(h) dh, \quad \varphi \in \pi,
\]
where \( Z \simeq \mathbb{G}_{m,F} \) is the center of \( G \). Jacquet proposed the following conjectural generalization of the theorem of Waldspurger.

**Conjecture 1.7** ([Guo96]). Let \( \pi' \) be the Jacquet–Langlands transfer of \( \pi \) to \( \text{GL}_{2n}(\mathbb{A}_F) \), \( \pi'_E \) its base change to \( \text{GL}_{2n}(\mathbb{A}_E) \). Let \( L(s, \pi'_E) \) be the standard \( L \)-function of \( \pi'_E \). If the period \( P_H(\varphi) \neq 0 \) for some \( \varphi \in \pi \), then \( \text{Hom}_{H(\mathbb{A}_F)}(\pi, \mathbb{C}) \neq 0 \) and \( L(\frac{1}{2}, \pi'_E) \neq 0 \). If \( n \) is odd, then the converse also holds.

Our motivation to study this particular generalization of Waldspurger’s theorem, comes from the work of Yun and Zhang [YZ17] on the Taylor expansion of \( L \)-functions on \( \text{PGL}_2 \) over function fields. Briefly, this is the “right” setting to generalize their work on automorphic representation of \( \text{PGL}_2 \) to all automorphic representations of \( \text{GL}_{2n} \) of symplectic type. In particular, over function fields and when \( D \) splits, there are moduli stacks of \( G \)-Shtukas and \( H \)-Shtukas with multiple modifications. There are analogues of “Heegner–Drinfeld” cycles in this generality, whose self-intersection number should be related to the Taylor expansion of the \( L \)-function \( L(s, \pi'_E) \) at its center. Therefore the above conjecture is the “step zero” before taking up the task of studying intersection theory of “Heegner–Drinfeld” cycles on the moduli space of \( \text{GL}_n \)-Shtukas.

Guo [Guo96] made some preliminary studies of Conjecture 1.7. In particular, he proposed a relative trace formula approach, analyzed the relevant double coset decomposition and proposed the corresponding fundamental lemma. The “fundamental lemma for units” is proved in [Guo96]. We are going to discuss this in Section 6. In addition to [Guo96], there are quite a lot of papers in the literature concerning the conjecture, e.g. [FMW18, Zha15]. With an eye on the work of Yun and Zhang, our focus is, however, different from theirs. Our ultimate goal is to prove an Ichino–Ikeda type formula of the form

\[
(1.3) \quad |P_H(\varphi)|^2 = C \times L(\frac{1}{2}, \pi'_E) \times \prod_v \alpha_v(\varphi_v, \varphi_v),
\]

\( C \) is an explicit nonzero constant involving some non-central \( L \)-values, and \( \alpha_v \) is a certain local invariant linear form. There are, however, a bunch of local problems, representation theoretic in nature, that need to be addressed. The first major problem is the definition of the local linear forms \( \alpha_v \). If one tries to define \( \alpha_v \)’s by mimicking Ichino–Ikeda using integration of matrix coefficients, the integral only converges when \( \pi_v \) is a discrete series representation. Thus new ideas...
are needed. It turns out that the recipes from [SV17], which relates linear forms of this type to
the Plancherel formula for \( H(F_v) \backslash G(F_v) \), is a plausible approach. Nevertheless obtaining such an
explicit Plancherel formula is a hard question, and one could only hope to establish such a formula
by comparing it with something else more manageable. Following ideas of Jacquet, the thing we are
going to compare \( H(F_v) \backslash G(F_v) \) to is the symmetric space \( \text{GL}_n \times \text{GL}_n \backslash \text{GL}_{2n} \) over \( F_v \). Among such
comparison, a crucial part is the comparison between the subset of tempered spectrum of \( \text{GL}_{2n}(F_v) \)
consisting of representations both \( \text{GL}_n(F_v) \times \text{GL}_n(F_v) \)-distinguished and \( (\text{GL}_n(F_v) \times \text{GL}_n(F_v), \eta) \)-
distinguished, with the subset of the tempered spectrum of \( G(F_v) \) consisting of \( H(F_v) \)-distinguished
representations. Theorem 1.1 achieves a large part of this comparison. Although the results are all
local in nature, the global relative trace formulae in [Guo96] play some fundamental roles in the
proof.

1.3. Methods. We develop a new general method of relating invariant linear forms to local root
numbers. The method is largely in the spirit of harmonic analysis à la Harish-Chandra, e.g.
nilpotent orbital integrals, Fourier transforms, Shalika germs, local character expansion of spherical
characters, etc. All these are done for the particular symmetric space \( H \backslash G \) and its split analogue.
As Rader and Rallis have shown in [RR96], most of these results do not hold for general symmetric
spaces, even some very simples ones. Thus the symmetric spaces appearing in this paper are indeed
of a particularly nice kind.

Apart from the small rank cases, previous approaches to problems of this kind are either “global
to local”, e.g. the method of Prasad [Pra07], or via “twisted local trace formulae”, e.g. the work of
Beuzart-Plessis and Waldspurger on the strong form of the local Gan–Gross–Prasad conjectures.
Both methods are not the best one for our particular situation. A simple global-to-local argument,
gives only very limited partial results on one implication, c.f. [FMW18, Theorem 1.7]. It in general
requires that one knows \textit{a priori} the relation between invariant linear form and local root numbers
for a large class of representations. This is not the case in the present situation. The trace formula
technique does not seem adequate either. As Beuzart-Plessis has explained to the author, one can
prove Corollary 1.2 via the method of local trace formula. However to obtain a twisted version of
this local trace formula so as to connect to the local root numbers seems very challenging. Moreover
the method of local trace formula is much more technically demanding than our method.

Let us switch back to the setting of Theorem 1.1 and describe our approach in some detail. Let
\( \mathfrak{s} \) be the tangent space at the identity of \( H \backslash G \) and \( H \) acts on \( \mathfrak{s} \) naturally. Assume that \( \pi \) is an
\( H \)-distinguished supercuspidal representation. We consider the distribution

\[
J_{\pi}(f) = \int_{Z \backslash H} \text{Trace}(\pi(h)\pi(f))dh, \quad f \in C_c^{\infty}(G).
\]

This integral is absolutely convergent. If \( \pi \) is not a discrete series representation, then this integral
is divergent but we can define an abstract spherical character attach to \( \pi \) which we again denote
by $J_\pi$. It can be proved that $J_\pi$ has a local character expansion

$$J_\pi(f) = \sum_{\mathcal{O}} c_\mathcal{O} \hat{\mu}_\mathcal{O}(f_\sharp).$$

Here

- $f$ is a test function supported in a small neighbourhood of identity;
- $f_\sharp \in C_c^\infty(\mathfrak{s})$ is obtained from $f$ via an integral transform and Cayley transform;
- the sum runs over an all nilpotent orbits of $\mathfrak{s}$;
- $c_\mathcal{O}$ are constants depending on $\pi$ but not on $f$,
- $\hat{\mu}_\mathcal{O}$ is the Fourier transform of the nilpotent orbital integral $\mu_\mathcal{O}$.

A key but technical point is that if $\pi$ is a supercuspidal $H$-distinguished representation, then $c_0 \neq 0$, where 0 is the minimal nilpotent orbit in $\mathfrak{s}$, consisting only the origin.

This local character expansion alone is not sufficient to prove the epsilon dichotomy, and it needs to be compared with something else. Thanks to the work of Jacquet and many others, it is now clear what $J_\pi$ needs to be compared with. Let

$$G' = \text{GL}_{2n,F}, \quad H' = \text{GL}_{n,F} \times \text{GL}_{n,F}.$$

Let $s' = H' \setminus G'$ be the symmetric space, $s'$ be the tangent space at the identity and $H'$ acts on $s'$ naturally. An element in $s'$ can be written as

$$\sigma \begin{pmatrix} A \\ B \end{pmatrix} \sigma, \quad A, B \in M_n(F).$$

Let $N'$ be the standard upper triangular unipotent subgroup of $G'$ and $P'$ be the mirabolic subgroup of $G'$. Let $W$ be the Whittaker model of $\pi'$. If $W \in W$, we put

$$l(W) = \int_{H' \cap N' \backslash H' \cap P'} W(p) dp, \quad l_\eta(W) = \int_{H' \cap N' \backslash H' \cap P'} W(p) \eta(\det p) dp.$$

Define the following distribution

$$I_{\pi'}(f') = \sum_W l(\pi'(f')W) \overline{l_\eta(W)}, \quad f' \in C_c^\infty(G'(F)).$$

where the sum runs over an orthonormal basis of $W$. It can be proved that $I_{\pi'}$ has a local character expansion

$$I_{\pi'}(f') = \sum_{\mathcal{O}} c'_\mathcal{O} \hat{\mu}'_\mathcal{O}(f'_\sharp).$$

Here

- $f'$ is a test function supported in a small neighbourhood of identity;
- $f'_\sharp \in C_c^\infty(s')$ is obtained from $f'$ via an integral transform and Cayley transform;
the sum runs over an all visible nilpotent orbits of \( \mathfrak{s}' \), where visible means that it supports an \((H',\eta)\)-invariant distribution;

- \( c'_O \) are constants depending on \( \pi' \) but not on \( f' \),

- \( \hat{\mu}'_O \) is the Fourier transform of the nilpotent orbital integral \( \mu'_O \).

There are two minimal visible nilpotent orbits, which are represented by

\[
\sigma \begin{pmatrix} 1_n \\ 0 \end{pmatrix} \sigma, \quad \sigma \begin{pmatrix} 0 \\ 1_n \end{pmatrix} \sigma,
\]

which we denote by \( \mathcal{O}^+_\text{min} \) and \( \mathcal{O}^-\text{min} \) respectively. The key observation is the following. There is an involution on \( \mathfrak{s}' \), given by conjugation of the longest elements \( w \) of \( G' \) that interchanges \( \mathcal{O}^+_\text{min} \) and \( \mathcal{O}^-\text{min} \). For any \( f' \in G' \), we may conjugate \( f' \) by \( w \), to get a new function \( t' \). A result of Lapid and Mao [LM17, Theorem 3.2] states that

\[
l(\pi'(w)W) = \epsilon(\pi')l(W), \quad l_\eta(\pi'(w)W) = \epsilon(\pi' \otimes \eta)\eta(-1)^n l_\eta(W),
\]

for all \( W \in \mathcal{W} \). Thus we have

\[
I_{\pi'}(t'f') = \epsilon(\pi'E)\eta(-1)^n I_{\pi'}(f').
\]

Combining this with the local character expansion, we arrive at the conclusion

\[
(1.4) \quad c'_O \mathcal{O}^+_\text{min} = \epsilon(\pi'E)\eta(-1)^n c'_O \mathcal{O}^-\text{min}.
\]

We can now compare \( I_{\pi'} \) and \( J_\pi \). As usual, there are notions of matching of regular semisimple orbits and matching of test functions. We are going to discuss them in Section 6. Via the global relative trace formula, we prove that there is a \textit{nonzero} constant \( \kappa \) so that for all matching \( f' \in C^\infty_c(G'(F)) \) and \( f \in C^\infty_c(G(F)) \) we have the following spherical character identity

\[
I_{\pi'}(f') = \kappa J_\pi(f).
\]

We do not try to compute the exact value of \( \kappa \) in this paper, though this is certainly possible with some additional work. This is the only place where we need the full fundamental lemma. Using this identity, the local character expansion and the homogeneity properties of the nilpotent orbital integrals, we conclude that there is a \textit{nonzero} constant \( C \) so that

\[
(1.5) \quad \epsilon(\pi'E)\eta(-1)^n \mu^+_{\mathcal{O}^+\text{min}}(f'_z) + \mu^-_{\mathcal{O}^-\text{min}}(f'_z) = C \mu_0(f_z).
\]

From here the epsilon dichotomy follows easily. It is clear that \( \mu_0 \) is a nonzero constant function. A little computation also shows that, up to some nonzero constant multiple, \( \mu^\pm_{\mathcal{O}^\pm\text{min}} \) are represented by locally integrable functions on \( \mathfrak{s}' \) where on the regular semisimple locus they are given by

\[
(1.6) \quad \mu^+_{\mathcal{O}^+\text{min}} \left( \sigma \begin{pmatrix} Y \\ X \end{pmatrix} \sigma \right) = \eta(\det X), \quad \mu^-_{\mathcal{O}^-\text{min}} \left( \sigma \begin{pmatrix} Y \\ X \end{pmatrix} \sigma \right) = \eta(\det Y).
\]
A basic observation is that \( \sigma \begin{pmatrix} X \\ Y \end{pmatrix} \sigma \) matches some elements in \( s \) if and only if \( \eta(\det XY) = \epsilon(D)^n \). It then follows that

\[
\epsilon(\pi'_E)\eta(-1)^n = \epsilon(D)^n,
\]

for otherwise the left hand side of (1.5) would be identically zero while the right hand side is not. This proves that if \( \pi \) is \( H \)-distinguished, then \( \epsilon(\pi'_E)\eta(-1)^n = \epsilon(D)^n \).

To obtain the implication in the converse direction, we argue as follows. Assume that \( \pi' \) is a supercuspidal representation of \( G' \) and is both \( H' \)-distinguished and \( (H', \eta) \)-distinguished. We need to prove that if \( G = \text{GL}_n(D) \) with \( \epsilon(D)^n = \epsilon(\pi'_E)\eta(-1)^n \) and \( \pi \) is the Jacquet–Langlands transfer of \( \pi' \), then \( \pi \) is \( H \)-distinguished. We first show, by a technical computation, that there is a nonzero constant \( C' \) such that if the test function \( f' \in C^\infty_c(G'(F)) \) is essentially a matrix coefficient of \( \pi' \), i.e. \( g \mapsto \int_{Z(G(F))} f'(zg)dz \) is a matrix coefficient of \( \pi' \), then

\[
\mu'_{O_{\min}^+} (f'_z) = C'I_{\pi'}(f').
\]

Again the result of Lapid and Mao [LM17, Theorem 3.2] implies that

\[
\mu'_{O_{\min}^+} (f'_z) + \epsilon(\pi'_E)\eta(-1)^n \mu'_{O_{\min}^-} (f'_z) = 2C'I_{\pi'}(f').
\]

Therefore there is an essential matrix coefficient \( f' \) so that

(1.7) \[
\mu'_{O_{\min}^+} (f'_z) + \epsilon(\pi'_E)\eta(-1)^n \mu'_{O_{\min}^-} (f'_z) \neq 0.
\]

By parabolic descent, we can prove that \( O(\gamma, \eta, f') = 0 \) if \( \gamma \) is regular semisimple but not elliptic. We can also prove that \( O(\gamma, \eta, f') = 0 \) if \( \gamma \) transfers to an element \( \delta \in G(F) \) with \( \epsilon(D)^n \neq \epsilon(\pi'_E)\eta(-1)^n \). Indeed consider the function on the elliptic locus of \( G'(F) \) given by

\[
\gamma \mapsto \Omega(\gamma)O(\gamma, \eta, f'),
\]

where \( \Omega(\gamma) \) is the transfer factor. By its very definition, this function is bi-\( H' \)-invariant. We now consider

\[
\gamma \mapsto \Omega(\gamma)O(\gamma, \eta, t f').
\]

On the one hand, we have

\[
\Omega(\gamma)O(\gamma, \eta, t f') = \Omega(\gamma)O(w\gamma w^{-1}, \eta, f') = \Omega(w^{-1}\gamma w)O(\gamma, \eta, f'),
\]

since \( w\gamma w^{-1} \) is in the same \( H' \times H' \) double coset as \( \gamma \). On the other hand we have

(1.8) \[
\Omega(\gamma)O(\gamma, \eta, t f') = \epsilon(\pi'_E)\eta(-1)^n \Omega(\gamma)O(\gamma, \eta, f').
\]

This can be seen as follows. Suppose that

\[
\int_{Z_G(F)} f'(zg)dz = \langle \pi'(g)W_1, W_2 \rangle
\]
where $W_1, W_2$ are in the Whittaker model $W$ of $\pi$. Then by the uniqueness of linear periods [JR96], we can find a constant $A$ (could be zero), depending on $\gamma$ and $\pi'$ but not on $W_1$ and $W_2$ so that

$$\Omega(\gamma)O(\gamma, \eta, f') = Al(W_1, f_\eta(W_2)).$$

Using the result of Lapid and Mao again, we get (1.8). We thus conclude that

$$\Omega(w^{-1}\gamma w) = \Omega(\gamma)\epsilon(\pi'_E)\eta(-1)^n,$$

if $O(\gamma, \eta, f') \neq 0$. By the property of the transfer factor $\Omega$, we conclude that $\gamma$ matches an elliptic element $\delta \in G$, where $G = \text{GL}_n(D)$ with $\epsilon(D)^n = \epsilon(\pi'_E)\eta(-1)^n$.

Thus let $f \in C_c^\infty(G(F))$ be the transfer of $f'$. By (1.6) and the fact that Fourier transform commutes with smooth transfer, there is a nonzero constant $B$ so that

$$\epsilon(D)^n \mu'_{O_{\min}^+}(f'_z) + \mu'_{O_{\min}^-}(f'_z) = B f(1).$$

We thus conclude that $f(1) \neq 0$ from (1.7) and the fact that $\epsilon(D)^n = \epsilon(\pi'_E)\eta(-1)^n$. Note that this is the geometric counterpart of the identity (1.5).

Using global methods, we can find an $H$-distinguished $\pi$ so that $J_\pi(f) \neq 0$ and again a spherical character identity

$$I_{JL(\pi)}(f') = \kappa J_\pi(f),$$

where $JL(\pi)$ stands for the Jacquet–Langlands transfer of $\pi$ to $G'(F)$ and $\kappa \neq 0$. We conclude that $I_{JL(\pi)}(f') \neq 0$. However by our very choice, $f'$ is essentially a matrix coefficient of $\pi'$. Thus we conclude that $\pi' = JL(\pi)$. This proves the implication in the other direction.

Let me finally mention that at this very last step, namely finding $\pi$ with $J_\pi(f) \neq 0$, we need to make use of a simple global relative trace formula so that the group that we integrate over is anisotropic. This forces us to consider an additional case where $G = A^\times$ where $A$ is a central division algebra instead of $G = \text{GL}_n(D)$. A byproduct of this consideration is Theorem 1.5 which takes nevertheless a better shape in dichotomy.

As we can seen from the above discussion, the crux of the matter is that there is an involution on the space of test functions on $G'(F)$. Applying this involution to the spherical characters, we obtain an identity involving $\epsilon(\pi'_E)$ and Fourier transform of minimal nilpotent orbital integrals. Applying this involution to the orbital integrals, we obtain an identity involving $\epsilon(D)$ and minimal nilpotent orbital integrals. These two identities are further connected by the fact that “Fourier transform commutes with smooth transfer”. We thus obtain an equality between $\epsilon(\pi'_E)$ and $\epsilon(D)$.

1.4. Organization of the paper. The paper is organized as follows. The second section is on the orbital integrals on $H \backslash G$. We develop the theory of Shalika germs for these orbital integrals and prove that Shalika germs are linearly independent. The third section is on the expansion of the spherical character $J_\pi$. Using linear independence of Shalika germs, we show that the coefficient attached to the minimal nilpotent orbit is not zero. The fourth section is devoted to the orbital integrals on $H' \backslash G'$ and the key point is to define nilpotent orbital integrals which turns out to be
elementary but very messy. The fifth section is on the local expansion of the spherical character $I_{\pi'}$ and we prove that local root number equals the ratio of suitable coefficients in the expansion. We also prove the nonvanishing result (1.7). Up to this point all arguments are local. The sixth section is devoted to the proof of Theorem 1.1. We first recall and slightly extend the global relative trace formula of Guo and use it to prove the identity between $J_{\pi}$ and $I_{\pi'}$. Then combining the results on the local character expansions we prove our main theorem follow the outline described above. The appendix by Beuzart-Plessis proves the existence of distinguished supercuspidal representations.

1.5. **Notation and conventions.** Throughout this paper, we make use of the following notation.

- Depending on the context, $E/F$ will always be a quadratic extension of global fields or local fields. The nontrivial Galois involution is denoted by $x \mapsto \overline{x}$. When in the local situation, we sometimes allow $E$ to be $F \times F$. We let $\eta$ be the quadratic character of $F^\times \backslash \mathbb{A}_F^\times$ or $F^\times$ attached to $E/F$ depending on $E/F$ being global or local.
- We fix a nontrivial additive character $\psi$ be nontrivial of $F \backslash \mathbb{A}_F$ or $F$ depending on $F$ being global or local. If $a \in F^\times$, we put $\psi_a(x) = \psi(ax)$.
- We write $F^n$ (resp. $F^m$) for the space of row (resp. column) vectors. We write $M_{m,n}$ for the space of $m \times n$ matrices and $M_n = M_{n,n}$.
- We denote by $1_r$ the $r \times r$ identity matrix. We denote by $0_{r,s}$ the $r \times s$ zero matrix. We also put $0_r = 0_{r,r}$. We simply write 0 if the dimension is clear.
- Let $f$ be a function on a vector space $V$ over $F$. We put $f_t(x) = f(t^{-1}x)$ for all $t \in F^\times$ and $x \in V$.
- Let $X$ be a set on which a group $G$ acts. Let $x \in X$. The stabilizer of $x$ in $G$ is denoted by $G_x$. We write $gx$ or $g.x$ or $x^g$ for the action of $g \in G$ on $x \in X$. If $U \subset X$, we denote by $U^G$ the set of elements $\{x^g \mid x \in U, g \in G\}$.
- We use capital letters to denote groups and symmetric spaces. We use Gothic letters to denote the corresponding Lie algebras. Thus if $G$ is a reductive group then $\mathfrak{g}$ stands for the Lie algebra of $G$, unless we explicitly say the contrary.
- Let $G$ be a reductive group. The notation $P = MN$ always means that $P$ is a parabolic subgroup, $M$ the Levi subgroup and $N$ the unipotent subgroup.
- When we say $G$ is an algebraic group over $F$, we mean that there is an algebraic group scheme over $F$ such that $G$ is the group of $F$-points. We sometime also write $G(F)$, $G(\mathbb{A}_F)$ etc. to avoid confusion.
- All integrals in this paper depend on the choice of measures. For most of the part, only the nonvanishing properties are concerned and hence the choice of the measures is not crucial. Thus we assume that we have prefixed some choice of the measures on each space we are integrating over. For some intermediate results we are going to specify the measure when needed. For integrations on the affine space, we are always going to use the self-dual measure.
1.6. Acknowledgement. I am grateful to W. Zhang for bringing my attention to the conjectures on linear periods. I would like to thank Beuzart-Plessis for many helpful discussions and writing an appendix for this paper. I thank Q. Li and M. Suzuki for some inspiring discussions. Part of this paper was written during my visit to the Morningside Center of Mathematics (MCM) in Beijing in the summer of 2018. I would like to thank MCM and in particular to my host Y. Tian for their hospitality and the excellent research environment. Finally I would like to thank S. Zhang for his interest in this project and the constant support. This work is partially supported by the Simons Collaboration Grant.

2. Orbital integrals: nonsplit case

We assume from now on till the Section 5 that we are in the local situation, i.e. $E/F$ is a quadratic extension of nonarchimedean local fields.

2.1. The symmetric space. Let $A$ be a central simple algebra (CSA) over $F$ of dimension $4n^2$ with a fixed embedding $E \to A$. Let $B$ be the centralizer of $E$ in $A$, which itself is a CSA over $E$. We fix a $\tau \in E^\times \setminus E^{\times, 2}$ so that $E = F[\sqrt{\tau}]$. Then conjugating by $\sqrt{\tau}$ is an involution on $A$ whose set of fixed points is $B$. We again denote this involution by $\theta$. We let $G = A^\times$ and $H = B^\times$, both viewed as algebraic groups over $F$. Let

$$S = H \setminus G = \{g^{-1}\theta(g) \mid g \in G\}, \quad \rho: G \to S, \; g \mapsto g^{-1}\theta(g),$$

as usual. We also let $s$ the $(-1)$-eigenspace of $\theta$, which is isomorphic to the tangent space of $S$ at 1. The group $H$ acts on $S$ and $s$ by conjugation. We say that an element $s \in S$ is $\theta$-semisimple (resp. $\theta$-regular, resp. $\theta$-elliptic) if it is semisimple (resp. regular, resp. elliptic) in $G$ in the usual sense. We say that an element $g \in G$ is $\theta$-semisimple (resp. $\theta$-regular, resp. $\theta$-elliptic) if its image in $S$ under $\rho$ is $\theta$-semisimple (resp. $\theta$-regular, resp. $\theta$-elliptic). We denote by $G_{\theta-ss}$, $G_{\theta-reg}$, $G_{\theta-ell}$ the locus of $\theta$-semisimple, $\theta$-regular, $\theta$-elliptic elements in $G$ respectively. Similarly we have $S_{\theta-ss}$, $S_{\theta-reg}$, $S_{\theta-ell}$. We say that an element $\xi \in s$ is $\theta$-semisimple if $\xi^2 \in B$ is semisimple in the usual sense. The (open) subset of $s$ consisting of semisimple elements are denoted by $s_{\theta-ss}$. Similarly we define $\theta$-regular and $\theta$-elliptic elements in $s$ and the corresponding loci in $s$ are denoted by $s_{\theta-reg}$ and $s_{\theta-ell}$ respectively. An element in $S$ (resp. $s$) is called $\theta$-unipotent (resp. $\theta$-unipotent) if it is unipotent in $G$ (resp. nilpotent in $g$) in the usual sense. We will denote by $N \subset s$ the set of $\theta$-nilpotent element in $s$ and call it the nilpotent cone. We sometimes even drop the prefix $\theta$ when there is no confusion.

Throughout this paper, we impose the following assumptions throughout.

Assumption 2.1. The central simple algebra $A$ is isomorphic to either $M_n(D)$ where $D$ is a quaternion algebra over $F$ (split or not) containing $E$ or a central division algebra over $F$. 
With this assumption $B$ is isomorphic to either $M_n(E)$ or a central division algebra over $E$ of dimension $n^2$. Our main results are all about the case where $A \simeq M_n(D)$. But we need the case of $A$ being a central division algebra at some point for technical reasons.

Let us consider the case $A = M_n(D)$ where things are more explicit. Let us put $\epsilon(D) = 1$ (resp. $-1$) if $D$ splits (resp. ramifies). In this case $G = \text{GL}_n(D)$ and $H = \text{GL}_n(E)$. We can realize elements in $G$ as matrices $\begin{pmatrix} X & \epsilon Y \\ Y & X \end{pmatrix}$, where $X, Y \in M_n(E)$, $\epsilon \in F^\times$ and equals in $\epsilon(D)$ in $F^\times / N_{E^\times}$. The involution $\theta$ is given by conjugating $\begin{pmatrix} 1_n \\ -1_n \end{pmatrix}$. An element $s = \begin{pmatrix} X & \epsilon Y \\ Y & X \end{pmatrix} \in G$ can be written as $s = s^+ + s^-$ where $s^+ = \begin{pmatrix} X \\ X \end{pmatrix} \in B$ and $s^- = \begin{pmatrix} Y & \epsilon Y \\ X & \epsilon X \end{pmatrix} \in \mathfrak{s}$. It is not hard to check that $s \in S$ if and only if $(s^+)^2 - 1 = (s^-)^2$ and $s^+ s^- = s^- s^+$, which is equivalent to $X^2 + 1 = \epsilon Y Y$ and $XY = Y X$. The elements in $\mathfrak{s}$ are of the form $\begin{pmatrix} \epsilon X \\ X \end{pmatrix}$ where $X \in M_{n,E}$.

The obvious morphism sending $\begin{pmatrix} \epsilon X \\ X \end{pmatrix}$ to $X$ identifies $\mathfrak{s}$ with $M_{n,E}$ and this isomorphism is $H$-equivariant if $H$ acts on $M_{n,E}$ via twisted conjugation $\text{Ad}_\theta(h)X = h^{-1}X h$.

The case $A$ being a central division algebra is less explicit. But still we have the decomposition of any element $g \in G$ as $g = g^+ + g^-$ where $g^+ \in B$ and $g^- \in \mathfrak{s}$ and $s \in S$ if and only if $(s^+)^2 - 1 = (s^-)^2$ and $s^+ s^- = s^- s^+$.

Let us analyze the $H$-orbits in $S$ and $\mathfrak{s}$. In the case $A = M_n(D)$, this is done in [Guo96]. We will summarise the results from [Guo96] and supplement the discussion when $A$ is a central division algebra.

**Lemma 2.2.** Suppose that $U \subset \mathfrak{s}$ is a neighbourhood of $0 \in \mathfrak{s}$. Then $U^H$ contains an $H$-invariant neighbourhood of the nilpotent cone $N$.

**Proof.** If $A$ is a central division algebra this is clear. If $A \simeq M_n(D)$, this is the twisted analogue of its counterpart for usual conjugation actions. The proofs are identical, see [Kot05, Lemma 15.3] for a proof for the usual conjugation action. \( \Box \)

**Lemma 2.3.** Two elements $\xi_1$ and $\xi_2$ in $\mathfrak{s}$ are conjugate by an element in $H$ if and only if $\xi_1^2$ and $\xi_2^2$ are conjugate by an element in $H$.

**Proof.** If $A \simeq M_n(D)$, then this is proved in [AC89, Lemma 1.3]. So we will be treating only the case where $A$ is a central simple algebra.

The “only if” direction is clear. Let us show the “if” direction. The case $n = 1$ is again covered by [AC89, Lemma 1.3]. We are going to reduce the general case to the case $n = 1$. By replacing $\xi_1$ by its $H$-conjugate, we may assume that we in fact have $\xi_1^2 = \xi_2^2 \in B$. Of course we may assume that $\xi_1$ and $\xi_2$ are both nonzero, thus are both invertible.
First we claim that the reduced characteristic polynomial of $\xi_1^2$ have coefficients in $F$, i.e. $\nu_B(\lambda - \xi_1^2) \in F[\lambda]$. We first note that $\xi_1$ (and hence $\xi_2$) must be invertible as they are both $\theta$-elliptic. We also note that $\xi_1$ and elements in $F$ commute and $\xi_1 \sqrt{\tau} = -\sqrt{\tau} \xi_1$. Thus conjugation by $\xi_1$ is an extension of the Galois conjugate of $E/F$. Therefore

$$\nu_B(\lambda - \xi_1^2) = \nu_B(\lambda - \xi_1^{-1} \xi_1^2 \xi_1) = \nu_B(\lambda - \xi_1).$$

Thus $\nu_B(\lambda - \xi_1^2) \in F[\lambda]$.

Put $L = F[\lambda]/(\nu_B(\lambda - \xi_1^2))$. This is a degree $n$ field extension of $F$. We embed $L$ in $A$ by sending $\lambda$ to $\xi_1^2$. Let $D$ be the centralizer of $L$ in $A$. Then $D$ is a quaternion algebra over $L$. Since $\theta(L) = L$, we have $\theta(D) = D$. We let $D^\circ = \mathfrak{s} \cap D$. Then $\xi_1, \xi_2 \in D^\circ$. We note that $L \cap E = F$ as elements in $E$ but not in $F$ do not commute with $\xi_1$ while elements in $L$ commute with $\xi_1$. But $E \subset D$ and hence $K = L \otimes E = E[\lambda]/(\nu_B(\lambda - \xi_1^2))$ is a quadratic field extension of $L$ and $D \cap B = K$. We are thus reduced to the $n = 1$ case. This proves the lemma. \hfill \Box

**Lemma 2.4.** Let $r$ be the split rank of $G$. If $A = M_n(D)$, then $r = n$ (resp. $2n$) if $D$ ramifies (resp. splits). If $A$ is a central division algebra, then $r = 1$. We have the following assertions.

1. Let $\xi \in \mathfrak{s}_{\theta,{\text{ell}}}$. Then the reduced characteristic polynomial of $\xi^2 \in B$ is an irreducible polynomial in $F[\lambda]$ and $\eta_{E/F}(\nu_B(\xi^2)) = (-1)^r$.

2. Conversely let $f(\lambda) \in F[\lambda]$ be an irreducible polynomial and $\eta(f(0)) = (-1)^r$. Then there is an $\xi \in \mathfrak{s}_{\theta,{\text{ell}}}$ so that $\nu_B(\lambda - \xi^2) = f(\lambda)$.

**Proof.** The case $A \simeq M_n(D)$ is covered by [Guo96, Lemma 1.8]. We are going to treat only the case $A$ being a central division algebra. In this case $r = 1$ and $\xi \in \mathfrak{s}$ is $\theta$-elliptic if $\xi \neq 0$. The case $n = 1$ this is covered by [Guo96, Lemma 1.8].

For the first assertions, as we have seen from the proof of the preceding lemma, $L = F[\lambda]/(\nu_B(\lambda - \xi^2))$ is a degree $n$ field extension of $F$. Thus the reduced characteristic polynomial of $\xi^2$ is irreducible. Again let $D$ be the centralizer of $L$ in $A$, then $D$ is a quaternion division algebra over $L$ and $\xi \in D \cap \mathfrak{s}$. It then follows from the $n = 1$ case that $\eta_{E/F}(\nu_B(\xi^2)) = \eta_{K/L}(\xi^2) = -1$.

Let us prove the second assertion. First note that $f$ is irreducible in $E[\lambda]$. Otherwise assume $f(\lambda) = p_1(\lambda) \cdots p_s(\lambda)$ with $p_1, \cdots, p_s$ irreducible polynomials in $E[\lambda]$. Then none of these $p_i$'s are in $F[\lambda]$ since $f$ is irreducible over $F$. Since $f$ has coefficient in $F$, by taking the nontrivial Galois conjugation, we see that $s = 2$ and $f(\lambda) = p_1(\lambda)\overline{p_1(\lambda)}$. Letting $\lambda = 0$ in this factorization we see that $f(0)$ is a norm from $E^\times$.

Let $L = F[\lambda]/(f(\lambda))$ and $K = L \otimes E$. Then $K$ is a field of degree $n$ over $E$. Therefore there is an embedding $K \to B$. We let $u$ be the image of $\lambda$, then $\nu_B(\lambda - u) = f(\lambda)$.

We need to find an $\xi \in \mathfrak{s}$ so that $u = \xi^2$. For this let again that $D$ be the centralizer of $L$ in $A$ then $D$ is quaternion division algebra over $L$ which is stable under $\theta$ and $D \cap \mathfrak{s}$ is the $-1$-eigenspace $\theta$ in $D$. We have $\eta_{K/L}(u) = \eta_{E/F}(\nu_B(u)) = -1$. It then follows from $n = 1$ case that there is an $\xi \in D \cap \mathfrak{s}$ so that $u = \xi^2$. \hfill \Box
Lemma 2.5. Let $g \in G$ be $\theta$-regular. Then there is an $\xi \in \mathfrak{s}_{\theta-\text{reg}}$ with $\nu_B(\xi^2 - 1) \neq 0$ such that $1 + \xi \in HgH$. Moreover $1 + \xi$ and $1 + \zeta$ where $\xi, \zeta \in \mathfrak{s}_{\theta-\text{reg}}$ are in the same $H \times H$ double coset if and only if $\xi^2, \zeta^2 \in B$ are conjugate by $H$.

Proof. Again the case $A \simeq M_n(D)$ has been covered by [Guo96, Lemma 1.7]. We provide a proof when $A$ is a central division algebra.

Let $s = g^{-1}\theta(g) = s^+ + s^- \in S$ where $s^+ \in B$ and $s^- \in \mathfrak{s}$. Put $X = -(1 + s^+)^{-1}s^- \in \mathfrak{s}$. As $s \in S$ we have $(s^+)^2 - 1 = (s^-)^2$ and $s^+s^- = s^-s^+$. Then if $s^+ = \pm 1$, then $s^- = 0$ and $g$ is not $\theta$-regular. Therefore $s^+ \neq \pm 1$. We put $\xi = (1 + s^+)^{-1}s^-$, a little computation gives

$$\xi^2 = (1 + s^+)^{-2}(s^-)^2 = (1 + s^+)^{-1}(s^+ - 1), \quad \xi^2 - 1 = -2(s^+ + 1)^{-1}.$$ 

Therefore $\nu_B((s^+)^2 - 1) \neq 0$. Simple computation also gives $s = (1 + \xi)^{-1}(1 - \xi)$. Therefore $g$ and $1 + \xi$ are in the same double coset. If $1 + \xi$ and $1 + \zeta$ are in the same double coset, then $\xi$ is conjugate to $\zeta$ by $H$, which is equivalent to $\xi^2$ is conjugate to $\zeta^2$ by $H$ by Lemma 2.3.  

2.2. Orbital integrals. Let $f \in C^\infty_c(G)$ and $g \in G$ be a $\theta$-semisimple element. Define the orbital integral

$$O(g, f) = \int_{(H \times H) \setminus H \times H} f(h_1^{-1}gh_2)dh_1dh_2.$$ 

This integral is absolutely convergent. We have a Lie algebra version of the $\theta$-semisimple orbital integrals. Let $f \in C^\infty_c(\mathfrak{s})$ and $\xi \in \mathfrak{s}$ be a $\theta$-semisimple element. Define

$$O(\xi, f) = \int_{H \xi \setminus H} f(\xi h)dh.$$ 

Let $\omega$ be an $H$-invariant neighbourhood of $0 \in \mathfrak{s}$ consisting of elements $\xi$ so that $1 + \xi$ is invertible. We have the usual Cayley map: $c : \omega \to S$ defined by

$$c(\xi) = (1 + \xi)^{-1}(1 - \xi) \in S.$$ 

If $f \in C^\infty_c(G)$, we define a function $f_\xi \in C^\infty_c(\omega)$ as follows. Let $\xi \in \omega$, we put

$$f_\xi(\xi) = \int_H f(h(1 + \xi))dh.$$ 

The function $f_\xi$ naturally extends to a function on $\mathfrak{s}$ which we again denote by $f_\xi$. If $g \in G$ and $\xi \in \mathfrak{s}$ and $\rho(g) = c(\xi)$, then it follows from the definition that

$$O(g, f) = O(\xi, f_\xi).$$ 

We now consider the nilpotent orbital integrals. First if $A$ is a central division algebra, then $\mathcal{N} = \{0\}$ is a single point and the nilpotent orbital integral is nothing but evaluation at $0$. We put $d_\mathcal{O} = d_0 = 0$ and $\mu_\mathcal{O}(f) = \mu_0(f) = f(0)$. Assume now that $A = M_n(D)$. Let $\mathcal{O} \subset \mathcal{N}$ be a nilpotent orbit. It follows from [Guo97, Lemma 2.3] that we can find an element $\xi = \left(\frac{\epsilon^{X}}{X}\right)$ with the
property that $X \in M_n(E)$ is nilpotent and in its Jordan canonical form. Of course $X$ acts on $E^n$ and we have the following filtration

$$0 = V_0 \subset V_1 \subset V_2 \subset \cdots \subset V_s = E^n,$$

where $V_i = \ker X^i$. We let $P^+ = M^+N^+$ be the parabolic subgroup of $H$ that stabilizes this filtration. Let $\mathfrak{p}^+ = \mathfrak{m}^+ + \mathfrak{n}^+$ be its Lie algebra. Put $d_O = \dim N^+$. Let us identify $\mathfrak{s}$ with $M_n(E)$ which is also the Lie algebra of $H$. The space $\mathfrak{n}^+$ then gives rise to a subspace of $\mathfrak{s}$ which we also denote by $\mathfrak{n}^+$. Guo [Guo98] showed that the $P^+$ orbit of $X$ is a Zariski open subset of a subspace $\mathfrak{n}^+$ in $\mathfrak{s}$. Choose a maximal compact subgroup $K$ of $H$ with $H = P^+K$. Let $f \in C^\infty_c(\mathfrak{s})$. We define

$$f_K(\xi) = \int_K f(k\xi k^{-1})dk, \quad \mu_O(f) = \int_{\mathfrak{n}^+} f_K(n)dn.$$

Guo [Guo98, Proposition 5.1] showed that these defining integrals are absolutely convergent and $f \mapsto \mu_O(f)$ is an $H$-invariant distribution on $\mathfrak{s}$ which is supported on $O$. It moreover follows from the Lebesgue dominated convergence theorem that the integral

$$\int_{H_c \backslash H} f(\xi^h)dh,$$

is convergent and equals $\mu_O(f)$. We call it the nilpotent orbital integral on $O$. To unify notation, we sometimes also denote this orbital integral $\mu_O(f)$ by $O(\xi, f)$.

**Lemma 2.6.** For any $t \in F^\times$ and $O \subset N$, we have

$$\mu_O(f_t) = |t|^{d_O} \mu_O(f).$$

**Proof.** This follows from directly from the definition of $\mu_O$. 

If $A$ is a central division algebra, then any $\xi \in \mathfrak{s}$ is $\theta$-semisimple so we have defined orbital integrals for any $\xi \in \mathfrak{s}$. Assume $A \simeq M_n(D)$. We now define the orbital integral for an arbitrary element $\xi \in \mathfrak{s}$. Let $\xi = \xi_s + \xi_n$ be the Jordan decomposition in $\mathfrak{g}$. The uniqueness of Jordan decomposition ensures that both $\xi_s$ and $\xi_n$ are in $\mathfrak{s}$. Let $H_{\xi_s}$ be the stabilizer of $\xi_s$ in $H$, which is reductive [AC89, Proof of Lemma 1.1]. Let $H_{\xi_s, \xi_n}$ be the stabilizer of $\xi_n$ in $H_{\xi_s}$. Let $f \in C^\infty_c(\mathfrak{s})$. For $y \in H$, put

$$f_1(y) = \int_{H_{\xi_s, \xi_n} \backslash H_{\xi_s}} f((\xi_s + \xi_n^h)y)dh.$$

This integral is convergent. We claim that as a function in $y \in H$, it is compactly supported on $H_{\xi_s} \backslash H$. Indeed, if for some $y \in H$, $f_1(y) \neq 0$. Then $\xi_s^y + \xi_n^{hy} \in \text{supp} f$ which is a compact set. Note that $\xi_s^y$ is semisimple in $\mathfrak{g}$ and $\xi_n^{hy}$ is nilpotent in $\mathfrak{g}$. So $\xi_s^y$ is the semisimple part of $\xi_s^y + \xi_n^{hy}$ and it thus lies in some compact subset $C$ of $\mathfrak{s}$. As the orbit of $\xi_s$ is closed, it follows that $y$ lies in some
compact subset of $H_\xi \setminus H$. This proves the claim. Note that $H_\xi = H_{\xi s, \xi n}$. Therefore
\[
\int_{H_\xi \setminus H} f_1(y) dy = \int_{H_\xi \setminus H} \int_{H_{\xi s, \xi n} \setminus H_{\xi s}} f((\xi_s + \xi_n^h)y) dh
\]
\[
= \int_{H_\xi \setminus H} f(\xi^h) dh,
\]
where all integrals are convergent. This defines the orbital integral $O(\xi, f)$ for any $\xi \in s$ and all $f \in C^\infty_c(s)$.

We now consider the Fourier transform of orbital integrals on $s$. Let us fix an $H$-invariant inner product on $s$, e.g.
\[
\langle \xi, \zeta \rangle = \text{Tr}_A \xi \zeta,
\]
where $\text{Tr}_A$ stands for the reduced trace on $A$. We have the Fourier transform of a function $f \in C^\infty_c(s)$, which we denote by $\hat{f}$. Then we have the Fourier transform of $H$-invariant distributions on $s$.

**Lemma 2.7.** Let $O \subset N$. Then the Fourier transform $\hat{\mu}_O$ is a locally integrable function on $s$. It is locally constant on $s_{\theta - \text{reg}}$.

**Proof.** If $A \simeq M_n(D)$, this is [Guo98, Theorem 2.3]. If $A$ is a central division algebra this is clear since the only nilpotent orbit is $O = \{0\}$ and $\hat{\mu}_O$ is a nonzero constant function (depending on the choice of the measures). □

**Lemma 2.8.** There is a function $\hat{j} \in C^\infty_c(s_{\theta - \text{reg}} \times s_{\theta - \text{reg}})$ that is locally integrable on $s \times s$, such that for any $f \in C^\infty_c(s)$, we have
\[
O(\xi, \hat{f}) = \int_s f(\zeta) \hat{j}(\xi, \zeta) d\zeta.
\]

**Proof.** If $A \simeq M_n(D)$, this is [Zha15, Theorem 6.6, 6.11]. If $A$ is a central division algebra and $\xi \in s_{\theta - \text{reg}}$, then we have
\[
O(\xi, \hat{f}) = \int_{H_\xi \setminus H} \int_s f(\zeta) \psi((\xi^h, \zeta)) d\zeta dh.
\]
This double integral is absolutely convergent as $H$ is compact modulo the center of $G$. We then switch the order of integration and conclude that the distribution $f \mapsto O(\xi, \hat{f})$ is represented by the function
\[
\hat{j}(\xi, \zeta) = \int_{H_\xi \setminus H} \psi((\xi^h, \zeta)) dh.
\]
This is clearly locally integrable in both variables. □
2.3. **Semisimple descent on the Lie algebra.** Assume that \( A = M_n(D) \) in this subsection.

Let \( \xi \in \mathfrak{s} \) be a \( \theta \)-semisimple element and \( G_\xi \) be its stabilizer in \( G \) and \( H_\xi = H \cap G_\xi \). They are invariant under conjugation by \( \epsilon \). Let \( \mathfrak{g}_\xi, \mathfrak{h}_\xi \) be the Lie algebras of them respectively. The involution \( \theta \) preserves \( G_\xi \), and hence \( \mathfrak{g}_\xi \). Let \( \mathfrak{s}_\xi \) be the \((-1)\)-eigenspace of \( \theta \) in \( \mathfrak{g}_\xi \). Then \( \mathfrak{g}_\xi = \mathfrak{h}_\xi \oplus \mathfrak{s}_\xi \) and \( H_\xi \) acts on \( \mathfrak{s}_\xi \). Up to conjugation by \( H_\xi \), the \( \theta \)-semisimple element \( \xi \) takes the following form

\[
\xi = \begin{pmatrix}
\epsilon X & 0_r \\
X & 0_r
\end{pmatrix},
\]

where \( X \in \text{GL}_{n-r}(E) \). It is not hard to check that the symmetric pair \((G_\xi, H_\xi)\) is of the form \((G_1, H_1) \times (G_2, H_2)\), where

\[
G_1 \simeq \left\{ g = \begin{pmatrix} a & eb \\ b & a \end{pmatrix} \in \text{GL}_{2n-2r}(E) \mid bX = Xb, \; aX = Xa \right\}, \quad H_1 \simeq \left\{ g = \begin{pmatrix} a \\ a \end{pmatrix} \in G_1 \right\},
\]

and \((G_2, H_2)\) is of the same shape as \((G, H)\) but of smaller size \( r \). The “Lie algebra” \( \mathfrak{s}_\xi \) is isomorphic to \( \mathfrak{s}_1 \times \mathfrak{s}_2 \) on which \( H_1 \times H_2 \) acts componentwise. The action of \( H_2 \) on \( \mathfrak{s}_2 \) is of the same shape as the action of \( H \) on \( \mathfrak{s} \), but of a smaller size. The space \( \mathfrak{s}_1 \) is of the form

\[
\mathfrak{s}_1 = \left\{ \begin{pmatrix} eb \\ b \end{pmatrix} \mid bX = Xb \right\},
\]

and \( H_1 \) acts on it by conjugation. We note that \( H_1 \) is an inner form of \( \text{GL}_{n,F} \) and \( \mathfrak{s}_1 \) is isomorphic to the Lie algebra of \( H_1 \) and the isomorphism is given by

\[
\begin{pmatrix} b \\ eb \end{pmatrix} \mapsto bX.
\]

This isomorphism is \( H_1 \) equivariant. In other words, the action of \( H_1 \) on \( \mathfrak{s}_1 \) is isomorphic to its usual adjoint action on its Lie algebra.

Let \( \mathfrak{s}_\xi' \) be the open subscheme of \( \mathfrak{s}_\xi \) consisting of all elements \( \gamma \) such that the morphism

\[
H \times_{H_\xi} \mathfrak{s}_\xi \to \mathfrak{s}, \quad (h, \gamma) \mapsto \gamma^h
\]
is etale at \((1, \gamma)\). The open subscheme \( (\mathfrak{s}_\xi//H_\xi)' \) consists of all elements in \( \mathfrak{s}_\xi//H_\xi \) whose inverse image in \( \mathfrak{s}_\xi \) lie in \( \mathfrak{s}_\xi' \).

As usual, the study of semisimple descent begins from the following compactness lemma.

**Lemma 2.9.** Let \( \omega_\mathfrak{s} \) be a compact subset of \( \mathfrak{s} \) and \( \omega_{\mathfrak{s}_\xi} \) be a compact subset of \((\mathfrak{s}_\xi//H_\xi)'(F)\). Then the closure of

\[
\{ h \in H \mid \omega_\mathfrak{s}^h \cap q_\xi^{-1}(\omega_{\mathfrak{s}_\xi}) \neq \emptyset \}
\]
is compact, where \( q_\xi : \mathfrak{s}_\xi \to \mathfrak{s}_\xi//H_\xi \) be the categorical quotient.
Proof. We have the morphisms

$$H \times_{H_\xi} S'_\xi \rightarrow s \times (s_\xi/H_\xi)'$$

The set in the lemma is contained in the compact set \(\eta^{-1}(\omega_\xi \times \omega_\xi')\). □

In the above lemma, let us put \(\omega_\xi = \text{supp} f\) and choose an arbitrary \(\omega_\xi'\). Let \(C\) be a compact subset of \(H/H_\xi\) such that the image of the subset in the above lemma is contained in \(C\). Let \(\alpha \in C^\infty_c(H)\) such that

\[
\int_{H_\xi} \alpha(gh)dh = 1_C.
\]

Put

\[
\phi(\gamma) = \int_H f(\gamma^h)\alpha(h)dh, \quad \gamma \in S'_\xi.
\]

Then \(\phi \in C^\infty_c(S'_\xi)\). Let \(\gamma \in q^{-1}_\xi(\omega_\xi')\). Then \((H_\xi)_\gamma = H_\gamma\). By its very construction we have

\[
\int_{(H_\xi)_\gamma \backslash H_\xi} \phi(\gamma^h)dh = \int_{H_\gamma \backslash H} f(\gamma^h)dh.
\]

2.4. The Shalika germ expansion. We prove the existence of Shalika germs in this subsection and prove that they are linearly independent in the next subsection. The argument works equally well in both cases \(A = M_n(D)\) and \(A\) is a central division algebra, though the results are almost trivial in the case \(A\) being a central division algebra.

**Proposition 2.10.** There is a unique \(H\)-invariant real valued function \(\Gamma_\mathcal{O}\) on \(s_{\theta-\text{reg}}\) for each nilpotent orbit \(\mathcal{O} \subset N\) with the following properties.

1. For any \(f \in C^\infty_c(s)\), there is an \(H\)-invariant neighbourhood \(U_f\) of \(0 \in s\) such that

\[
O(\xi, f) = \sum_{\mathcal{O} \subset N} \Gamma_\mathcal{O}(\xi)\mu_\mathcal{O}(f).
\]

for all \(\theta\)-regular \(\xi \in U_f\).

2. For all \(t \in F^\times\) and all \(\xi \in s_{\theta-\text{reg}}\), we have

\[
\Gamma_\mathcal{O}(t\xi) = |t|^{-d_\mathcal{O}}\Gamma_\mathcal{O}(\xi).
\]

**Proof.** It follows from [RR96, Proposition 1.2] that for each there are functions \(\Gamma'_\mathcal{O}\) on \(s_{\theta-\text{reg}}\) with property (1). Note that if \(\Gamma''_\mathcal{O}\) is another function satisfying (1), then \(\Gamma'_\mathcal{O}\) and \(\Gamma''_\mathcal{O}\) have the same germ at \(\xi = 0\) (i.e. they equal in a small neighbourhood of \(0\)). We first explain that \(\Gamma'_\mathcal{O}\) can be chosen to be real valued, at least when \(\xi\) is close to \(0 \in s\). In fact, for each \(\mathcal{O} \subset N\) we can find a function \(f_\mathcal{O}\) so that \(\mu_\mathcal{O}(f_\mathcal{O}) = \delta_{\mathcal{O}, \mathcal{O}'}\) (Kronecker delta). It is obvious that \(f_\mathcal{O}\)’s can be chosen to be real valued. For this particular choice, we have \(O(\xi, f_\mathcal{O}) = \Gamma'_\mathcal{O}(\xi)\) when \(\xi\) lies in a small neighbourhood of \(0\). Indeed, this can be taken as the definition of \(\Gamma'_\mathcal{O}(\xi)\). As \(f_\mathcal{O}\) is real, it follows
that \( \Gamma'_{\mathcal{O}}(\xi) \) can be taken to be real. We now need to prove that among these functions, we can choose a unique \( \Gamma_{\mathcal{O}} \) for each \( \mathcal{O} \subset \mathcal{N} \) with property (2).

Let \( t \in F^\times \) be fixed. We claim that as a function of \( \xi \), \( \Gamma_{\mathcal{O}}(t\xi) \) and \( |t|^{-d_{\mathcal{O}}} \Gamma_{\mathcal{O}}(\xi) \) have the same germ at \( \xi = 0 \). Indeed, on the one hand, we have

\[
O(\xi, f_i) = \sum_{\mathcal{O} \subset \mathcal{N}} \Gamma'_{\mathcal{O}}(\xi)|t|^{d_{\mathcal{O}}} \mu_{\mathcal{O}}(f)
\]

when \( \xi \) lies in a small neighbourhood (depending on \( f \) and \( t \)) of \( 0 \in s \). On the other hand,

\[
O(\xi, f_i) = O(t^{-1}\xi, f) = \sum_{\mathcal{O} \subset \mathcal{N}} \Gamma'_{\mathcal{O}}(t^{-1}\xi)\mu_{\mathcal{O}}(f).
\]

when \( \xi \) lies in a small neighbourhood (depending on \( f \) and \( t \)) of \( 0 \in s \). Comparing these two, we conclude that \( \Gamma'_{\mathcal{O}}(t\xi) \) and \( |t|^{-d_{\mathcal{O}}} \Gamma'_{\mathcal{O}}(\xi) \) have the same germ at \( \xi = 0 \).

Thus we put

\[
\Gamma_{\mathcal{O}}(\xi) = \lim_{k \to \infty} |\varpi|^{kd_{\mathcal{O}}} \Gamma'_{\mathcal{O}}(\varpi^k\xi).
\]

It is straightforward to check that \( \Gamma_{\mathcal{O}}(\xi) \) does satisfy property (2). Of course, in order that \( \Gamma_{\mathcal{O}} \) satisfies property (2), it has to be of this form. Thus this function is unique. \( \Box \)

We now consider the Shalika germ expansion around an arbitrary \( \theta \)-semisimple element \( \xi \in s \). Let \( H_\xi \) be the stabilizer of \( \xi \) in \( H \). Note that \( \xi \in s_\xi \) and by definition \( \xi \) is fixed by \( H_\xi \). As we have discussed in the previous subsection, the action of \( H_\xi \) on \( s_\xi \) is isomorphic to the action of \( H_1 \times H_2 \) on \( s_1 \times s_2 \) where the action of \( H_1 \) on \( s_1 \) is isomorphic to the conjugation of \( H_1 \) on its Lie algebra and the action of \( H_2 \) on \( s_2 \) is of the same shape as the action of \( H \) on \( s \) but of a smaller size. A nilpotent orbit in \( s \) is of the form \( \mathcal{O}_1 \times \mathcal{O}_2 \) where \( \mathcal{O}_i \) is a nilpotent orbit in \( s_i \) under the action of \( H_i \) (\( i = 1, 2 \)). Let \( \{ \mathcal{O}_1, \cdots, \mathcal{O}_r \} \) be the set of nilpotent orbits in \( s_\xi \). We thus have the Shalika germs on \( s_\xi \), indexed by the \( \theta \)-nilpotent orbits in \( s_\xi \), which on \( s_1 \) is given by the one defined in [Kot05, Section 17] and on \( s_2 \) is given the one we have just defined. Let \( \{ \eta_1, \cdots, \eta_r \} \) be a complete set of representatives of \( \theta \)-nilpotent elements in \( s_\xi \) and \( \eta_i \in \mathcal{O}_i \). We denote the Shalika germ on \( s_\xi \) indexed by \( \mathcal{O}_i \) by \( \Gamma^\xi_i \).

\textbf{Corollary 2.11.} Let \( f \in C_c^\infty(s) \). Then there is a neighbourhood \( U_f \) of \( \xi \) in \( s_\xi \) so that for any \( \theta \)-regular \( \eta \in U_f \), we have

\[
O(\eta, f) = \sum_{i=1}^r \Gamma^\xi_i(\eta)O(\xi + \eta_i, f).
\]

\textbf{Proof.} By the semisimple descent of the orbital integrals, we can find a \( \phi \in C_c^\infty(s_\xi) \) so that

\[
O(\eta, f) = O^\xi(\eta, \phi),
\]

if \( q_\xi(\eta) \) lies in a small neighbourhood of \( q_\xi(\xi) \) in \( s_\xi \). Apply Proposition 2.10 (germ expansion on \( s_2 \) near \( 0 \)) and [Kot05, Theorem 17.5] (germ expansion on \( s_1 \) near a central element), we have

\[
O^\xi(\eta, \phi) = \sum_{i=1}^r \Gamma^\xi_{\mathcal{O}_i}(\eta)\mu_{\mathcal{O}_i}(\phi),
\]
where \( \mu_{\mathcal{O}}^{\xi} \) stands for the nilpotent orbital integral on \( \mathfrak{s}_{\xi} \). Using semisimple descent of orbital integrals again we have \( \mu_{\mathcal{O}}^{\xi}(\phi) = O(\xi + \eta, f) \). This proves the corollary. \( \square \)

2.5. **Linear independence of Shalika germs.** The goal of this subsection is to prove that the Shalika germs \( \Gamma_{\mathcal{O}} \) are linearly independent. We follow the argument of [Kot05, Section 27] closely.

The argument is essentially an application of the homogeneity of the nilpotent orbital integrals and the representability of the Fourier transform of the orbital integrals. We have already established all the necessary ingredients that we need. So we will be sketchy at various points, referring the readers to the argument in [Kot05].

The first step is the following lemma.

**Lemma 2.12.** The set of all orbital integrals is weakly dense in \( D(\mathfrak{s})^H \).

*Proof.* The proof is identical to [Kot05, Proposition 27.1], making use of the finiteness of the nilpotent orbits. We omit the detailed arguments. \( \square \)

The second step is to prove the following.

**Lemma 2.13.** The functions \( \Gamma_{\mathcal{O}} \)'s for \( \mathcal{O} \subset \mathcal{N} \) are linearly independent if and only if their restriction to an arbitrary small neighbourhood of \( 0 \in \mathfrak{s} \) are still linearly independent.

*Proof.* The proof is identical to [Kot05, Lemma 27.2], making use of the homogeneity of \( \Gamma_{\mathcal{O}} \)'s. We omit the details. \( \square \)

The next step is to relate the linear independence of the Shalika germs to the density of regular semisimple orbital integrals.

**Lemma 2.14.** The following assertions are equivalent.

1. The Shalika germs \( \Gamma_{\mathcal{O}} \), \( \mathcal{O} \subset \mathcal{N} \) are linearly independent.
2. The nilpotent orbital integrals \( \mu_{\mathcal{O}} \)'s lie in the weak closure of the set of \( \theta \)-regular orbital integrals in the space of \( H \)-invariant distributions on \( \mathfrak{s} \).

*Proof.* (1) \( \Rightarrow \) (2). Assume that the \( \theta \)-regular orbital integrals \( O(\xi, f) \) are all zero. Let \( f \in C_{c}^{\infty}(\mathfrak{s}) \). Then it follows from the Shalika germ expansion that

\[
\sum_{\mathcal{O} \subset \mathcal{N}} \mu_{\mathcal{O}}(f) \Gamma_{\mathcal{O}}(\xi) = 0
\]

for any \( \theta \)-regular \( \xi \in U_f \) where \( U_f \) is a small neighbourhood of \( 0 \in \mathfrak{s} \). Since \( \Gamma_{\mathcal{O}} \)'s linear independent, by the previous lemma, they are linearly independent even when restricted to \( U_f \). Thus we conclude that \( \mu_{\mathcal{O}}(f) = 0 \) for all \( \mathcal{O} \).

(2) \( \Rightarrow \) (1). Suppose that we have a linear relations

\[
\sum_{\mathcal{O} \subset \mathcal{N}} a_{\mathcal{O}} \Gamma_{\mathcal{O}}(\xi) = 0, \text{ for all } \xi \in \mathfrak{s}_{\theta-\text{reg}}.
\]
As $\mu_O$’s form a basis of the space of $H$-invariant distributions on $\mathcal{N}$, we may choose a test function $f \in C_c^\infty(\mathfrak{s})$ so that $\mu_O(f) = a_O$ for all $O \subset \mathcal{N}$. Thus using the Shalika germ expansion, we conclude that there is a small neighbourhood $U_f$ of $0 \in \mathfrak{s}$ so that

$$O(\xi, f) = \sum_{O \subset \mathcal{N}} \mu_O(f) \Gamma_O(\xi) = \sum_{O \subset \mathcal{N}} a_O \Gamma_O(\xi) = 0$$

for all $\theta$-regular $\xi \in U_f$. By Lemma 2.2, the set $U_f^H$ contains an open and closed neighbourhood $V$ of $\mathcal{N}$. Let $f' = f 1_V$. Then we have that $O(\xi, f') = 0$ for all $\theta$-regular $\xi$. Moreover, since $V$ is an open and closed neighbourhood of $\mathcal{N}$, we have that $\mu_O(f) = \mu_O(f')$ for all $O \subset \mathcal{N}$. Now by assertion (2), the nilpotent orbital integrals $\mu_O$’s all lie in the weak closure of the $\theta$-regular orbital integrals. Since $O(\xi, f') = 0$ for $\theta$-regular $\xi$, we conclude that $a_O = \mu_O(f) = \mu_O(f') = 0$. This proves (1).

The following lemma allows us to use induction.

**Lemma 2.15.** Let $\xi$ be a $\theta$-semisimple element in $\mathfrak{s}$ and $H_\xi$ its centralizer in $H$. Let $\mathfrak{s}_\xi$ be the induced symmetric space and $\{\Gamma^\xi_O\}$ the Shalika germs on $\mathfrak{s}_\xi$ indexed by the nilpotent orbits $O$ in $\mathfrak{s}_\xi$. Suppose that $\Gamma^\xi_O$’s are linearly independent. Then for all $\eta$ whose $\theta$-semisimple part is $\xi$ the orbital integral $O(\eta, f)$ lies in the weak closure of the set of all $\theta$-regular orbital integrals.

**Proof.** Let $\mathcal{N}_\xi$ be the nilpotent cone of $\mathfrak{s}_\xi$ and for each nilpotent orbit $O \subset \mathcal{N}_\xi$, we fix an element $\eta_O \in O$. Then by the Shalika germ expansion at $\xi$, there is a small neighbourhood $U_f$ of $\xi$ in $\mathfrak{s}_\xi$, so that for all $\theta$-regular $\eta \in U_f$,

$$O(\eta, f) = \sum_{O \subset \mathcal{N}_\xi} \Gamma^\xi_O(\eta) O(\xi + \eta_O, f).$$

As $\Gamma^\xi_O$’s are linearly independent and they remain linearly independent when restricted to $U_f$, we conclude that if $O(\eta, f) = 0$ for all $\theta$-regular $\eta \in U_f$, we have $O(\xi + \eta_O, f) = 0$ for all $O$. This proves the lemma.

We now prove the linear independence of the Shalika germs and the density of regular semisimple orbital integrals simultaneously.

**Proposition 2.16.** The following assertions holds.

1. The Shalika germs $\Gamma_O$’s, $O \subset \mathcal{N}$, are linearly independent.

2. The set of $\theta$-regular orbital integrals are weakly dense in the space of $H$-invariant distributions on $\mathfrak{s}$.

**Proof.** Let $\xi$ be a $\theta$-semisimple element in $\mathfrak{s}$ and let $\mathfrak{s}_\xi$ and $H_\xi$ be as before. As $H_\xi$ is isomorphic to $H_1 \times H_2$, $\mathfrak{s}_\xi$ is isomorphic to $\mathfrak{s}_1 \times \mathfrak{s}_2$, and the action of $H_1$ on $\mathfrak{s}_1$ is isomorphic to its adjoint action on the Lie algebra of $H_1$, the action of $H_2$ on $\mathfrak{s}_2$ takes the same shape as the action of $H$ on $\mathfrak{s}$, we may use induction and assume that the proposition holds for all $\mathfrak{s}_\xi$ for $\theta$-semisimple $\xi \neq 0$. 

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First we show that, under the inductive hypothesis, the two assertions in the proposition are equivalent. In fact, if the second assertion holds, then the first holds by Lemma 2.14. If the first assertion holds, then the nilpotent orbital integrals lie in the weak closure of $\theta$-regular orbital integrals. When combined with the induction hypothesis, this implies that all orbital integrals lie in the weak closure of the $\theta$-regular orbital integrals. This proves that two assertions in the proposition are equivalent.

Put
$$C_1 = \{ f \in C_c^\infty(s) \mid \text{all orbital integrals of } f \text{ vanish} \};$$
$$C_2 = \{ f \in C_c^\infty(s) \mid \text{all } \theta\text{-regular orbital integrals of } f \text{ vanish} \}.$$ 
Now by the induction hypothesis, we conclude that $C_2$ consists of all functions $f \in C_c^\infty(s)$ such that all orbital integrals, except the nilpotent orbital integrals, vanish. Thus $C_2/C_1$ is spanned by all $\mu_O$'s, $O \subset N$.

By Lemma 2.12, $C_1$ consists of all $f \in C_c^\infty(s)$ such that $I(f) = 0$ for all $H$-invariant distribution $I$. Thus it is clear that $C_1$ is closed under the Fourier transform. Since the Fourier transform of $\theta$-regular orbital integrals are represented by $H$-invariant integrable functions on $s_{\theta\text{-reg}}$ by Lemma 2.8, we conclude that $C_2$ is also preserved under the Fourier transform. Thus Fourier transform induces an isomorphism of $C_2/C_1$ onto itself. Therefore $C_2/C_1$ is also spanned by $\widehat{\mu}_O$.

We note that $\mu_O$ and $\widehat{\mu}_O$ have the homogeneity properties
$$\mu_O(f_t) = |t|^{d_O} \mu_O(f), \quad \widehat{\mu}_O(f_t) = |t|^{n^2-d_O} \widehat{\mu}_O(f).$$

The key point is that $d_O$ is the dimension of a unipotent subgroup of $H$ and we have $2d_O < n^2$. Therefore $d_O \neq n^2 - d_O'$ for any $O, O' \subset N$. We thus have two spanning sets of vectors of $C_2/C_1$, all being homogeneous, but with different scaling factors from each set. Therefore $C_2/C_1 = 0$ and this proves the proposition.

3. Spherical characters: nonsplit case

3.1. Definitions and first properties. Let us keep the notation from the previous section. To unify notation for later use, we temporarily allow $E = F \times F$, in which case $G = \text{GL}_{2n,F}$ and $H = \text{GL}_{n,F} \times \text{GL}_{n,F}$. Recall that $Z$ is the center of $G$. It is also the split center of $H$ if $E$ is a field.

Let $\pi$ be an irreducible discrete series representation of $G$. Assume that
$$\text{Hom}_H(\pi, \mathbb{C}) \neq 0.$$ 

We call such a $\pi$ $H$-distinguished. By [Guo97, JR96], this Hom space is one dimensional. Let us define a spherical character on $G$ by
$$J_\pi(f) = \int_{Z \backslash H} \text{Tr}(\pi(h)\pi(f)) dh, \quad f \in C_c^\infty(G).$$

Lemma 3.1. The defining integral of $J_\pi$ is absolutely convergent. Moreover $J_\pi$ is not identically zero if and only if $\pi$ is $H$-distinguished.
Proof. If $A$ is a central division algebra, then the lemma is clear as $Z \setminus H$ is compact. If $A = M_n(D)$, the first assertion is in [GO16]. They did the case $D$ being split, but the general case can be obtained with similar technique. Once we have the first assertion, the second assertion follows from [BPa, Proposition 3.2.1].

Proposition 3.2. The spherical character $J_\pi$ is represented by a locally integrable function on $G$ that is locally constant on $G_{\theta-\text{reg}}$.

Proof. If $A$ is a central division algebra, then the lemma is clear as $Z \setminus H$ is compact and $\pi$ is finite dimensional. If $A = M_n(D)$, this is the main result of [Guo98]. It is a simple consequence of the character expansion of $J_\pi$ (cf. [RR96, Theorem 7.11] and below) and the representability of the Fourier transform of $\mu_O$ (cf. Lemma 2.7).

We denote this locally integrable function by $\Phi_\pi$. It is clear that $\Phi_\pi$ is bi-$H$-invariant.

In the case $\pi$ is not a discrete series representation, the integral is not absolutely convergent. We will make an alternative definition as follows. Assume that $\pi$ is unitary and $H$-distinguished. Then again we have $\text{Hom}_H(\pi, \mathbb{C})$ is one dimensional. We fix a nonzero element $l \in \text{Hom}_H(\pi, \mathbb{C})$ and put

$$J_\pi(f) = \sum_{\varphi \in \pi} l(\pi(f) \varphi) \widehat{l(\varphi)},$$

where the sum runs over an orthonormal basis of $\pi$. This definition depends on the choice of $l$ and we sometimes denote it by $J_{\pi,l}$ to stress this dependence. Again by [Guo96], it is representable by a bi-$H$-invariant locally integrable function on $G$, which we again denote by $\Phi_\pi$.

3.2. The local character expansion. The following proposition is one of the main results of [RR96].

Proposition 3.3. Let $\pi$ be an irreducible discrete series representation of $G$. There are constants $c_O$, $O \subset N$, depending on $\pi$, such that

$$\Phi_\pi(1 + \xi) = \sum_{O \in N} c_O \widehat{\mu_O}(\xi),$$

for all $\xi \in \mathfrak{s}_{\theta-\text{reg}}$ which are sufficiently close to $0 \in \mathfrak{s}$. In terms of the distributions, this means that there is a small neighbourhood $\omega$ of $0 \in \mathfrak{s}$ so that for all $f \in C_c^\infty(H(1 + \omega)H)$, we have

$$J_\pi(f) = \sum_{O \in N} c_O \mu_O(\tilde{f}_\omega).$$

Proof. This is [RR96, Theorem 7.11]. Note that the nilpotent orbital integrals provide a canonical basis of the space of distributions supported on $N$. 

We say that $f$ is an essential matrix coefficient of $\pi$ if

$$g \mapsto \int_Z f(zg) dz$$
is a matrix coefficient of \( \pi \). Recall that if \( g \) is \( \theta \)-elliptic, the stabilizer of \( g \) in \( H \times H \) is an anisotropic torus modulo \( Z \), thus up to some constant depending only on the measure, for all \( f \in C_c^\infty(G) \), we have

\[
O(g, f) = \int_{Z \setminus (H \times H)} f(h_1gh_2) dh_1 dh_2.
\]

**Lemma 3.4.** Let \( \pi \) be an \( H \)-distinguished supercuspidal representation of \( G \) and \( f \) be a \( K \)-finite essential matrix coefficient of \( \pi \). Then for all \( \theta \)-elliptic \( g \in G \), we have

\[
O(g, f) = \Phi_\pi(g) \int_H f(h) dh.
\]

**Proof.** It is enough to prove that for any \( K \)-finite \( \varphi \in C_c^\infty(G_{\theta-\text{ell}}) \), we have

\[
\int_G \varphi(g) O(g, f) dg = J_\pi(\varphi) \int_H f(h) dh.
\]

As \( \varphi \) is supported on the \( \theta \)-elliptic locus, we have

\[
\int_G \varphi(g) O(g, f) dg = \int_G \int_{Z \setminus (H \times H)} \varphi(g) f(h_1gh_2) dh_1 dh_2 dg.
\]

The right hand of this integral is absolutely convergent and we can change the order of integration. By assumption, we can pick \( v, w \in \pi \) and \( \int_Z f(zg) dz = \langle \pi(g)v, w \rangle \). Thus this integral equals

\[
\int_{Z \setminus H \times Z \setminus H} \langle \pi(h_1)\pi(\varphi)\pi(h_2)v, w \rangle dh_1 dh_2.
\]

As \( \varphi \) is \( K \)-finite, we may find \( v_1, \ldots, v_r, w_1, \ldots, w_r \) so that

\[
\pi(\varphi)v = \sum_{i=1}^r \langle v, v_i \rangle w_i.
\]

It then follows that

\[
(3.1) = \int_H f(h) dh \int_{Z \setminus H} \text{Trace}(\pi(h)\pi(\varphi)) dh.
\]

This proves the lemma. \( \square \)

**Lemma 3.5.** Let 0 be the minimal nilpotent orbit \( \{0\} \subset \mathfrak{s} \).

1. If \( \xi \) is \( \theta \)-elliptic, then \( \Gamma_0(\xi) \neq 0 \).

2. Let \( \pi \) be an irreducible \( H \)-distinguished supercuspidal representation of \( G \). Let \( \xi \) be any \( \theta \)-elliptic element in \( \mathfrak{s} \). Then \( c_0 = \Gamma_0(\xi) \neq 0 \).

**Proof.** As \( \pi \) is \( H \)-distinguished, we can find a \( K \)-finite essential matrix coefficient \( f \) of \( \pi \) such that

\[
\int_H f(h) dh \neq 0.
\]

Let \( \xi \in \mathfrak{s} \) be sufficiently close to 0 (depending on \( f \)), then

\[
O(1 + \xi, f) = O(\xi, f_\xi).
\]

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Then we have the Shalika germ expansion

\[ O(1 + \xi, f) = \sum_{\mathcal{O} \subset N} \Gamma_{\mathcal{O}}(\xi) \mu_{\mathcal{O}}(f). \]

On the other hand, by Proposition 3.3, we have the local character expansion

\[ \Phi(1 + \xi) = \sum_{\mathcal{O} \in \mathcal{N}} c_{\mathcal{O}} \hat{\mu}_{\mathcal{O}}(\xi). \]

By Lemma 3.4, for all \( \xi \) which are \( \theta \)-elliptic and is sufficiently close to \( 0 \in \mathfrak{s} \), we have

\[ \sum_{\mathcal{O} \subset \mathcal{N}} \Gamma_{\mathcal{O}}(\xi) \mu_{\mathcal{O}}(f) = \left( \int_{H} f(\mathfrak{n}) \, dh \right) \sum_{\mathcal{O} \in \mathcal{N}} c_{\mathcal{O}} \hat{\mu}_{\mathcal{O}}(\xi). \]

By homogeneity of \( \Gamma_{\mathcal{O}} \) and \( \hat{\mu}_{\mathcal{O}} \), for all \( \xi \) which are \( \theta \)-elliptic and is sufficiently close to \( 0 \in \mathfrak{s} \), we have

\[ \Gamma_{\mathcal{O}}(0) \mu_{\mathcal{O}}(f) = c_{\mathcal{O}} \left( \int_{H} f(\mathfrak{n}) \, dh \right) \hat{\mu}_{\mathcal{O}}(\xi). \]

Note that \( \hat{\mu}_{\mathcal{O}} \) equals a nonzero constant function. Moreover by definition

\[ \mu_{\mathcal{O}}(f) = \int_{H} f(\mathfrak{n}) \, dh \neq 0. \]

We thus conclude that for all \( \xi \) that is \( \theta \)-elliptic and sufficiently close to \( 0 \in \mathfrak{s} \), we have \( c_{0} = \Gamma_{\mathcal{O}}(\xi) \).

By the linear independence of the Shalika germs, cf. Proposition 2.16, \( \Gamma_{\mathcal{O}} \) is not identically zero when restricted to any small neighbourhood of \( 0 \in \mathfrak{s} \). Thus there is a \( \theta \)-elliptic element \( \xi \in \mathfrak{s} \) so that \( \Gamma_{\mathcal{O}}(\xi) \neq 0 \). This proves both assertions of the lemma.

**Remark 3.6.** The above two lemmas should hold for all \( H \)-distinguished discrete series representations. In order to extend these lemma to this discrete series representations, we need to address some convergence issues. More precisely we need to prove that for all Harish-Chandra Schwartz test functions \( f \) on \( G \), all orbital integrals are absolutely convergent. To achieve this one needs the Howe’s finiteness theorem for the symmetric space \( S \).

**4. Orbital integrals: split case**

We study the orbital integrals in the split case. This is parallel to the study in the nonsplit case, except that the analysis of the nilpotent cone and the definition of nilpotent orbital integrals are much more technical. In particular, as we shall see, the nilpotent orbital integrals are not absolutely convergent in general, but they can be regularized and hence still form a basis of the space of invariant distributions supported on the nilpotent cone. We discuss these issues in detail. For results that are parallel to the ones in the nonsplit case, we usually just state the results and give little details on the proof.
4.1. The symmetric space and semisimple orbital integrals. Let $G' = \text{GL}_{2n,F}$ and $H' = \text{GL}_{n,F} \times \text{GL}_{m,F}$ with an embedding

$$(h_1, h_2) \mapsto \begin{pmatrix} h_1 & h_2 \\ h_1 & h_2 \end{pmatrix}, \quad h_1, h_2 \in \text{GL}_{n,F}.$$ 

Both are considered as algebraic groups over $F$. Note that $H'$ is the fixed point of the involution $\theta(g) = \text{Ad}\left(\begin{smallmatrix} 1_n \\ & -1_n \end{smallmatrix}\right)$, on $G'$. We are using the same letter $\theta$ to denote the involution on $G$ of the previous section and on $G'$ here, but the meaning will either be explained or be clear from the context. Let

$$S' = \{g^{-1}\theta(g) \mid g \in G'\} \subset G'$$

This is a closed subvariety of $G'$ over $F$. Elements of $S'$ are all of the form

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}, \quad A^2 = D^2 = 1_n + BC, \quad AB = BD, \quad DC = CA.$$ 

The group $H' \times H'$ acts on $G'$ by left and right multiplication and the group $H'$ acts on $S'$ by conjugation. We say that an element $\gamma \in S'$ is $\theta$-semisimple etc. if it is semisimple etc. (in the usual sense) in $G'$. We say that an element $g \in G'$ is $\theta$-semisimple (resp. $\theta$-regular, resp. $\theta$-elliptic) if its image in $S'$ is so. By [Guo96, Lemma 1.3], every $\theta$-regular element in $G'$ can be written in the form

$$\begin{pmatrix} 1_n & a \\ 1_n & 1_n \end{pmatrix},$$

and $a \in \text{GL}_{n,F}$ is regular semisimple in $\text{GL}_{n,F}$ in the usual sense and $\det(a - 1_n) \neq 0$. Moreover it is $\theta$-elliptic if $a$ is elliptic in $\text{GL}_{n,F}$.

Let $f \in C_c^\infty(G')$ and $g \in G'$ be a $\theta$-semisimple element. By the determinant function on $H'$, we mean $\det h = \det h_1 \det h_2$ where $h = (h_1, h_2) \in H'$. We define the orbital integral

$$O(g, \eta, f) = \int_{(H' \times H')_g \setminus H' \times H'} f(h_1gb_2)\eta(\det h_2)dh_1dh_2.$$ 

As the orbit is closed, this integral is absolutely convergent. For a given $f$, $O(\cdot, \eta, f)$ is left $H'$-invariant and right $(H', \eta)$-invariant.

We also consider the Lie algebra of $S'$. Let $s' = M_{n,F} \times M_{n,F}$. We always consider $s'$ as a subspace of $M_{2n,F}$ consisting of matrices of the form

$$\begin{pmatrix} 0 & X \\ Y & 0 \end{pmatrix}, \quad X, Y \in M_{n,F}.$$ 

The group $H'$ acts on $s'$ by conjugation, i.e. $\gamma^h = h^{-1}\gamma h$, $\gamma \in s'$ and $h \in H'$. We define an element in $s'$ to be $\theta$-semisimple etc. if it is so in $M_{2n,F}$. By [JR96, Proposition 2.1], any $\theta$-semisimple
element in $s'$ is $H'$ conjugate to an element of the form
\[
\begin{pmatrix}
0 & 0 & 1_m & 0 \\
0 & 0 & 0 & 0 \\
A & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix},
\]
where $A \in \text{GL}_{m,F}$ is semisimple (in the usual sense). It is $\theta$-regular if $m = n$ and $A \in \text{GL}_{n,F}$ is regular semisimple (in the usual sense). It is elliptic if $A$ is moreover elliptic.

Let $f' \in C_c^\infty(G')$. We put $\tilde{f}' \in C_c^\infty(S')$ as
\[
\tilde{f}'(g^{-1}\theta(g)) = \int_{H'} f'(hg)dh, \quad g \in G'.
\]
We fix a $H'$-invariant neighbourhood $\omega'$ of $0 \in s'$ so that the Cayley transform
\[
c : \omega' \to S', \quad \xi \mapsto (1 + \xi)^{-1}(1 - \xi)
\]
is defined and is a homeomorphism, and denote by $\Omega'$ be its image in $G'$. For any $f' \in C_c^\infty(G')$ we define $f'_\xi \in C_c^\infty(s')$ as
\[
f'_\xi(\xi) = \begin{cases}
\tilde{f}'(c(\xi)), & \xi \in \omega'; \\
0, & \xi \notin \omega'.
\end{cases}
\]
Let $\xi \in s'$ be $\theta$-semisimple and $f' \in C_c^\infty(s')$, we define an orbital integral
\[
O(\xi, \eta, f'') = \int_{H'_\xi \backslash H'} f''(\xi h)\eta(\det h)dh.
\]
The integral is absolutely convergent as the orbit of $\xi$ is closed.

Let us put an $H'$-invariant inner product on $s'$ by
\[
\langle \gamma, \delta \rangle = \text{Tr} \gamma \delta,
\]
where on the right hand side the product and the trace are taken in $M_{2n,F}$. Thus we can speak of the Fourier transform of elements in $C_c^\infty(s')$ and hence the Fourier transform of distributions on $s'$. We denote the Fourier transform by $\hat{\cdot}$.

**Lemma 4.1.** Let $\gamma \in s'$ be $\theta$-regular. Then the Fourier transform of the distribution
\[
f' \mapsto O(\gamma, \eta, \hat{f}')
\]
is represented by a locally integrable $(H', \eta)$-conjugation invariant function on $s'$. This function is locally constant on $s'_{\theta-\text{reg}}$.

**Proof.** This is [Zha15, Theorem 6.1].

\[\square\]
4.2. **Parabolic descent.** We explain the parabolic descent of \( \theta \)-regular semisimple orbital integrals in this subsection.

Let \( P = MN \) be the standard upper triangular parabolic subgroup of \( GL_n \) corresponding to the partition \( n = r_1 + \cdots + r_s \). Let \( p = m + n \) be its Lie algebra. Let \( P' = M'N' \) be the parabolic subgroup of \( G' \) consisting of matrices of the form
\[
\begin{pmatrix} A & B \\ C & D \end{pmatrix}, \quad A, B, C, D \in p.
\]
Then \( N' \) and \( M' \) consist of matrices of the form
\[
N' = \begin{pmatrix} \ast & \ast & 0 & \ast & \ast \\ \ast & \ast & \ast & \ast & \ast \\ \ast & \ast & \ast & \ast & \ast \\ 0 & \ast & \ast & \ast & \ast \\ 0 & \ast & \ast & \ast & \ast \end{pmatrix}, \quad M' = \begin{pmatrix} \ast & \ast & \ast \\ \ast & \ast & \ast \\ \ast & \ast & \ast \\ \ast & \ast & \ast \\ \ast & \ast & \ast \end{pmatrix}.
\]
Note that the pair \( (M', M' \cap H') \) is a product of the symmetric spaces of the same shape as \( (G', H') \) but of smaller sizes. Of course \( H' \cap P' \) is a parabolic subgroup of \( H' \). Let us put \( P'_{H'} = P' \cap H' \), \( M'_{H'} = M' \cap H' \), \( N'_{H'} = N' \cap H' \).

Let \( g \in M' \) be a \( \theta \)-regular semisimple element. Let \( f' \in C_c^\infty(G') \). By definition
\[
O(g, \eta, f') = \int_{(H' \times H')_0 \backslash H' \times H} f(h_1 g h_2) \eta(\det h_2) dh_1 dh_2.
\]
Let \( K \subset H' \) be a maximal open compact subgroup in good position with \( P \cap H' \). Put
\[
f'_K(g) = \int_K f'(k_1 g k_2) \eta(\det k_2) dk_1 dk_2, \quad f'(P')(g) = \delta_{P'}(g)^{\frac{1}{2}} \int_{N'} f'_K(g n) dn.
\]
Let \( h_1 = k_1 m_1 n_1 \) and \( h_2 = n_2 m_2 k_2 \) be the Iwasawa decompositions of \( h_1 \) and \( h_2 \) respectively, where \( m_1, m_2 \in M'_{H'} \), \( n_1, n_2 \in N'_{H'} \) and \( k_1, k_2 \in K \). Then, up to some nonzero constant depending only on the choice of the measures, we have
\[
O(g, \eta, f') = \int_{(M'_{H'} \times M'_{H'})_0 \backslash M'_{H'} \times M'_{H'}} \int_{N'_{H'} \times N'_{H'}} f'_K(m_1 n_1 g n_2 m_2) \delta_{P'}^{-1}(m_1 m_2^{-1}) \eta(\det m_2) dn_1 dn_2 dm_1 dm_2.
\]

**Lemma 4.2.** Let the notation be as above. Then map
\[
\delta_g : N'_{H'} \times N'_{H'} \to N', \quad (n_1, n_2) \mapsto g^{-1} n_1 g n_2
\]
is bijective and submersive everywhere.

**Proof.** The bijectivity of \( \delta_g \) can be checked directly. The tangent map at the point \( (n_1, n_2) \) is given by
\[
\left. d\delta_g \right|_{(n_1, n_2)} : (n' \cap h) \times (n' \cap h) \to n^+, \quad (\xi_1, \xi_2) \mapsto g^{-1} n_1 \xi_1 g n_2 + g^{-1} n_1 g n_2 \xi_2.
\]
As $n_1$ and $n_2$ are both unipotent, the determinant of $d\delta_g$ is independent of $n_1$ and $n_2$, and equals $d\delta_g = d\delta_g|_{(1,1)}$ which we now compute. First note that if $m_1, m_2 \in M' \cap H'$, then we have

$$|\det d\delta_{m_1gm_2}| = \delta_{P_{H'}}(m_1m_2)^{-1}|\det d\delta_g|.$$ 

As $g$ is $\theta$-regular, we may assume that

$$g = \begin{pmatrix}
1_n & a \\
1_n & 1_n
\end{pmatrix},$$

where $a$ is regular semisimple in the usual sense and $\det(1_n - a) \neq 0$. Direct computation shows that the determinant of $d\delta_g$ equals $\det(1 - a)^{-\dim N'}$ times the determinant of

$$n \times n \mapsto n \times n, \quad (X_1, X_2) \mapsto (-X_1 + X_2, X_1a - aX_2).$$

The later is nonzero as $a$ is regular semisimple and 1 is not an eigenvalue. This proves that $\delta_g$ is submersive at any $\theta$-regular element $g$. \hfill $\square$

Let us put

$$\Delta(g) = \delta_{P'}(g)^{-\frac{1}{2}}|\det d\delta_g|^{-1},$$

where $\delta_g$ is as in the above lemma. As $\delta_{P'}(m) = \delta_{P'}(m)^{\frac{1}{2}}$ if $m \in M'_{H'}$, it follows from (the proof of) the above lemma that $\Delta(g)$ is bi-$M'_{H'}$-invariant. Then by making a change of variable $u = g^{-1}n_1gn_2$ in the integral (4.1), we see that

$$\begin{align*}
O(g, \eta, f') &= \int_{(M'_{H'} \times M'_{H'})_n \setminus M'_{H'} \times M'_{H'}} \int_{N'} \Delta(g)\delta_{P'}(g)^{-\frac{1}{2}}f'_K(m_1gm_2)\delta_{P_{H'}\cap H'}(m_1m_2^{-1})\eta(\det m_2)du dm_1 dm_2 \\
&= \int_{(M'_{H'} \times M'_{H'})_n \setminus M'_{H'} \times M'_{H'}} \int_{N'} \Delta(g)\delta_{P'}(m_1gm_2)^{\frac{1}{2}}f'_K(m_1gm_2u)\eta(\det m_2)du dm_1 dm_2 \\
&= \Delta(g) \int_{(M'_{H'} \times M'_{H'})_n \setminus M'_{H'} \times M'_{H'}} f'^{(P')} (m_1gm_2) \eta(\det m_2) dm_1 dm_2.
\end{align*}$$

This last integral is an orbital integral on $M'$ of the function $f'^{(P')}$. 

4.3. A summary of results on nilpotent orbital integrals. The discussion in the next two subsections is elementary but very technical. So let us summarise the main results here. The readers can (and are encouraged to) take these results for granted and skip the next two subsections for the first reading.

An element $\xi \in \mathfrak{g}'$ or the orbit $O$ it represents is called $\theta$-nilpotent if $\xi$ is nilpotent as an element in $M_{2n}(F)$. The nilpotent cone $\mathcal{N}'$ is the set of all $\theta$-nilpotent element in $\mathfrak{g}'$. The main results are as follows.

(1) We classify all $\theta$-nilpotent orbits in $\mathfrak{g}'$ and describe a necessary and sufficient condition that a $\theta$-nilpotent orbit $O$ supports an $(H', \eta)$-invariant distribution. A $\theta$-nilpotent orbit supporting such a distribution is called visible. The subset of $\mathcal{N}'$ consisting of $\theta$-nilpotent
elements is denoted by $N_0'$. There are two minimal (in the sense of dimension) visible $\theta$-nilpotent orbits

$$
O_{\min}^+ = \begin{pmatrix} 0 & 1_n \\ 1_n & 0 \end{pmatrix}, \quad O_{\min}^- = \begin{pmatrix} 1_n & 0 \\ 0 & 1_n \end{pmatrix}.
$$

(2) We show that the $(H', \eta)$ invariant distribution on a visible $\theta$-nilpotent orbit $O$ extends to an $(H', \eta)$ invariant distribution on $s'$. Such a distribution will be called a nilpotent orbital integral $\mu'_O$. This is the most technical part of the argument.

(3) We prove the homogeneity property for $\mu'_O$. There is an integer $d'_O \geq 0$ for each visible $\theta$-nilpotent orbit $O$. For any $f \in C_c^\infty(s')$, we have

$$
\mu'_O(f_t) = |t|^{d'_O} \eta(t)^n \mu'_O(f).
$$

Moreover $d'_O = 0$ if and only if $O$ is minimal. This follows directly from the definition of the nilpotent orbital integrals.

4.4. The nilpotent cone. To analyse the $\theta$-nilpotent orbits, it is better to use a more canonical formulation. Let $V = V^+ \oplus V^-$ be a $\mathbb{Z}/2\mathbb{Z}$-graded vector space with homogeneous components $V^\pm$ and $\dim V^\pm = n$. Then we have

$$
s' \simeq \text{Hom}(V^+, V^-) \oplus \text{Hom}(V^-, V^+), \quad H' \simeq \text{GL}(V^+) \times \text{GL}(V^-).
$$

The nilpotent cone in $s'$ consists of pairs of endomorphism $\xi = (X, Y) \in \text{End}(V)$, $X \in \text{Hom}(V^+, V^-)$ and $Y \in \text{Hom}(V^-, V^+)$ such that $XY$ and $YX$ are both nilpotent.

Let $\theta \in H$ be the element which acts on $V^\pm$ by $\pm 1$. Then $\theta$ acts on $\mathfrak{gl}(V)$ by sending $Z \in \mathfrak{gl}(V)$ to $\text{ad}(\theta) = \theta Z \theta$. It is clear that $h'$ and $s'$ are eigenspaces of $\text{ad}(\theta)$ of eigenvalue $1$ and $-1$ respectively.

Let $\xi = (X, Y) \in N'$. Then we have a filtration on $V$ given by

$$
0 = W_0 \subset W_1 \subset W_2 \subset \cdots \subset W_r \subset W_{r+1} = V, \quad W_i = \text{Ker} \xi^i.
$$

We may view $V$ as an $F[\xi]$-module and $V$ is a direct sum of indecomposable $F[\xi]$-submodules. By [KP79, Section 4], one can choose the generators of these submodules to be homogeneous. More concretely, let $U$ be such an indecomposable submodule of dimension $a$ over $F$. Then we can choose a homogeneous element $u \in U$ so that

$$
u, \xi^2 \nu, \cdots \xi^{a-1} \nu
$$

form a $F$-basis of $U$. It follows that for each $i$, we have

$$
W_i = W_i^+ \oplus W_i^-, \quad W_i^\pm = W_i \cap V^\pm.
$$

Therefore we have two filtrations

$$
0 = W_0^\pm \subset W_1^\pm \subset W_2^\pm \subset \cdots \subset W_r^\pm \subset W_{s-1}^\pm \subset W_s^\pm = V^\pm.
$$

Note that while the filtration (4.3) is strictly increasing, these two filtration might not be strictly increasing.
We put \( r_i^? = \dim W_i^? / W_i^? \) where ? stands for +, −, or empty. Note that \( \xi \) induces an injective map \( W_{i+1} / W_i \to W_i / W_{i-1} \) for \( i = 1, \cdots, s - 1 \). It follows that \( r_i \geq r_{i+1} \) for all \( i \). Moreover since \( \xi \) induces injective maps \( W_{i+1}^? / W_i^? \to W_i^? / W_{i-1}^? \), we conclude that \( r_i^? \geq r_{i+1}^? \) for all \( i \). By suitably choosing the basis of these successive quotients and lifting them to \( V^\pm \), we may assume that the maps \( W_{i+1}^? / W_i^? \to W_i^? / W_{i-1}^? \) induced by \( \xi \) are all of the form

\[
\begin{pmatrix}
1 & r_i^? - r_{i+1}^? \\
0 & 1
\end{pmatrix},
\]

where 0 stands for the zero matrix of size \((r_i^? - r_{i+1}^?) \times r_i^?\).

Let \( P = MN \) be the parabolic subgroup of \( \text{GL}(V) \) stabilizing the filtration (4.3), and \( P^+ = M^+ N^+ \) be the parabolic subgroup of \( H' \) stabilizing both filtrations (4.4). We have

\[
M^+ \simeq \prod_{i=0}^{s-1} \text{GL}(W_{i+1}^+ / W_i^+) \times \prod_{i=0}^{s-1} \text{GL}(W_{i+1}^- / W_i^-).
\]

**Lemma 4.3.** We have

\[ P \cap H' = P^+, \quad M \cap H' = M^+, \quad N \cap H' = N^+. \]

**Proof.** It follows from the definition that \( P \cap H' \supset P^+ \). If \( h \in H' \cap P \), then \( h(W_i^?) \subset W_i \). But \( h(W_i^?) \subset V^\pm \). It follows that \( h(W_i^?) \subset W_i \cap V^\pm = W_i^\pm \). This proves \( P \cap H' = P^+ \). One can similarly prove the other two equalities. \( \square \)

**Lemma 4.4.** The following assertions hold.

1. We have

\[
\text{Ad}(N^+)\xi = \xi + [n, n] \cap s',
\]

where \([-,-]\) stands for the Lie algebra bracket of \( n \).

2. For any \( h \in H \), if \( \text{Ad}(h)(n \cap s') \subset n \cap s' \), then \( h \in P^+ \).

**Proof.** By [How74, Lemma 2(b)], \( \text{Ad}(N)\xi = \xi + [n, n] \). Note that \( \text{ad}(\theta)\xi = -\xi \). Then both sides of (4.5) are \((-1\)-eigenspaces of \( \text{ad}(\theta) \). This proves the first assertion.

By [How74, Lemma 2(d)], if \( g \in G' \) and \( \text{Ad}(g)\xi \subset n \), then \( g \in P \). Note that \( \xi \in n \cap s' \). Then the second assertion follows from Lemma 4.3. \( \square \)

**Lemma 4.5.** The \( P^+\)-orbit of \( \xi \) in \( s' \) is an (Zariski) open subset of \( n \cap s' \) consisting of elements \( Z \) with the properties that

\[
Z|_{W_{i+1}^\pm / W_i^\pm} : W_{i+1}^\pm / W_i^\pm \to W_i^\mp / W_{i-1}^\mp, \quad i = 1, \cdots, s - 1
\]

is injective.
Proof. Since $\text{Ad}(N^+)\xi$ is the coset $\xi + [n, n] \cap s'$ in $n \cap s'$, it is enough to consider the image of $\text{Ad}(M^+)\xi$ in
\[ n \cap s'/[n, n] \cap s', \]
which is isomorphic to
\[ \bigoplus_{i=1}^{s-1} \text{Hom}(W^+/W^-_{i+1}/W^-_{i}) \oplus \bigoplus_{i=1}^{s-1} \text{Hom}(W^-_{i+1}/W^-_{i}, W^+/W^-_{i}). \]

As explained before, $\xi$ induces an injective map $W^+/W^-_{i+1}/W^-_{i} \to W^+/W^-_{i}$ for all $i$ and with suitable choice of basis, this map is represented by the matrix $\begin{pmatrix} 1 & \pm 1 \\ 0 & 0 \end{pmatrix}$. Moreover by choosing suitable bases, any injective map $W^+/W^-_{i+1}/W^-_{i} \to W^+/W^-_{i}$ can be represented by a matrix of this form. It follows that the image of $\text{Ad}(P^+)\xi$ in $\text{Hom}(W^+/W^-_{i+1}/W^-_{i}, W^+/W^-_{i})$ is the subset of all injective maps. This proves the lemma. \qed

We now study the stabilizer $M^+_\xi$ of $\xi$ in $M^+$. If the $H'$-orbit represented by $\xi$ were to support an $(H', \eta)$-invariant distribution, then $\eta \circ \det$ would have to be trivial on $M^+_\xi$.

We have two chains of injective maps induced by the element $\xi$:
\begin{equation}
W^\epsilon_{s-1}/W^\epsilon_s \hookrightarrow \cdots \hookrightarrow W^\epsilon_3/W^\epsilon_2 \hookrightarrow W^\epsilon_2/W^\epsilon_1 \hookrightarrow W^\epsilon_1,
\end{equation}
where $\epsilon = +$ or $-$ according to the parity of $r$. For each $i$, the map $W^\pm_{i+1}/W^\pm_i \to W^\pm_i/W^\pm_{i-1}$ is either an isomorphism or (genuine) injective and it is an isomorphism if and only if $\dim W^\pm_{i+1}/W^\pm_i = \dim W^\pm_i/W^\pm_{i-1}$. We call the integer $i$ a jump if $\dim W^\pm_{i+1}/W^\pm_i < \dim W^\pm_i/W^\pm_{i-1}$ (either the $+$ one or the $-$ one, the inequality does not have to hold for both filtrations). To unify treatment, we call $s$ a jump if $\dim W^\epsilon_s/W^\epsilon_{s-1} \neq 0$.

**Lemma 4.6.** Suppose that the orbit represented by $\xi$ supports an $(H, \eta)$-invariant distribution. Then all jumps are even integers.

**Proof.** Let $i$ be the smallest jump in one of the chains of injective maps (4.6), say the one ends with $W^+_1$. Assume that $i$ is odd. The last a few terms in the filtration looks like
\[ W^-_{i+1}/W^-_i \hookrightarrow W^+_i/W^+_i \sim W^-_{i-1}/W^-_{i-2} \sim \cdots \sim W^+_1, \]
where the leftmost arrow is injective but not an isomorphism. Let us choose a vector $w \in W^+_1$ which is not in the image of $W^-_{i+1}/W^-_i$. We may choose a basis of $V$ so that it contains a basis of the image of $W^-_{i+1}$ and $w$. Choose $\lambda \in F^\times$ with $\eta(\lambda) = -1$ and let $h \in \text{GL}(M^+)$ be an element such that it acts as multiplication by $\lambda$ on $w$ and trivially on all elements of the basis (see Example 4.7 below for a concrete example). Then by construction $h \in M^+_\xi$. But $\eta(\det h) = (-1)^i = -1$ since $i$ is odd, which contradicts the fact that $\eta \circ \det$ is trivial on $M^+_\xi$. Thus $i$ is even. We may repeat this process for all other jumps. \qed
Summarising the discussion above, we arrive at the following conclusion. To each $\xi \in \mathcal{N}'$, we have constructed two sequences of numbers $r_\pm^i$, $i = 1, \cdots, s$, such that

\[ n = r_1^+ + \cdots + r_s^+, \quad r_1^+ \geq r_2^+ \geq r_3^+ \geq \cdots. \]  

Moreover a necessary condition that a $\theta$-nilpotent orbit supports an $(H', \eta)$ invariant distribution is that the strict inequality $r_i^\epsilon > r_{i+1}^-\epsilon$ ($\epsilon = +$ or $-$) in (4.7) only happens when $i$ is even. Conversely, given any two sequences of integers $r_i^\pm$ satisfying the properties (4.7), one can find an element $\xi \in \mathcal{N}'$ so that $\dim W_i^\pm / W_{i-1}^\pm = r_i^\pm$. This can be achieved as follows. We are going to write $s'$ explicitly as matrices of the form $\begin{pmatrix} X \\ Y \end{pmatrix}$ as before. First write $X$ as a blocked matrix where rows correspond to the partition $n = r_1^+ + \cdots + r_s^+$ and columns correspond to the partition $n = r_1^- + \cdots + r_s^-$. Similarly write $Y$ as a blocked matrix where rows correspond to the partition $n = r_1^- + \cdots + r_s^-$ and columns correspond to the partition $n = r_1^+ + \cdots + r_s^+$. Then $\xi$ is the matrix of following form. All the block entries of $X$ and $Y$ are zero except for the $(i, i+1)$ entry. The $(i, i+1)$ entry of $X$ and $Y$ are of size $r_i^+ \times r_{i+1}^-$ and $r_i^- \times r_{i+1}^+$ respectively and we have $r_i^+ \geq r_{i+1}^-$. The $(i, i+1)$ entry of $X$ and $Y$ are of the form $\begin{pmatrix} 1_{r_{i+1}^+}^\pm \\ 0 \end{pmatrix}$ where $1_{r_{i+1}^+}^\pm$ stands for the identity matrix of size $r_{i+1}^+$ in $X$ and size $r_{i+1}^-$ in $Y$, and $0$ stands for the zero matrix. It is not hard to check that this $\xi$ is the desired $\theta$-nilpotent matrix.

**Example 4.7.** To help the reader understand the discussion in this subsection, we suggest the following example. Let us consider the case $n = 4$ and the nilpotent element $\xi \in s'$ given by the following matrix.

\[
\xi = \begin{pmatrix}
0 & 1 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}, \quad V^+ = \begin{pmatrix}
\ast \\
\ast \\
\ast \\
0
\end{pmatrix}, \quad V^- = \begin{pmatrix}
0 \\
0 \\
\ast \\
\ast
\end{pmatrix}.
\]
Moreover \((r_1^+, r_2^-, r_3^+) = (2, 2, 1)\) and \((r_1^-, r_2^+, r_3^-) = (1, 1, 1)\). The element \(w\) and \(h\) that we chose in the proof of Lemma 4.6 is

\[
W_1^+ = \begin{pmatrix}
\ast & 0 \\
0 & \ast \\
0 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0
\end{pmatrix}, \quad W_2^+ = \begin{pmatrix}
\ast & 0 \\
0 & \ast \\
0 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0
\end{pmatrix}, \quad W_3^+ = \begin{pmatrix}
\ast & 0 \\
0 & \ast \\
0 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0
\end{pmatrix}; \quad W_1^- = \begin{pmatrix}
0 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0
\end{pmatrix}, \quad W_2^- = \begin{pmatrix}
0 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0
\end{pmatrix}, \quad W_3^- = \begin{pmatrix}
0 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0
\end{pmatrix}.
\]

It is straightforward to check that \(h\) commutes with \(\xi\). According to our previous terminology, in the sequence \(r_1^+ = r_2^- > r_3^+\), 2 and 3 are jumps, which are not all even. The orbit represented by \(\xi\) does not support any \((H', \eta)\) invariant distribution.

4.5. Nilpotent orbital integrals. In this subsection, we are going to show that the necessary condition in Lemma 4.6 that a nilpotent orbital integral supports an \((H', \eta)\) invariant distribution is also sufficient. Moreover these \((H', \eta)\) invariant distributions extend to an \((H', \eta)\) invariant distribution on \(s'\).

Let \(\mathcal{O}\) be a nilpotent orbit in \(s'\) represented by an element \(\xi\). Then attached to \(\xi\) is a parabolic subgroup \(P^+ = M^+N^+\) of \(H'\). We also have two sequences of integers \(r_1^+ \geq r_2^+ \geq r_3^+ \geq \cdots\). We assume that all the jumps in these two sequences are even integers. By Lemma 4.6, this is a necessary condition for \(\mathcal{O}\) to support an \((H', \eta)\) invariant distribution.

Let \(2i_1 < \cdots < 2i_a\) be the set of all jumps in the sequence \(r_1^+ \geq r_2^- \geq \cdots\). Let \(2j_1 < \cdots < 2j_b\) be the set of all jumps in the sequence \(r_1^- \geq r_2^+ \geq \cdots\). Note that we either have \(2i_a = s + 1\) and \(W_{s+1}^-/W_s^- \neq 0\), or \(2i_a < s + 1\) and all \(W_{s+i}^-/W_s^- = 0\) if \(i \geq 2i_a\) where \(\epsilon\) is an appropriate sign. We have a similar assertion for the jump \(2j_b\). Then the space \(n \cap s'/[n, n] \cap s'\) is isomorphic to

\[
\bigoplus_{i=1}^{2i_a} \text{Hom}(W_{i+1}^{(-1)}/W_i^{(-1)}, W_i^{(-1)i-1}/W_{i-1}^{(-1)i-1}) \oplus \bigoplus_{i=1}^{2j_b} \text{Hom}(W_{i+1}^{(-1)i+1}/W_i^{(-1)i+1}, W_i^{(-1)}/W_{i-1}^{(-1)})
\]
Let us define some determinant functions. Let us write an element in \( n \cap s_{2n}/[n,n] \cap s_{2n} \) as a sequence

\[
m = (x_1, \cdots, x_{2j_a}; y_1, \cdots, y_{2j_b}),
\]

with

\[
x_i \in \text{Hom}(W_{i+1}^{(-1)^i}/W_i^{-1}, W_i^{(-1)^{i-1}}/W_{i-1}^{(-1)^{i-1}}), \quad y_i \in \text{Hom}(W_{i+1}^{(-1)^{i+1}}/W_i^{(-1)^i}, W_i^{(-1)^{i-1}}/W_{i-1}^{(-1)^{i-1}}).
\]

Note that if \( i \) is odd, then both \( r_{i+1}^\pm = r_i^\mp \) by the assumption that all jumps are even integers. Moreover

\[
\xi|_{W_{i+1}^+/W_i^\pm} : W_{i+1}^+/W_i^\pm \to W_i^+/W_{i-1}^\pm
\]

is an isomorphism. To shorten notation, we put \( \tilde{\xi}_i = \xi|_{W_{i+1}^+/W_i^\pm} \). Define

\[
det_{2i-1}^+(x_{2i-1}) = \det x_{2i-1}(\xi_{2i-1})^{-1}, \quad \det_{2i-1}^-(y_{2i-1}) = \det y_{2i-1}(\xi_{2i-1})^{-1},
\]

and

\[
det_n(m) = det_1^+(x_1) \cdots det_{2j_a-1}^-(x_{2j_a-1}) \cdots det_{2j_b-1}^+(y_{2j_b-1}) \cdots det_{2j_b-1}^-(y_{2j_b-1}).
\]

**Lemma 4.8.** For \( p \in P^+ \) and \( u \in n \cap s' \), we have

\[
\eta(det_n(pup^{-1})) = \eta(det p)\eta(det_n u).
\]

**Proof.** This follows from the definition of \( det_n \). \( \Box \)

Let \( n' \) be the subspace of \( n \cap s' \) generated by \([n,n] \cap s' \) and

\[
\bigoplus_{i \text{ even}} \text{Hom}(W_{i+1}^{+}/W_i^{+}, W_i^{-}/W_{i-1}^{-}) \oplus \bigoplus_{i \text{ even}} \text{Hom}(W_{i+1}^{-}/W_i^{-}, W_i^{+}/W_{i-1}^{+}).
\]

Let \( f \in C_c^\infty(s) \), we define a function \( \tilde{f} \in C_c^\infty(n \cap s'/n') \) as

\[
\tilde{f}(m) = \int_{n'} f(m + u)du.
\]

Before we proceed, let us recall the following result due to Godement and Jacquet [GJ72, Theorem 3.3] (taking the representation \( \pi \) to be \( \eta \circ \det \)). Note that the holomorphy is a consequence of the fact that \( E/F \) is a quadratic extension of nonarchimedean local fields and \( \eta \) is nontrivial.

**Lemma 4.9.** Let \( \varphi \in C_c^\infty(M_n(F)) \). Put

\[
Z(s, \eta, \varphi) = \int_{\text{GL}_n(F)} \varphi(h) |\det h|^s \eta(\det h)dh,
\]

where \( dh \) stands for the multiplicative measure on \( \text{GL}_n(F) \). Then this integral is convergent if \( \Re s \gg 0 \) and has a meromorphic continuation to the whole complex plane. It is holomorphic at all \( s \in \mathbb{R} \).
The function $\tilde{f}$ is a function in the variables

$$m = (x_1, x_3, \ldots, x_{2j_h-1}; y_1, y_3, \ldots, y_{2j_h-1})$$

Let $\underline{s} = (s_1, s_3, \ldots, s_{2j_h-1})$ and $\underline{t} = (t_1, t_3, \ldots, t_{2j_h-1})$ be complex numbers. Put

$$\det_{n, \underline{s}, \underline{t}}(m) = |\det_1^+ (x_1)|^{s_1} |\det_3^+ (x_3)|^{s_3} \cdots |\det_{2j_h-1}^+ (x_{2j_h-1})|^{s_{2j_h-1}}$$

$$|\det_1^- (y_1)|^{t_1} |\det_3^- (y_3)|^{t_3} \cdots |\det_{2j_h-1}^- (y_{2j_h-1})|^{t_{2j_h-1}}.$$

Consider the integral

$$Z(\underline{s}, \underline{t}, \eta, \tilde{f}) = \int \tilde{f}(m) \eta(\det_n(m)) \det_{n, \underline{s}, \underline{t}}(m) dm,$$

where the domain of integration is $n \cap s'/n'$, which is identified with

$$\bigoplus_{i \text{ odd}} \text{Hom}(W_{i+1}^-/W_i^-, W_i^+/W_i^+) \oplus \bigoplus_{i \text{ odd}} \text{Hom}(W_{i+1}^+/W_i^+, W_i^-/W_i^-).$$

By Lemma 4.9, the integral $Z(\underline{s}, \underline{t}, \eta, \tilde{f})$ is convergent when the real part of $s_i$ and $t_i$'s are large enough and $Z(\underline{s}, \underline{t}, \eta, \tilde{f})$ has meromorphic continuation to $\mathbb{C}^{j_0 + j_h}$, which is holomorphic at the points where all $s_i$ and $t_i$'s are integers. We define

$$\tilde{\mu}_\mathcal{O}(f) = Z(\underline{s}, \underline{t}, \eta, \tilde{f}) \bigg|_{s_i = r_i^-, \text{ for all } i \atop t_i = r_i^+, \text{ for all } i}.$$

The point is that for the variable coming from one of the decreasing sequences, we evaluate this integral at the point given by the corresponding integer in the other sequence.

**Lemma 4.10.** For any $f \in C_c^\infty(s')$, and any $p \in P^+$, we have

$$(4.8) \quad \tilde{\mu}_\mathcal{O}(\text{Ad}(p)f) = \delta_{P^+}(p) \eta(\det p) \tilde{\mu}_\mathcal{O}(f).$$

**Proof.** The invariance by elements in $N^+$ is straightforward to check. One has to prove (4.8) for elements in $M^+$. We may even assume that $m \in \text{GL}(W_{i+1}^+/W_i^+)$. The other cases can be derived from this one or follow from the same argument.

Elementary computation shows that

$$\delta_{P^+}(m) = |\det m|^{-(r_i^+ + \cdots + r_{i+2}^+ + \cdots + r_s^+)}.$$

If $i$ is odd, then in computing the integration over $n'$, after changing variables, we obtain

$$|\det m|^{-(r_{i+1}^- + \cdots + r_i^- + r_{i+2}^- + \cdots + r_s^-)}.$$

In computing the integration $Z(\underline{s}, \underline{t}, \eta, \tilde{f})$, by changing the variable, we obtain another term

$$|\det m|^{-r_i^+} \eta(\det m).$$

Note that we have $r_i^\pm = r_i^\pm$ for $i = 1, 3, \ldots$. Thus we conclude

$$-(r_i^+ + \cdots + r_1^+) + r_{i+2}^+ + \cdots + r_s^+ = -(r_{i-1}^- + \cdots + r_1^-) + r_{i+2}^- + \cdots + r_s^- + r_i^+.$$
This proves (4.8) when \( i \) is odd. The case \( i \) being even is similar. □

Let us now choose an open compact subgroup \( K \) of \( H' \) so that \( H' = P^+ K \). Let us put

\[(4.9) \quad f_K(\gamma) = \int_K f(\gamma k) \eta(\det k) dk, \quad \mu'_O(f) = \tilde{\mu}_O(f_K).\]

It follows from [How74, Proposition 4] that the distribution on \( C_c^\infty(s') \) given by \( f \mapsto \mu'_O(f) \) is \((H', \eta)\) invariant. Even though the statement of [How74, Proposition 4] does not involve the extra character \( \eta \), the same argument goes through without change. It is also clear that the linear form \( \mu'_O \) extends the \((H', \eta)\)-invariant distribution on \( O \) to an \((H', \eta)\)-invariant distribution on \( s' \) supported on \( O \).

To summarize, for any nilpotent orbit \( O \) satisfying the necessary condition specified in Lemma 4.6, we have constructed an \((H', \eta)\) invariant distribution \( \mu'_O \) on \( s' \) supported on \( O \). Moreover these distributions, by their very construction, when restricted to \( O \), equal (up to some nonzero constant) the \((H', \eta)\)-invariant distribution on \( O \). In the following, we call such nilpotent orbits visible. We let \( N'_0 \) be the subset of \( N' \) consisting of visible nilpotent orbits. Of course from the discussion above, the set

\[\{\mu'_O \mid O \subset N'_0\}\]

is a natural basis of the space of \((H', \eta)\) invariant distributions on \( N' \).

Let us put \( d'_O = \dim N^+ \).

**Lemma 4.11.** For any \( t \in F^\times \) and \( O \subset N' \) we have

\[\mu'_O(f_t) = |t|^{d'_O} \eta(t)^n \mu'_O(f).\]

Recall that \( f_t(X) = f(t^{-1}X) \).

**Proof.** Suppose that \( O \) is represented by \( \xi \) and gives rise to the sequences of numbers \( r_1^+ \geq r_2^+ \geq \cdots \).

It follows from the definition of \( \mu_O \) that

\[\mu'_O(f_t) = |t|^{\dim n' + 2(r_1^+ r_1^- + r_3^+ r_3^- + \cdots)} \eta(t)^n \mu'_O(f).\]

It is thus enough to prove that

\[(4.10) \quad \dim N^+ = \dim n' + 2(r_1^+ r_1^- + r_3^+ r_3^- + \cdots).\]

We have

\[(4.11) \quad \dim N^+ = \sum_{i=1}^{n} \sum_{j \geq i+1} r_i^+ r_j^+ + r_i^- r_j^- .\]

To organize the terms on the right hand side of (4.10) into a better form, let us write \( 2(r_1^+ r_1^- + r_3^+ r_3^- + \cdots) \) as

\[r_1^+ r_2^+ + r_3^+ r_4^+ + \cdots + r_1^- r_2^- + r_3^- r_4^- + \cdots\]
Then the right hand side becomes

\begin{equation}
\sum_{i \text{ odd}} \left( r_i^+ r_{i+1}^+ + r_i^- r_{i+1}^- + \sum_{j \geq i+2} (r_i^+ r_j^- + r_i^- r_j^+) \right) + \sum_{i \text{ even}} \sum_{j \geq i+1} (r_i^+ r_j^- + r_i^- r_j^+). \tag{4.12}
\end{equation}

Let $i$ be an integer. In computing the dimension of $N^+$, the terms involving $r_i^+$ are $r_i^+ (r_{i+1}^+ + r_{i+2}^+ r_{i+3}^+ + \cdots)$. If $i$ is odd, then on the right hand side of (4.10), the terms involving $r_i^+$ are

$$r_i^+ r_{i+1}^+ + r_i^+ (r_{i+2}^- + r_{i+3}^- + \cdots).$$

If $i$ is even, then we have

$$r_i^- (r_{i+1}^+ + r_{i+2}^+ + \cdots).$$

Note that we have $r_1^+ = r_2^-$, $r_3^+ = r_4^-$ etc. So we conclude that for a fixed $i$, the terms in (4.11) and in (4.12) involving $r_i^+$ coincide. Similarly we can conclude that the terms involving $r_i^-$ coincide. Thus we conclude that (4.11) and (4.12) are the same, i.e. the identity (4.10) holds. This proves the lemma.

\begin{flushright}
\begin{small}
□
\end{small}
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**Example 4.12.** To facilitate understanding, the reader is encouraged to look at the following example. Let $O$ be the nilpotent orbit represented by

$$\xi = \begin{pmatrix} 0 & 1 \\ 0 & 1 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}$$

We have $r_1^+ = r_2^- = r_3^+ = r_4^- = 1$ and $r_1^- = r_2^+ = 2 > r_3^- = r_4^+ = 0$. So this orbit is visible. The spaces $[n, n] \cap s'$, $n \cap s'/[n, n] \cap s'$, and $n'$ look like the following respectively

$$\begin{pmatrix} 0 & * \\ 0 & * \\ 0 & * \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & * \\ 0 & * \\ 0 & * \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & * \\ 0 & * \\ 0 & * \\ 0 & 0 \end{pmatrix}$$

In this case, direct computation shows that we have $\mu_O(f_t) = |t|^{10} \mu_O(f)$. This is compatible with Lemma 4.11.
5. The Spherical character: split case

5.1. Definitions and first properties. Let $\sigma \in \text{GL}_{2n}(F)$ be the permutation matrix corresponding to the permutation that sends $123\cdots(2n)$ to $135\cdots(2n-1)246\cdots(2n)$. If $g \in M_{2n}(F)$, then we put $g^\sigma = \sigma g \sigma$. This is an involution on $M_{2n}(F)$. If $C$ is a subset of $M_{2n}(F)$, we put $C^\sigma = \sigma C \sigma$.

Recall that we have put $G' = \text{GL}_{2n,F}$ and $H' = \text{GL}_{n,F} \times \text{GL}_{n,F}$ and $H'$ embeds in $G'$ via diagonal blocks. Let $Z'$ be the center of $G'$, $T'$ be the diagonal torus of $G'$, $B' = T'N'$ be the standard upper triangular Borel subgroup of $G'$ and $P'$ be the mirabolic subgroup of $G'$, i.e. the subgroup of elements whose last row is $(0, \ldots, 0, 1)$.

Let $\pi'$ be an irreducible generic representation of $G'$. We say that $\pi'$ is of $H'$-distinguished if $\text{Hom}_{H'}(\pi', \mathbb{C}) \neq 0$. Note that this is equivalent to $\text{Hom}_{H'}(\pi', \mathbb{C}) \neq 0$. This is because if $l$ is an $H'$-invariant linear form, then $l \circ \pi'(\sigma)$ is an $H'^\sigma$-invariant linear form. It is known that $\pi$ is self-dual and has trivial central character, and moreover $\text{Hom}_{H'}(\pi', \mathbb{C})$, hence $\text{Hom}_{H'}(\pi', \mathbb{C})$, are at most one dimensional [JR96]. Assume that $\pi'$ is $H'$-distinguished. Recall that we have fixed a nontrivial additive character $\psi$ and it naturally defines a generic character of $N'$ as usual. Let $W = W(\pi', \psi)$ the corresponding Whittaker model of $\pi'$. Define

$$l(W) = \int_{H'^\sigma \cap N' \setminus H'^\sigma \cap P'} W(p) dp.$$ 

Then this integral is absolutely convergent and $l$ defines a nonzero element in $\text{Hom}_{H'}(\pi', \mathbb{C})$, cf. [LM15a, Proposition 3.2]. We say $\pi'$ is $(H', \eta)$-distinguished if $\text{Hom}_{H'}(\pi' \otimes \eta, \mathbb{C}) \neq 0$. We also put

$$l_\eta(W) = \int_{H'^\sigma \cap N' \setminus H'^\sigma \cap P'} W(p) \eta(\det p) dp.$$ 

We denote by $\epsilon(s, \pi', \psi)$ the standard epsilon factor attached to $\pi'$ and put $\epsilon(\pi') = \epsilon(1/2, \pi', \psi) = \pm 1$. The second equality follows from the fact $\pi'$ is self-dual. Let $\pi'_E$ be the base change of $\pi'$ to $G'(E)$. Then $\epsilon(\pi'_E) = \epsilon(\pi') \epsilon(\pi' \otimes \eta)$.

Let $w \in G'$ be the longest Weyl element.

**Proposition 5.1.** We have $l(\pi'(w)W) = \epsilon(\pi')l(W)$ and $l_\eta(\pi'(w)W) = \epsilon(\pi' \otimes \eta) \eta(-1)^n l_\eta(W)$.

**Proof.** The first equality is [LM17, Theorem 3.2]. The second one follows from the first one because $l_\eta$ is the linear form $l$ for $\pi' \otimes \eta$. \hfill $\square$

Define a spherical character as follows. For any $f' \in C_c^\infty(G')$, put

$$I_{\pi'}(f') = \sum_{W \in W(\pi', \psi)} l(\pi'(f')W) \overline{l_\eta(W)},$$

where the sum runs over an orthonormal basis of $W(\pi', \psi)$.

Define an involution on $t_{\pi'}$ on $C_c^\infty(G')$ as follows. Let $f' \in C_c^\infty(G')$ and put $t_{\pi'}f'$ be the function on $G'$ given by $t_{\pi'}f'(g) = f'(wgw)$.

**Corollary 5.2.** We have $I_{\pi'}(t_{\pi'}f') = \epsilon(\pi'_E) \eta(-1)^n I_{\pi'}(f')$. 

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Proof. We have
\[ I_{\pi'}(f') = \sum_{W \in \mathcal{W}(\pi', \psi)} l(\pi'(w)\pi'(f')\pi'(w)W)l_\eta(W) \]
\[ = \sum_{W \in \mathcal{W}(\pi', \psi)} l(\pi'(w)\pi'(f')W)l_\eta(\pi'(w)W) \]
\[ = \epsilon(\pi')\epsilon(\pi' \otimes \eta)\eta(-1)^n \sum_{W \in \mathcal{W}(\pi', \psi)} l(\pi'(f')W)l_\eta(W). \]

This proves the corollary. \qed

5.2. The local character expansion. Recall that for any test function \( f' \in C_c^\infty(G') \), we have defined \( f'_\sharp \in C_c^\infty(s') \), cf. Subsection 4.1. For any \( f' \in C_c^\infty(G') \), we put \( f'^{\sigma}(g) = f'(\sigma g \sigma) \). For brevity, we put \( f'^{\sigma}_\sharp = (f'^{\sigma})_\sharp \).

We have a natural bilinear pairing in \( s' \) given by \( (\gamma, \delta) \mapsto \text{Tr} \gamma \delta \) where \( \gamma, \delta \in s' \). Using this pairing and the self-dual measure on \( s' \), we have a Fourier transform on \( C_c^\infty(s') \), which we denote by \( \hat{\cdot} \). Moreover we have the Fourier transform of distributions on \( s' \), which is again denoted by \( \hat{\cdot} \).

In particular, we have the Fourier transform of the orbital integrals.

**Proposition 5.3.** There are uniquely determined constants \( c'_O \) depending on \( \pi \) and the visible nilpotent orbit \( O \subset N' \), but not on the test function, so that if the support of \( f' \in C_c^\infty(G') \) is a small neighbourhood around identity, then
\[ I_{\pi'}(f') = \sum_{O \subset N'_0} c'_O \mu'^{\sigma}_O(f'^{\sigma}). \]

**Proof.** This follows from the [RR96, Theorem 7.11]. Even though our spherical character \( I_{\pi'} \) involves a twist \( \eta \), making it \( (H'^{\sigma}, \eta) \) invariant from the right, the argument in [RR96] goes through without change. The essential ingredient, Howe’s Finiteness Theorem for the Lie algebra,, holds in this context. \qed

Conjugation by \( w \) is an involution on \( s' \). A notable feature of this involution is that it preserves the nilpotent cone and defines an involution on the set of nilpotent orbits. Moreover it is not hard to check that \( O \) is visible, then \( {}^tO = \text{Ad}(w)O \) is also visible. It is clear that \( {}^tO_\min^\pm = O_\min^\pm \).

**Lemma 5.4.** We have \( c'_{O_\min^\pm} = \epsilon(\pi'_E)\eta(-1)^n c'_{O_\min^\pm} \) for all \( O \subset N' \).

**Proof.** We only need to prove that if \( f' \) has small support around the identity, then
\[ \mu'^{\sigma}_O(\mu'^{\sigma}_{O}(f'^{\sigma})) = \mu'^{\sigma}_O(f'^{\sigma}). \]
This can be achieved via a simple change of variable. \qed
5.3. Minimal nilpotent orbital integrals of matrix coefficients. Let $\pi'$ be a supercuspidal representation of $G'$, both $H'$-distinguished and $(H', \eta)$-distinguished. We say that $f' \in C_c^\infty(G')$ is an essential matrix coefficient of $\pi'$ if the function on $G'$ given by

$$g \mapsto \int_{Z_c(F)} f(zg) \, dz$$

is a matrix coefficient of $\pi'$. The goal of this subsection is to prove the following result.

**Proposition 5.5.** There is a nonzero constant $C'$ so that for any $f' \in C_c^\infty(G')$ that is essentially a matrix coefficient of $\pi'$, we have

$$\mu'_{\mathcal{O}^+_{\min}}(f'^{\sigma}) = C'I_{\pi'}(f').$$

The proof of Proposition 5.5 is a rather involved computation. Before we embark on the proof of it, we note the following corollary.

**Corollary 5.6.** There is an essential matrix coefficient $f' \in C_c^\infty(G')$ so that

$$\mu'_{\mathcal{O}^+_{\min}}(f'^{\sigma}) + \epsilon(\pi'_E)\eta(-1)^n\mu'_{\mathcal{O}^-_{\min}}(f'^{\sigma}) \neq 0.$$

**Proof.** It is straightforward to check that

$$\mu'_{\mathcal{O}^+_{\min}}(f'^{\sigma}) = \mu'_{\mathcal{O}^+_{\min}}(t' f' \sigma) = C'I_{\pi'}(t' f') = \epsilon(\pi'_E)\eta(-1)^n C'I_{\pi'}(f').$$

Therefore

$$\mu'_{\mathcal{O}^+_{\min}}(f'^{\sigma}) + \epsilon(\pi'_E)\eta(-1)^n\mu'_{\mathcal{O}^-_{\min}}(f'^{\sigma}) = 2C'I_{\pi'}(f').$$

There is an $f'$ which is essentially a matrix coefficient of $\pi'$ so that $I_{\pi'}(f') \neq 0$ as $\pi'$ is $H'$-distinguished and $(H', \eta)$-distinguished. The corollary then follows. \qed

**Remark 5.7.** These results should again hold for discrete series representation of $G'$. To handle the case of discrete series representation, we need to address the convergence issues. What we need is that the orbital integrals (including the $\theta$-unipotent ones which we haven’t defined) extend continuously to the space of Harish-Chandra Schwartz functions. This again requires Howe’s finiteness theorem for the symmetric space $H' \backslash G'$.

Let us now begin the proof of Proposition 5.5. We are going to deduce it from a general unfolding identity. We let $\xi_{1,-}$ be the matrix in $M_n(F)$ with $(i, i - 1)$ ($i = 2, \cdots, n$) entries being all 1’s and all other entries being zero. Let $\xi_- = \begin{pmatrix} 1_n \\ \xi_{1,-} \end{pmatrix} \in \mathfrak{s}'$. Note that $\xi'^{\sigma} \in M_{2n}(F)$ is the matrix with all $(i, i - 1)$ entries being 1, $i = 2, \cdots, 2n$, and all other entries being zero.

Let $f' \in C_c^\infty(G')$ and we define a function on $G' \times G'$ by

$$W_{f'}(g_1, g_2) = \int_{N'} f'(g_1^{-1} u g_2) \psi(\text{Tr} \xi'^{\sigma} u) \, du.$$
Proposition 5.8. Let $f' \in C^\infty_c(G')$ and $W_{f'}$ as above. Then there is a nonzero constant $C_1$ independent of $f'$ and we have

$$\mu'_{C_{\min}}(f'^\sigma) = C_1 \int_{H' \cap N'} W_{f'}(h_1, h_2) \eta(h_2) dh_1 dh_2,$$

where the right hand side is absolutely convergent.

Proof of Proposition 5.8. To facilitate understanding, we proceed in steps.

Step 1: We check that the right hand side of (5.1) is absolutely convergent.

We are going to make use of the following notation in this step. Let $T'$ be the diagonal torus of $G'$. Let $T_1$ be diagonal torus in $P'$, $N_1 = N' \cap H'$ and $K_1$ be a maximal compact subgroup in $\text{GL}_n(F) \times \text{GL}_{n-1}(F)$. Then we have the Iwasawa decomposition $P' \cap H' = N_1 T_1 K_1$. Similarly let $T_2$ be the diagonal torus in $H'$, $N_2 = N' \cap H'$ and $K_2$ be a maximal compact subgroup in $H'$. Then we have the Iwasawa decomposition $H' = N_2 T_2 K_2$. We let $\delta$ be the modulus character of $T'$ with respect to $N'$ and $\delta_1, \delta_2$ be the modulus character of $T_1$ and $T_2$ with respect to $N_1 \cap M_{P'}$ and $N_2$ respectively, where $M_{P'} \simeq \text{GL}_n(F) \times \text{GL}_{n-1}(F)$ is the reductive part of $P'$. If $a$ is a diagonal matrix in $G'$, we let $a_i$ be its $i$-th diagonal entry. Let us also put $\zeta(x) = \max\{\log|x|, \log|x|^{-1}\}$ and

$$\zeta(a) = \max_{1 \leq i \leq n} \zeta(a_i).$$

Let $r$ and $\nu$ be the function on $G'$ defined by $r(g) = |\det g|^{\frac{1}{4}}$ and $\nu(g) = 1 + \|e_2 g\|$ where $\|\cdot\|$ stands for the $L^\infty$ norm on $F_{2n}$. Choose a large integer $N$ which will be determined later, and apply [BPb, Lemma 2.4.3] to $r^N \nu f$, we end up with the estimate

$$|W_f(a_1 k_1, a_2 k_2)| \ll |\det a_1^{-1} a_2|^{\frac{1}{2}} \left(1 + |a_{1,2n-1}|\right)^{-N} \left(1 + |a_{2,2n}|\right)^{-N} \prod_{i=1}^{2n-2} \left(1 + \left|\frac{a_{1,i}}{a_{1,i+1}}\right|\right)^{-N} \prod_{i=1}^{2n-1} \left(1 + \left|\frac{a_{2,i}}{a_{2,i+1}}\right|\right)^{-N} \delta(a_1 a_2)^\frac{1}{2} \delta_1(a_1)^{-1} \delta_2(a_2)^{-1} \zeta(a_1)^d \zeta(a_2)^d,$$

for some integer $d > 0$, where $a_i \in T_i$, $k_i \in K_i$, $i = 1, 2$.

Therefore to prove the convergence of the right hand side of (5.1), we need to prove that

$$\int_{T_1} \int_{T_2} |\det a_1^{-1} a_2|^{\frac{1}{2}} \left(1 + |a_{1,2n-1}|\right)^{-N} \left(1 + |a_{2,2n}|\right)^{-N} \prod_{i=1}^{2n-2} \left(1 + \left|\frac{a_{1,i}}{a_{1,i+1}}\right|\right)^{-N} \prod_{i=1}^{2n-1} \left(1 + \left|\frac{a_{2,i}}{a_{2,i+1}}\right|\right)^{-N} \delta(a_1 a_2)^\frac{1}{2} \delta_1(a_1)^{-1} \delta_2(a_2)^{-1} \zeta(a_1)^d da_1 da_2$$

is absolutely convergent for sufficiently large $N$. Note that

$$\delta(a_1)^{-\frac{1}{2}} \delta_1(a_1)^{-1} = |\det a_1|^{\frac{1}{2}}, \quad \delta(a_2)^{-\frac{1}{2}} \delta_2(a_2)^{-1} = \prod_{i=1}^{n} \left|\frac{a_{2,2i-1}}{a_{2,2i}}\right|^{\frac{1}{2}}.$$

Thus we need to prove that both integrals

$$\int_{T_1} |\det a_1|^{\frac{1}{2}} \left(1 + |a_{1,2n-1}|\right)^{-N} \prod_{i=1}^{2n-2} \left(1 + \left|\frac{a_{1,i}}{a_{1,i+1}}\right|\right)^{-N} \zeta(a_1)^d da_1,$$
and
\[
\int_{T^2} \prod_{i=1}^n \left( a_{2i-1} \right)^{3/2} \left( a_{2i} \right)^{-3/2} \left( 1 + \left| a_{1,2n-1} \right| \right)^{-N} \prod_{i=1}^{2n-2} \left( 1 + \left| \frac{a_{2,i}}{a_{2,i+1}} \right| \right)^{-N} \zeta(a_2)^d da_2
\]
are absolutely convergent for sufficiently large \( N \). Both of these are implied by the following claim, which can be proved via a simple change of variable.

**Claim.** Let \( s_1, \ldots, s_n \) be real numbers. Fix \( d > 0 \). If \( s_1 + \cdots + s_k > 0 \) for all \( k = 1, \ldots, n \), then we can find a large \( N \) so that the integral
\[
\int_{F^x} (1 + |x_n|)^{-N} \prod_{i=1}^{n-1} \left( 1 + \left| \frac{x_i}{x_{i+1}} \right| \right)^{-N} \prod_{i=1}^n |x_i|^{s_i}(x_i)^d dx_1 \cdots dx_n
\]
is absolutely convergent.

This proves the absolute convergence of the right hand side of (5.1).

**Step 2:** We reduce (5.1) to an equality on the Lie algebra.

The left hand side of (5.1) is already on the Lie algebra and we make it more explicit. Let us define the partial Fourier transform of \( f' \in C^\infty_c(\mathfrak{g}') \) by
\[
\mathcal{F}_{\psi - \frac{1}{2}} f' \left( \begin{array}{c} Y \\ X \end{array} \right) = \int_{M_n(F)} f' \left( \begin{array}{c} Y \\ X' \end{array} \right) \psi_{-\frac{1}{2}}(\text{Tr} X'X) dX'.
\]
Then according to our definition of the nilpotent orbital integrals, we have
\[
\mu'_{\text{orb}, \text{min}} (f_2^0) = \gamma_{\text{GL}_n}(s, 1, \eta, \psi_{-\frac{1}{2}}) \bigg|_{s=0} \int_{\text{GL}_n(F)} \mathcal{F}_{\psi - \frac{1}{2}} f_2^0 \left( \begin{array}{c} h \\ 0 \end{array} \right) \eta(\text{det } h) dh,
\]
where \( dh \) stands for the restriction to \( \text{GL}_n(F) \) the additive measure on \( M_n(F) \), and \( \gamma_{\text{GL}_n}(s, 1, \eta, \psi_{-\frac{1}{2}}) \) is the Godement–Jacquet standard gamma factor attached to the trivial representation of \( \text{GL}_n(F) \), which is holomorphic and nonzero at \( s = 0 \), c.f. Lemma 4.9.

Let us now compute the right hand side of (5.1). Plugging in the definition of \( W' \), we have
\[
\text{RHS of (5.1)} = \int_{H^\sigma \cap N' \cap H'^\sigma \cap P'} \int_{H^\sigma \cap N' \cap H'^\sigma \cap N'} \int_{H^\sigma \cap H'^\sigma} \int_{H^\sigma \cap H'^\sigma} \int_{H^\sigma \cap H'^\sigma} f'(h_1^{-1}uh_2) \overline{\psi(\text{Tr } \xi u)} \eta(\text{det } h_2) dudh_1 dh_2
\]
\[
= \int_{H^\sigma \cap N' \cap H'^\sigma \cap P'} \int_{H^\sigma \cap N' \cap H'^\sigma \cap N'} \int_{H^\sigma} \int_{H^\sigma} \int_{H^\sigma} f'(h_1^{-1}uh_2) \overline{\psi(\text{Tr } \xi u)} \eta(\text{det } h_2) dudh_1 dh_2
\]
\[
= \int_{H^\sigma \cap N' \cap H'^\sigma \cap P'} \int_{N' \cap H'^\sigma \cap N'} \int_{H^\sigma} \int_{H^\sigma} \int_{H^\sigma} \int_{H^\sigma} \overline{\psi(\text{Tr } \xi u)} \eta(\text{det } h_2) dudh_1 dh_2.
\]
The second equality is valid as the inner two integrals are absolutely convergent. The third identity is the definition of \( \bar{f}' \). Note that the Cayley transform \( \varsigma \) is submersive and a bijection and of determinant one when restricted to \( \mathfrak{n}' \cap \mathfrak{s}' \). Moreover the map
\[
N'^\sigma \cap H' \cap N'^\sigma \rightarrow N' \cap S', \quad u \mapsto u^{-1} \theta(u)
\]
is submersive everywhere and a bijection and of determinant \( |2|^{\dim N'^\sigma \cap H' \cap N'^\sigma} \). Moreover
\[
\psi(\text{Tr } \xi u) = \psi_{-\frac{1}{2}}(\text{Tr } \xi u), \quad \text{if } u' \in \mathfrak{n}' \cap \mathfrak{s}', \quad \varsigma(u') = u^{-1} \theta(u).
\]
We thus conclude that the right hand side of (5.1) equals \(2^{\dim N' \cap H' \backslash N' \cap H'}\) times
\[
(5.2) \quad \int_{H' \cap N' \backslash H' \cap P''} \int_{P'' \cap H'} f_2'(h_2^{-1}uh_2) \overline{\varphi_{-\frac{1}{2}}(u)} \eta(\det h_2) dh \, du.
\]
As \(\mathcal{F}_{-\frac{1}{2}}\) is a bijection from \(C_c^\infty(s')\) to itself, to prove Proposition 5.8, it is enough to prove that for any \(f_1' \in C_c^\infty(s')\), we have
\[
(5.3) \quad \int_{GL_n(F)} f_1'(a) \|h\| \eta(\det h) dh = \int_{H' \cap N' \backslash H' \cap P''} \int_{P'' \cap H'} \mathcal{F}^{-1}_{-\frac{1}{2}} f_1'(h^{-1}uh) \overline{\varphi_{-\frac{1}{2}}(u)} \eta(\det h) dh \, du,
\]
where the measure \(dh\) on the left hand side is the additive measure. Observe that the right hand side of this equality is independent of the choice of the additive character \(\varphi\). Thus it is enough to prove the same identity with \(\varphi_{-\frac{1}{2}}\) replaced by \(\varphi\). This observation simplifies the notation a little bit in the following. In what follows, by (5.3), we mean the identity with \(\varphi_{-\frac{1}{2}}\) replaced by \(\varphi\).

**Step 3:** Computing the right hand side of (5.3) via Fourier inversion formula.

For the rest of the proof, we are going to temporarily use the following notation. We let \(G_1 = GL_n(F), B_1\) be the upper triangular Borel subgroup, \(N_1\) its unipotent subgroup, \(B_{1,-}\) and \(N_{1,-}\) be the opposite of \(B_1\) and \(N_1\) respectively. We let \(b_1, n_1, b_{1,-}, n_{1,-}\) be the Lie algebra of \(B_1, N_1, B_{1,-}\) and \(N_{1,-}\) respectively. Similarly we put \(G_2 = GL_{n-1}(F), B_2, N_2, B_{2,-}, N_{2,-}\) and their Lie algebras \(b_2, n_2, b_{2,-}, n_{2,-}\). The group \(G_2\) is embedded in \(G_1\) via \(a \mapsto \begin{pmatrix} a & 0 \\ 1 & 1 \end{pmatrix}\) and so are their subgroups.

**Claim:** For any \(h = \begin{pmatrix} h_1 & \hfill \\ h_2 & \hfill \end{pmatrix} \in H'\), we have
\[
\int_{n' \cap g'} \mathcal{F}^{-1}_{-\frac{1}{2}} f_1'(h^{-1}uh) \varphi(u) du = |\det h_2^{-1}h_1|^n \int_{n_2} \int_{N_1} f'(h_2^{-1}u_2 h_1) \psi(Tr \xi_{1,-} u_2) du_1 du_2.
\]

The right hand side is absolutely convergent.

In fact, this is an application of the Fourier inversion formula. Explicitly the left hand side of the lemma equals
\[
\int_{n_1} \int_{b_1} \int_{M_n(F)} f'(h_2^{-1}u_2 h_1) X \psi(-Tr X h_1^{-1} u_1 h_2) \psi_2(Tr u_1) \psi(Tr \xi_{1,-} u_2) dX du_1 du_2.
\]
The integral is convergent in this order. Make a change of variable \(X \mapsto h_2^{-1} X h_1\). Then the integral above equals
\[
\int_{n_1} \int_{b_1} \int_{M_n(F)} f'(h_2^{-1}u_2 h_1) X h_1 \psi(-Tr X u_1) \psi_2(Tr u_1) \psi_2(Tr \xi_{1,-} u_2) |\det h_2^{-1} h_1|^n dX du_1 du_2.
\]
Applying Fourier inversion formula to the inner two integrals, we obtain that this integral equals
\[
\int_{n_1} \int_{N_1} f'(h_2^{-1}u_2 h_1) \psi(Tr \xi_{1,-} u_2) |\det h_2^{-1} h_1|^n du_3 du_2.
\]
This proves the claim.

Thus to prove (5.3), we only need to compute

\[
(5.4) \int_{N_1 \setminus G_1} \int_{N_2 \setminus G_2} \int_{n_1} \int_{N_1} f'(h_1^{-1}u_2h_1) \psi_2(\text{Tr} \xi_{1,-}u_2)|\det h_1^{-1}h_1|^n\eta(\det h_1h_2)du_3du_2dh_2dh_1.
\]

It is straightforward to see that we can (and will) change the order of the inner two integrals or the out two integrals. Combining the integral against \(u_3\) and \(h_1\) and making a change of variable \(h_1 \mapsto h_2h_1\), we have that

\[
(5.4) = \int_{N_2 \setminus G_2} \int_{G_1} \int_{n_1} f(h_1^{-1}u_2h_2h_1) \psi_2(\text{Tr} \xi_{1,-}u_2)|\det h_1|^n\eta(\det h_1)du_2dh_1dh_2.
\]

**Step 4.** An unfolding argument.

It is clear from Step 3 that to prove Proposition 5.8, it is enough to prove the following claim.

**Claim:** For any \(f \in C^\infty_c(M_n(F))\), we have

\[
f(0) = \int_{N_2 \setminus G_2} \int_{n_1} f(h^{-1}uh)\psi_2(\text{Tr} \xi_{1,-}u)dudh.
\]

The proof of the claim is an unfolding argument very similar to that of [LM15b, Lemma 4.4]. We replace the integration over \(N_2 \setminus G_2\) with the integration over \(B_{2,-}\) and recall that we are using the right invariant Haar measure on \(B_{2,-}\). We temporarily introduce the following notation. We let \(A_i, i = 0, \ldots, n - 1\) be the subgroup of \(B_{2,-}\) consisting of elements whose upper left \(i \times i\) block is the identity matrix. We let \(L_i, i = 0, \ldots, n - 1\), be the subspace of \(n_1\) consisting of matrices whose upper left \((i + 1) \times (i + 1)\) block is zero. Let us introduce the following auxiliary integral

\[
I_i = \int_{A_i} \int_{L_i} f(h_i^{-1}u_ih_i)\psi(\text{Tr} \xi_{1,-}u_i)du_idh_i.
\]

The measure \(dh_i\) is the right invariant measure on \(A_i\). Of course, \(I_0\) is the right hand side of the equality in the claim, while the \(I_{n-1} = 0\). We are going to prove that \(I_i = I_{i+1}\) for all \(i\) and this will prove the claim.

Let \(h_i \in A_i\). We write \(h_i = ah_{i+1}\) where \(h_{i+1} \in A_{i+1}\) and \(a\) takes the following form

\[
a = \begin{pmatrix} 1_i & v_i & x_i \\ v_i & x_i & 1_{n-i-2} \end{pmatrix}, \quad v_i \in F_i, \ x_i \in F^x.
\]

The measure \(dh_i\) decomposes as

\[
dh_i = |x_i|^{-(n-i-2)}dx_idv_idh_{i+1},
\]

where \(dx_i\) is the multiplicative measure on \(F^x\).
Let \( u_i \in L_i \). We write \( u_i = c + u_{i+1} \) where \( u_{i+1} \in L_{i+1} \) and \( c \) takes the following form
\[
c = \begin{pmatrix} 0_{i+1} & w_{i+1} \\ 0 & 0_{n-i-2} \end{pmatrix}, \quad w_{i+1} \in F^{i+1}.
\]

Then we have
\[
I_i = \int_{A_{i+1}} \int_{F_i} \int_{F} \int_{F^{i+1}} \int_{L_{i+1}} f(h^{-1}_{i+1} a^{-1}(c + u_{i+1})a h_{i+1}) \psi(\text{Tr} \xi_{1,-}(c + u_{i+1}))
\]
\[
|x_i|^{-(n-i-2)} du_{i+1} dc dx_i dv_i dh_{i+1}.
\]

Let us make change of variables \( c \mapsto ac^{-1} \) and \( u_{i+1} \mapsto au_{i+1}a^{-1} \). Let us note that
\[
\text{Tr} \xi_{1,-}aca^{-1} = (v_i, x_i) w_{i+1}, \quad \text{Tr} \xi_{1,-}au_{i+1}a^{-1} = \text{Tr} \xi_{1,-}u_{i+1},
\]
and
\[
daca^{-1} = |x_i| dc, \quad dau_{i+1}a^{-1} = |x_i|^{n-i-2} du_{i+1}.
\]
Then we have
\[
I_i = \int_{A_{i+1}} \int_{F_i} \int_{F} \int_{F^{i+1}} \int_{L_{i+1}} f(h^{-1}_{i+1}(c + u_{i+1})h_{i+1}) \psi((v_i, x_i) w_{i+1}) \psi(\text{Tr} \xi_{1,-}u_{i+1})
\]
\[
|x_i| du_{i+1} dc dx_i dv_i dh_{i+1}.
\]

Note that \(|x_i| dx_i \) gives the additive measure on \( F \). We apply the Fourier inversion formula to the integration over \( w_{i+1} \) and \((v_i, x_i) \). It follows that
\[
I_i = \int_{A_{i+1}} \int_{L_{i+1}} f(h^{-1}_{i+1} u_{i+1} h_{i+1}) \psi(\text{Tr} \xi_{1,-}u_{i+1}) du_{i+1} dh_{i+1}.
\]

The right hand side is precisely the definition of \( I_{i+1} \). This proves the claim.

This finishes the proof of Proposition 5.8. \( \square \)

**Remark 5.9.** We present a slightly more conceptual proof of the claim in step 4. In fact, it is enough to prove the same identity with \( f \) replaced by its Fourier transform \( \hat{f} \) (defined by \( \psi_2 \)). We also replace the integration over \( N_2 \backslash G_2 \) by integration over \( B_{2,-} \) (recall that we are using the right invariant measure). Then we have
\[
\int_{B_{2,-}} \int_{B_{2,+}} \hat{f}(h^{-1} u h) \psi_2(\text{Tr} \xi_{1,-}u) du dh = \int_{B_{2,-}} \int_{B_{2,+}} f(h^{-1}(\xi_{1,-} + b)h) db dh,
\]

Consider the map
\[
\rho : B_{2,-} \times B_1 \to M_n(F), \quad (b_2, X) \mapsto b_2^{-1}(\xi_{1,-} + X)b_2.
\]
The claimed identity then follows from the fact that \( \rho \) is injective and has Zariski dense image and the Jacobian of \( \rho \) equals \( \pm |x_1|^{-(n-2)} |x_2|^{-(n-3)} \cdots |x_{n-2}|^{-1} \) if the diagonal entries of \( b \) are \( x_1, \ldots, x_{n-2} \). We leave this last fact to the interested reader.

We now prove the main result of this subsection.
Proof of Proposition 5.5. Let \( f' \in C_c^\infty(G') \) be an essential matrix coefficient of a supercuspidal representation \( \pi' \). Then by a result of Lapid and Mao [LM15b, Lemma 4.4], if

\[
\int_{Z'} f'(zg) dz = \langle \pi'(g) W_1, W_2 \rangle,
\]

where \( W_1, W_2 \in W \), then

\[
\int_{Z'} W f'(zg_1, g_2) dz = W_1(g_1) W_2(g_2).
\]

Thus by Proposition 5.8, Proposition 5.5 will follow if we can prove that there is a nonzero constant \( c \) so that

\[
(5.5) \quad l_\eta(W) = c \cdot \int_{Z'(H' \cap N') \setminus H'} W(h) \eta(\det h) dh,
\]

where the right hand side is absolutely convergent. This is a consequence of the functional of equation of Matringe [Mat15, Mat17] which we now explain.

For the moment, let \( \pi' \) be a (unitary) generic representation \( G' \). For any \( t \in \mathbb{C} \), we define a character \( \chi_t(h) = |\det h_1 h_2^{-1}|^t \) where \( h = \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} \in H' \). Let \( W \in W(\pi', \psi) \) and \( \phi \in C_c^\infty(F_n) \).

Put

\[
\Psi(s, t, W, \phi) = \int_{N' \cap H' \setminus H'} W(h) \phi(e_n h_2) \chi_{t+\frac{1}{2}}(h) |\det h|^s dh, \quad h = \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}^\sigma.
\]

We will only make use of this integral when \( t = 0 \) or \( t = -\frac{1}{2} \). By [Mat15, Proposition 4.16], for a fixed \( t \), this integral is convergent when the real part of \( s \) is large, and it has a meromorphic continuation (in the variable \( s \)) to the whole complex plane. Moreover there is a unit \( e^{lin}(s, \pi, \psi) \) in \( \mathbb{C}[q^\pm s] \) that does not depend on \( W \) and \( \phi \) so that

\[
(5.6) \quad \frac{\Psi \left( \frac{1}{2} - s, -\frac{1}{2}, \hat{W}, \hat{\phi} \right)}{L \left( \frac{1}{2} - s, \tilde{\pi}' \right) L(1 - 2s, \pi', \Lambda^2)} = e^{lin}(s, \pi', \psi) \frac{\Psi(s, 0, W, \phi)}{L(s + \frac{1}{2}, \pi') L(2s, \pi', \Lambda^2)}.
\]

where \( \hat{\phi} \) is the Fourier transform of \( \phi \), and \( \hat{W}(g) = W(w^t g^{-1}) \), \( w \) being the longest Weyl element in \( G' \).

Let us now get back to the setup of Proposition 5.5. We apply the functional equation (5.5) to the representation \( \pi' \otimes \eta \), which is \( H' \)-distinguished. We are going to evaluate both sides of (5.6) when \( s \to 0^+ \).

First the right hand side. By [Mat15, Corollary 4.10], as \( \pi' \) is supercuspidal, the defining integral of \( \Psi(s, 0, W, \phi) \) is convergent when \( \Re s > 0 \) (this is in fact the case if \( \pi' \) is only assumed to be a discrete series representation). Moreover as \( \pi' \otimes \eta \) is \( H' \)-distinguished, \( L(2s, \pi', \Lambda^2) \) has a simple pole at \( s = 0 \). We have

\[
\Psi(s, 0, W, \phi) = \int_{Z'(N' \cap H')} \int_{H'} W(h) \eta(\det h) \phi(e_n z h_2) |z|^{2ns} |h|^s \chi_{\frac{1}{2}}(h) dz dh.
\]
Here integration over \( Z'(N' \cap H'^\sigma) \setminus H'^\sigma \) means that we integrate over a set of representatives of \( Z'(N' \cap H'^\sigma) \setminus H'^\sigma \) in \( (N' \cap H'^\sigma) \setminus H'^\sigma \). Note that \( |\det h| \) in this set is bounded both above and below away from zero. As \( s \) approaches \( 0^+ \), the inner integral has a simple pole and the residue equals

\[
\frac{1}{2n} \phi(0) \Res_{s=0} \gamma(s, 1, \psi),
\]

which is independent of \( h \). Thus

\[
\Res_{s=0} \Psi(s, 0, W, \phi) = \frac{1}{2n} \phi(0) \Res_{s=0} \gamma(s, 1, \psi) l_{\eta}^\#(W),
\]

where \( l_{\eta}^\#(W) \) stands for the integral appearing on the right hand side of (5.5).

Now let us evaluate the left hand side of (5.6). We have

\[
\Psi \left( \frac{1}{2}, -\frac{1}{2}, \hat{W}, \hat{\phi} \right) = \int_{N' \cap H'^\sigma \setminus H'^\sigma} \hat{W}(h) \hat{\phi}(\epsilon_n h_2) \eta(\det h) |\det h_2| dh_1 dh_2.
\]

It is not hard to see that the right hand side is absolutely convergent. We decompose

\[
h_2 = pzu, \quad u = \begin{pmatrix} 1_{n-1} \\ u_{n-1} \\ 1 \end{pmatrix},
\]

where \( p \) is in the mirabolic subgroup of \( \GL_n(F) \), \( u_{n-1} \in F_{n-1} \) and \( z \) is in the center of \( \GL_n(F) \). Then we have

\[
\Psi \left( \frac{1}{2}, -\frac{1}{2}, \hat{W}, \hat{\phi} \right) = \int_{F_{n-1}} \int_{F^\times} l_\eta(\pi(u^+) \hat{W}) \hat{\phi}(u_{n-1}, z) du_{n-1} dz,
\]

where \( u^+ = \begin{pmatrix} 1_n \\ u \end{pmatrix}^\sigma \) and the measures are all additive. As \( l_\eta \) is \( (H', \eta) \)-invariant, we conclude that

\[
\Psi \left( \frac{1}{2}, -\frac{1}{2}, \hat{W}, \hat{\phi} \right) = \phi(0) l_\eta(\hat{W}) = \phi(0) \eta(-1)^n l_\eta(W).
\]

The last inequality follows from [Off11, Corollary 7.2].

The upshot of the above computation is that for all \( W \in W \) and \( \phi \in C_\infty_c(F_n) \), we have

\[
\frac{\phi(0) \Res_{s=0} \gamma(s, 1, \psi) l_{\eta}^\#(W)}{2n \Res_{s=0} L(2s, \pi', \Lambda^2)} c_{\text{lin}}(0, \pi', \psi) = \frac{\phi(0) \eta(-1)^n l_\eta(W)}{L(1, \pi', \Lambda^2)}.
\]

Of course we can choose \( \phi \) with \( \phi(0) \neq 0 \). All terms in this identity not involving \( W \) are nonzero. We thus have proved the desired identity (5.5) and hence Proposition 5.5.

\[\square\]

6. The relative trace formulae

6.1. Simple relative trace formulae. We now assume that \( E/F \) are global fields of characteristic not two. We will be working with the following general setup. This is slightly more general than the setting in the literature, e.g. [FMW18], but the relative trace formulae in this setting can be dealt with using the same technique.
(1) The nonsplit side. Let $A$ be a CSA over $F$ of dimension $4n^2$ with a fixed embedding $E \to A$. Let $B$ be the centralizer of $E$ in $A$. It is a CSA over $E$ of dimension $n^2$. Put $G = A^\times$ and $H = B^\times$ with a natural inclusion $H \to G$. Let $Z$ be the center of $G$, which is also the split center of $H$. We assume that for all places $v$ of $F$, Assumption 2.1 is in effect, i.e. $A(F_v)$ is either a central division algebra, or $M_n(D_v)$ where $D_v$ is a quaternion division algebra over $F_v$ (split or not).

(2) The split side. Let $G' = \text{GL}_{2n,F}$ and $H' = \text{GL}_n,F \times \text{GL}_{n,F}$ with a natural embedding
\[
(h_1, h_2) \mapsto \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}.
\]
Let $Z'$ be the center of $G$. Let $\sigma \in G'$ be the permutation matrix as in Subsection 5.1. We have another embedding $H'^\sigma \to G'$. If $f'$ is a function on $G'$, we put $f'^\sigma(g) = f'(\sigma g \sigma)$.

We could define the notion of $\theta$-regular etc. elements of $G$ and $G'$ respectively, in exactly the same manner as in the local situation. Moreover we have the global orbital integrals. If $\gamma \in G$ is $\theta$-regular, then we define for $f \in C_c^\infty(G(\mathbb{A}_F))$ the orbital integral
\[
O(g, f) = \int_{(H \times H)_\gamma(\mathbb{A}_F) \setminus (H \times H)(\mathbb{A}_F)} f(h_1gh_2)dh_1dh_2.
\]
This integral is absolutely convergent. If moreover $f$ is factorizable, i.e. $f = \otimes_v f_v$ where $f_v \in G(F_v)$ then we have
\[
O(g, f) = \prod_v O(g, f_v),
\]
where the product ranges over all places of $F$. Similarly if $g \in G'$ if $\theta$-regular, then we define for $f' \in C_c^\infty(G'(\mathbb{A}_F))$ the global orbital integral
\[
O(g, \eta, f') = \int_{(H' \times H')_\gamma(\mathbb{A}_F) \setminus (H' \times H')(\mathbb{A}_F)} f'(h_1gh_2)\eta(\det h_2)dh_1dh_2.
\]
This integral is absolutely convergent. This orbital integral factorizes as a product of local orbital integrals if $f'$ is factorizable.

Let us define some global distributions. Let $\pi$ be an irreducible cuspidal automorphic representation of $G(\mathbb{A}_F)$. Let $\varphi \in \pi$, we put
\[
P_H(\varphi) = \int_{Z(\mathbb{A}_F)H(F)_\gamma \setminus H(\mathbb{A}_F)} \varphi(h)dh.
\]
We say that $\pi$ is $H(\mathbb{A}_F)$-distinguished if $P_H$ is not identically zero. For $f \in C_c^\infty(G(\mathbb{A}_F))$, we define
\[
J_\pi(f) = \sum_{\varphi \in \pi} P_H(\pi(f)\varphi)P_H(\varphi),
\]
where the sum runs over an orthonormal basis of $\pi$.

Let $\pi'$ be an irreducible cuspidal automorphic representation of $G'(\mathbb{A}_F)$. Let $\varphi \in \pi$, we put
\[
P_{H'}(\varphi) = \int_{Z'(\mathbb{A}_F)H'^\sigma(F)_\gamma \setminus H'^\sigma(\mathbb{A}_F)} \varphi(h)dh, \quad P_{H', \eta}(\varphi) = \int_{Z'(\mathbb{A}_F)H'^\sigma(F)_\gamma \setminus H'^\sigma(\mathbb{A}_F)} \varphi(h)\eta(\det h)dh.
\]
We say that $\pi$ is $H'(A_F)$-distinguished (resp. $(H'(A_F), \eta)$-distinguished) if $P'_{H'}$ (resp. $P'_{H',\eta}$) is not identically zero. For $f' \in C_c^\infty(G'(A_F))$, we define

$$I_{\pi'}(f') = \sum_{\varphi \in \pi'} P'_{H'}(\pi'(f')\varphi) \overline{P'_{H',\eta}(\varphi)},$$

where the sum runs over an orthonormal basis of $\pi'$.

Let us now state the simple relative trace formulae that we are going to make use of.

(1) The nonsplit side. Assume the following conditions.

(a) For almost all places $f'_v = 1_{G'(A_v)}$;
(b) At some finite place $v_1$ of $F$, $f'_{v_1}$ is a essentially supercuspidal, i.e. $\int_{Z(F_v)} f'_{v_1}(zg)dz$ is a matrix coefficient of a supercuspidal representation.
(c) At some finite place $v_2$ of $F$, $f_{v_2}$ is supported in the $\theta$-elliptic locus of $G(F_{v_2})$.

Then we have

$$\sum_{g \in G(F)} O(g) = \sum_{\pi \text{ cuspidal}} J_{\pi}(f).$$

Both sides are absolutely convergent.

Instead of condition (c), we could also have following (c') instead.

(c') At some finite place $v_2$ of $F$, $A(F_{v_2})$ is a central division algebra. In particular $Z(F_{v_2}) \setminus G(F_{v_2})$ is compact.

In this case we have

$$\sum_{g \in G(F)} O(g) = \sum_{\pi \text{ cuspidal}} J_{\pi}(f).$$

Both sides are absolutely convergent. The convergence of the left hand side follows from the fact that $(H \times H)_g$ is anisotropic for all $g \in G(F)$. Note also that every $g \in G(F)$ is $\theta$-semisimple.

(2) The split side. Assume the following conditions.

(a) For almost all places $f'_v = 1_{G'(A_v)}$;
(b) At some finite place $v_1$ of $F$, $f''_{v_1}$ is a essentially supercuspidal, i.e. $\int_{Z'(F_v)} f''_{v_1}(zg)dz$ is a matrix coefficient of a supercuspidal representation.
(c) At some finite place $v_2$ of $F$, $f'_{v_2}$ is supported in the $\theta$-elliptic locus of $G'(F_{v_2})$.

Then we have

$$\sum_{g \in G'(F)_{\theta-\text{ell}}} O(g, \eta, f''_{v_2}) = \sum_{\pi' \text{ cuspidal}} I_{\pi'}(f).$$

Both sides are absolutely convergent.
6.2. Comparison. We now recall the smooth transfer of orbits and orbital integrals in our setting. Again this is a slight extension of [Guo96, Zha15]. The smooth transfer of orbits needs a little bit of extra work which we explain. The smooth transfer of orbital integrals, however, can be proved in exactly the same way as [Zha15]. Details of a full treatment of smooth transfer, including all CSAs (not just central division algebras) will not be given here but in a companion paper.

First we consider matching orbits on the symmetric spaces. Assume that \( F \) is local or global. Let \( \gamma \in G(F) \) be \( \theta \)-regular. By Lemma 2.5, up to left and right translation of \( H \), we may assume that \( \gamma = 1 + \xi \) where \( \xi \in \mathfrak{s} \). Moreover \( \xi \in \mathfrak{s} \) is \( \theta \)-regular and \( \nu_B(\xi^2 - 1) \neq 0 \). Let \( \gamma' \in G'(F) \) be a \( \theta \)-regular element. Then up to left and right \( H'(F) \) translation, we may assume that

\[
\gamma' = \begin{pmatrix} 1_n & X' \\ 1_n & 1_n \end{pmatrix},
\]

where \( X' \in \text{GL}_n(F) \) is regular semisimple and \( \det(1_n - X') \neq 0 \). Moreover \( \gamma' \) is \( \theta \)-elliptic if \( X' \) is elliptic in \( \text{GL}_n(F) \) (in the usual sense).

In this notation, we say that \( \gamma \) and \( \gamma' \) (or rather the orbits in \( G \) and in \( G' \) represented by them) match if \( \xi^2 \) and \( X' \) have the same (reduced) characteristic polynomial. Given any \( \gamma \in G_{\theta-\text{reg}} \), one can find a \( \gamma' \in G'(F) \) that matches it. If we write \( \gamma' = \begin{pmatrix} 1_n & X' \\ 1_n & 1_n \end{pmatrix} \), then \( X' \) is elliptic (in the usual sense) and \( \eta(\det X') = (-1)^r \) where \( r \) is the split rank of \( G(F) \) by Lemma 2.4. By Lemma 2.3, the map \( \gamma \mapsto \gamma' \) defines an injective map

\[
H(F) \backslash G(F)_{\theta-\text{reg}}/H(F) \to H'(F) \backslash G'(F)_{\theta-\text{reg}}/H'(F).
\]

Conversely, assume that \( F \) is local. If \( \gamma' = \begin{pmatrix} 1_n & X' \\ 1_n & 1_n \end{pmatrix} \) with \( X' \) being elliptic (in the usual sense) and \( \eta(\det X') = (-1)^r \) where \( r \) is the split rank of \( G(F) \), again by Lemma 2.4, there is a \( \gamma \in G(F) \) so that \( \gamma \) and \( \gamma' \) match.

If \( F \) is global, we have the following.

**Lemma 6.1.** Assume that \( F \) is global. Then the image of \( H(F) \backslash G(F)_{\theta-\text{ell}}/H(F) \) is precisely the orbits represented by \( \gamma' = \begin{pmatrix} 1_n & X' \\ 1_n & 1_n \end{pmatrix} \) where \( X' \in \text{GL}_n(F) \) has the properties of (1) being elliptic (in the usual sense), (2) \( \det(X' - 1) \neq 0 \), (3) \( \eta_v(\det X') = (-1)^{r_v} \) for all places \( v \) of \( F \) where \( r_v \) is the split rank of \( G(F_v) \), (4) the characteristic polynomial of \( X' \) is irreducible in \( F_v \) if \( G(F_v) \) is the multiplicative group of a central division algebra over \( F_v \).

**Proof.** If for all places \( v \) of \( F \), \( G(F_v) = \text{GL}_n(D) \) where \( D \) is a quaternion algebra over \( F_v \), this is proved by [Guo96]. If there is a place \( v \) of \( F \) such that \( G(F_v) \) is the multiplicative group of a central division algebra over \( F_v \), then \( G = A^\times \) where \( A \) is a central division algebra over \( F \) and \( f(\lambda) \) is irreducible over \( E \) (since it is irreducible over \( E_v \), c.f. the argument in Lemma 2.4). Let \( f(\lambda) \) be the characteristic polynomial of \( X', L = F[\lambda]/(f(\lambda)) \) and \( K = LE \) which is a field of degree \( n \) over \( E \). We note that \( B \otimes_E K \) is split as it is split over all places \( v \) of \( E \). Thus there is an embedding
Let us consider the matching of orbits on the Lie algebras. We always assume that $F$ is local as this is the only case that we need.

Let $\xi \in \mathfrak{s}(F)_{\theta\text{-reg}}$ and $\xi' \in \mathfrak{s}'(F)_{\theta\text{-reg}}$. Up to conjugation by $H'(F)$, we may assume that

$$\xi' = \begin{pmatrix} X' \\ 1_n \end{pmatrix},$$

where $X' \in \text{GL}_n(F)$ is regular semisimple (in the usual sense). Moreover $\xi'$ is $\theta$-elliptic if and only if $X'$ is elliptic in the usual sense. We say that $\xi$ and $\xi'$ (or the orbits in $\mathfrak{s}$ and in $\mathfrak{s}'$ represented by them) match if $\xi^2 \in B$ and $X'$ are conjugate have the same (reduced) characteristic polynomial.

Given $\xi \in \mathfrak{s}_{\theta\text{-reg}}$ we can find an $\xi' \in \mathfrak{s}'_{\theta\text{-reg}}$ which matches it. The map $\xi \mapsto \xi'$ defines an injective map

$$H(F) \backslash \mathfrak{s}(F)_{\theta\text{-reg}} / H(F) \to H'(F) \backslash \mathfrak{s}'(F)_{\theta\text{-reg}} / H'(F).$$

Conversely given $\xi' \in \mathfrak{s}'_{\theta\text{-reg}}$ with $X'$ being elliptic in $\text{GL}_n(F_v)$, $\det(X' - 1) \neq 0$ and $\eta_{E_v/F_v}(\det X') = (-1)^{r_v}$ where $r_v$ is the split rank of $G(F_v)$, by Lemma 2.3 and 2.4 there is a $\xi \in \mathfrak{s}_{\theta\text{-ell}}$ which matches $\xi'$.

We now define the transfer factor. Assume that $F$ is global. Let $v$ be a place of $F$ and $\gamma' \in G'(F_v)_{\theta\text{-reg}}$ and $\gamma'^{-1}\theta(\gamma') = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in S'(F_v)$. We define the transfer factor

$$\Omega_v(\gamma') = \eta_{E_v/F_v}(\det B).$$

If $\gamma' \in \mathfrak{s}'_{\theta\text{-reg}}(F_v) = \begin{pmatrix} X \\ Y \end{pmatrix}$, then we define the transfer factor

$$\omega_v(\gamma') = \eta_{E_v/F_v}(\det X).$$

**Lemma 6.2.** Let $w$ be the longest Weyl group element in $G'(F)$. Let $r_v$ be the split rank of $G(F_v)$.

1. Assume that $\gamma' \in G'(F_v)_{\theta\text{-ell}}$ and $\gamma \in G(F_v)_{\theta\text{-ell}}$ match. Then $\Omega_v(w^{-1}\gamma'w)\Omega_v(\gamma')^{-1} = (-1)^{r_v}$.

2. Assume that $\gamma' \in \mathfrak{s}'(F_v)_{\theta\text{-ell}}$ and $\gamma \in \mathfrak{s}(F_v)_{\theta\text{-ell}}$ match. Then $\omega_v(w^{-1}\gamma'w)\omega_v(\gamma)^{-1} = (-1)^{r_v}$.

**Proof.** This follows from a direct calculation using the definition. □

Now we consider matching of test functions. Assume that $F$ is a global field. Let $f \in C^\infty_c(G(\mathbb{A}_F))$ and $f' \in C^\infty_c(G'(\mathbb{A}_F))$. Let $v$ be a place of $F$. We say that $f_v$ and $f'_v$ match if

$$\Omega_v(\gamma')O(\gamma', \eta, f'_v) = \begin{cases} O(\gamma; f_v), & \text{if } \gamma' \in G'(F_v)_{\theta\text{-reg}} \text{ and } \gamma \in G(F_v)_{\theta\text{-reg}} \text{ match} \\ 0, & \text{if } \gamma' \in G'(F_v)_{\theta\text{-reg}} \text{ does not match any } \gamma \in G(F_v)_{\theta\text{-reg}}. \end{cases}$$

We say that $f = \otimes f_v$ and $f' = \otimes f'_v$ match if $f_v$ and $f'_v$ match at all places $v$. 

K → B over E and we let $u$ be the image of $\lambda$ in $B$. Let $R$ be the centralizer of $L$ in $A$ which is a quaternion division algebra over $L$ which is stable under $\theta$ and $R \cap B = K$. Using the case $n = 1$ we find an $\xi \in R \cap s$ so that $u = \xi^2$. Then $\gamma = 1 + \xi \in G(F)$ matches $\gamma'$.
Proposition 6.3. Let $v$ be a finite place or a split archimedean place of $F$. Let $f_v \in \mathcal{C}_c^\infty(G(F_v))$, then there is an $f'_v \in \mathcal{C}_c^\infty(G'(F_v))$ such that $f_v$ and $f'_v$ match. Conversely, let $f'_v \in \mathcal{C}_c^\infty(G(F_v))$ such that $O(\gamma', \eta, f'_v) = 0$ if $\gamma' \in G'(F_v)_{\theta-reg}$ does not match any $\gamma \in G(F_v)_{\theta-reg}$, then there is an $f_v \in \mathcal{C}_c^\infty(G(F_v))$ such that $f_v$ and $f'_v$ match.

Proof. This is the main result of [Zha15]. Though our setup includes the case of central division algebras over $F_v$, the proof of [Zha15] can be verbatimly copied.

Now we consider matching of test functions on the Lie algebra. We only need the case $F$ being local. Let $f \in \mathcal{C}_c^\infty(s(F))$ and $f' \in \mathcal{C}_c^\infty(s'(F))$. We say that $f$ and $f'$ match if

$$\omega(\xi')O(\xi, \eta, f') = \begin{cases} O(\xi, f), & \text{if } \xi' \in s'(F)_{\theta-reg} \text{ and } \xi \in s(F)_{\theta-reg} \text{ match} \\ 0, & \text{if } \xi' \in s'(F)_{\theta-reg} \text{ does not match any } \xi \in s(F)_{\theta-reg}. \end{cases}$$

Proposition 6.4. For any $f \in \mathcal{C}_c^\infty(s)$, there is an $f' \in \mathcal{C}_c^\infty(s')$ so that $f$ and $f'$ match. For any $f' \in \mathcal{C}_c^\infty(s')$ with the properties that

$$O(\xi', \eta, f') = 0 \text{ if } \xi' \in s'(F)_{\theta-reg} \text{ does not match any } \xi \in s(F)_{\theta-reg},$$

then there is an $f \in \mathcal{C}_c^\infty(s(F))$ such that $f$ and $f'$ match.

Proof. This is [Zha15, Theorem 5.14].

Proposition 6.5. Suppose that $f \in \mathcal{C}_c^\infty(s(F))$ and $f' \in \mathcal{C}_c^\infty(s'(F))$ match. Then there is a nonzero constant $c$ which does not depend on $f$ or $f'$ so that $\hat{f}$ and $c\hat{f}'$ match.

Proof. This is [Zha15, Theorem 5.16].

We have the corresponding fundamental lemma. This is the working hypothesis (FL) in the Introduction. The fundamental lemma for units is proved in [Guo96]. It is expected that the full fundamental lemma can be proved using global argument via a similar argument as [AC89, Section 1.4].

Conjecture 6.6 (Hypothesis (FL)). Assume the following conditions.

1. The place $v$ of $F$ is unramified in $E$.
2. $G(F_v) = \text{GL}_{2n}(F_v)$.
3. The measures on $H$ and $H'$ are chosen so that $\text{vol} H(o_v) = \text{vol} H'(o_v)$.

The functions $f_v = f'_v \in \mathcal{H}_v$ match where

$$\mathcal{H}_v = \mathcal{C}_c^\infty(\text{GL}_{2n}(o_v) \backslash \text{GL}_{2n}(F_v) / \text{GL}_{2n}(o_v))$$

stands for the unramified Hecke algebra.

The main result of [Guo96] confirms this for the units.

Proposition 6.7. In the situation of Conjecture 6.6, the functions $f_v = f'_v = 1_{\text{GL}_{2n}(o_v)}$ match.
6.3. Weak spherical character identities. From now on let us assume the working hypothesis (FL), i.e. the full fundamental lemma, Conjecture 6.6.

Let us retain the notation from Subsection 6.1. First we have the following consequence of the simple relative trace formulae.

**Lemma 6.8.** Assume the following conditions.

1. $E/F$ splits at the archimedean places.
2. $\pi$ is $H(\mathbb{A}_F)$ distinguished.
3. There is a finite place $v_1$ which splits in $E$ such that $\pi_{v_1}$ is supercuspidal.
4. There is a finite place $v_2$ such that $J_{\pi_{v_2}}$ is not identically zero when restricted to the $\theta$-elliptic locus.

Then there are nice test functions $f = \otimes f_v \in C_c^\infty(G(\mathbb{A}_F))$ and $f' = \otimes f'_v \in C_c^\infty(G'(\mathbb{A}_F))$ so that $f$ and $f'^\sigma$ match, $f_{v_1} = f'_{v_1}$ are essentially a matrix coefficient of $\pi_{v_1}$, $f_{v_2}$ and $f'_{v_2}$ are supported in the $\theta$-elliptic locus, and

$$J_{\pi}(f) = I_{\pi'}(f') \neq 0.$$  

**Proof.** This is a slight extension of [FMW18, identity (6.3)] and the proofs are identical. Note that we assume the full fundamental lemma so that we can further distinguish $\pi$ and $\pi \otimes \eta$. □

**Remark 6.9.** This is the only place where we make use of the full fundamental lemma.

Assume that $E/F$ splits at all archimedean places. Let $\pi$ be an $H(\mathbb{A}_F)$-distinguished cuspidal automorphic representation so that there is a split place $v_1$ with $\pi_{v_1}$ being supercuspidal and there is a place $v_2$ with $J_{\pi_{v_2}} \neq 0$ when restricted to the $\theta$-elliptic locus. For each place $v$, the space $\text{Hom}_{H(F_v)}(\pi_v, \mathbb{C})$ is one dimensional (c.f. [AG09,BM,Guo97]) and let us fix a generator $l_v$ in $\text{Hom}_{H(F_v)}(\pi_v, \mathbb{C})$. It follows that if $f = \otimes f_v \in C_c^\infty(G(\mathbb{A}_F))$, then we have a factorization

$$J_{\pi}(f) = C(\pi) \prod_v J_{\pi_v, l_v}(f_v),$$

where $C(\pi)$ is a nonzero constant. Let $\pi'$ be the Jacquet–Langlands transfer of $\pi$. Then the simple relative trace formula in Lemma 6.8 implies that $\pi'$ is $H'(\mathbb{A}_F)$-distinguished and $(H'(\mathbb{A}_F), \eta)$-distinguished. For each place $v$, by [JR96, Theorem 1.1] and [AG09, Theorem 8.2.4], the space $\text{Hom}_{H'(F_v)}(\pi_v, \mathbb{C})$ and $\text{Hom}_{H'(F_v)}(\pi_v \otimes \eta_v, \mathbb{C})$ are both one dimensional. It follows that if $f' = \otimes f'_v \in C_c^\infty(G'(\mathbb{A}_F))$, we have a factorization

$$I_{\pi'}(f') = C'(\pi') \prod_v I_{\pi'_v}(f'_v),$$

where $C'(\pi')$ is a nonzero constant. It follows from these factorizations that there is a nonzero constant $\kappa(\pi_v)$ so that

$$I_{\pi'_v}(f'_v) = \kappa(\pi_v)J_{\pi_v, l_v}(f_v)$$

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whenever $f'_v$ and $f_v$ match. This constant $\kappa(\pi_v)$, which presumably depends on the global representation $\pi$, indeed does not, for that both $I_{\pi'_v}(f'_v)$ and $J_{\pi_v,l_v}(f_v)$ are defined solely in terms of $\pi_v$. It is also clear that $\kappa(\pi_v) \neq 0$.

We summarize this discussion in the following lemma.

**Lemma 6.10.** Suppose that $\pi_v$ is the local component at $v$ of a globally $H(\mathbb{A}_F)$-distinguished cuspidal automorphic representation $\sigma$ of $G(\mathbb{A}_F)$. Assume that $E/F$ splits at all archimedean places, and there are two nonarchimedean places $\{v_1, v_2\}$, $v_1$ being split, such that $\sigma_{v_1}$ is supercuspidal and $J_{\sigma_{v_2}}$ is nonzero when restricted to the $\theta$-elliptic locus. Then there is a nonzero constant $\kappa(\pi_v)$ so that for any matching test functions $f_v$ and $f'_v$ we have

$$I_{\pi'_v}(f'_v) = \kappa(\pi_v) J_{\pi_v,l_v}(f_v),$$

where $\pi'_v$ is the Jacquet–Langlands transfer of $\pi_v$ to $GL_{2n}(F_v)$, $l_v \in \text{Hom}_{H(F_v)}(\pi_v, \mathbb{C})$ is a fixed nonzero $H(F_v)$-invariant linear form on $\pi_v$.

In particular the identity (6.4) holds for any $H(F_v)$-distinguished supercuspidal representations.

**Proof.** The identity (6.4) follows from the discussion prior to the lemma. The last assertion holds by a globalization result of Prasad and Schulze-Pillot, c.f. [PSP08, Theorem 4.1].

The next result roughly says that the space spanned by spherical characters is weakly dense in the space of all invariant distributions.

**Proposition 6.11.** Let $f_v \in C_c^\infty(G(F_v))$ with $\int_{H(F_v)} f(h)dh \neq 0$, and let $f'_v \in C_c^\infty(G'(F_v))$ be a test functions that matches it. Then there is an irreducible $H(F_v)$-distinguished representation $\pi_v$ of $G(F_v)$ and a nonzero constant $\kappa_v$ so that

$$I_{\pi'_v}(f'_v) = \kappa_v J_{\pi_v,l_v}(f_v) \neq 0,$$

where $\pi'_v$ is the Jacquet–Langlands transfer of $\pi_v$ to $GL_{2n}(F_v)$, and $l_v \in \text{Hom}_{H(F_v)}(\pi_v, \mathbb{C})$ is a fixed nonzero $H(F_v)$-invariant linear form on $\pi_v$.

**Proof.** We prove it by a global argument. Consider the following data.

1. Let $E/F$ be a quadratic extension of global fields which splits at all archimedean places and there is a place $v$ of $F$ so that $E/F \simeq E_v/F_v$.
2. Let $A$ be a CSA over $F$ containing $E$, such that there is a place $v_0$ of $F$ such that $A \otimes F_{v_0}$ is a central division algebra. Let $B$ be the centralizer of $E$ in $A$.
3. Let $G = A^\times$, $H = B^\times$.
4. Let $\{v_1, v_2\}$ be two nonarchimedean places of $F$ (different from $v_0$), $v_1$ being split, and $\pi_{v_i}$ be an $H(F_{v_i})$-distinguished supercuspidal representations of $G(F_{v_i})$, $i = 1, 2$. In particular $J_{\pi_i}$ does not vanish identically on the $\theta$-elliptic locus.
5. Let $v_3$ be a nonarchimedean place of $F$ different from $\{v_0, v_1, v_2\}$.
6. Let $f = \otimes f_v \in C_c^\infty(G(\mathbb{A}_F))$ be a test function so that
(a) \( f_v = f_v; \)
(b) for \( i = 1, 2, f_{v_i} \) is essentially a matrix coefficient of \( \pi_{v_i}, \int_{H(F_{v_i})} f_{v_i}(h)dh \neq 0; \)
(c) for all other places \( w \neq v_i, i = 1, 2, 3, \) we choose an arbitrary test function \( f_w \) in \( C_c^\infty(G(F_w)) \) with \( \int_{H(F_w)} f_w(h)dh \neq 0; \)
(d) for the place \( v_3, \) we choose a test function \( f_{v_3} \) supported in a small neighbourhood of identity with \( \int_{H(F_{v_3})} f_{v_3}(h)dh \neq 0 \) so that if \( \gamma \in G(F) \) and \( \mathbb{H}(A_F)\gamma \mathbb{H}(A_F) \cap \text{supp} f \neq \emptyset, \) then \( \gamma \in \mathbb{H}(F). \)

We only need to explain how to achieve the choice of \( f_{v_3}, \) i.e. the test function in (6)(d). For \( A \) and \( B, \) we have the involution \( \theta \) and the space \( S \) and \( s \) as before. Let \( \gamma \in G(A_F) \) and \( \mathbb{H}(A_F)\gamma \mathbb{H}(A_F) \cap \text{supp} f \neq \emptyset. \) Consider \( s = \gamma^{-1}\theta(\gamma) \) and the coefficients of the reduced characteristic polynomial of \( (s - 1)^2. \) We viewed it as an element in \( A_F^n. \) Of course \( \mathbb{H}(A_F)\gamma \mathbb{H}(A_F) \) contains an element in \( G(F) \) if and only if the coefficients of the reduced characteristic polynomial of \( (s - 1)^2 \) lie in \( F^n. \) Moreover for any place \( w \neq v_3, \) these coefficients lie in some fixed compact subset \( \Omega_w \) of \( F^n_w, \) containing zero (because we have assume that \( \int_{H(F_w)} f_w(h)dh \neq 0). \) Therefore by the product formula we can choose a sufficiently small neighbourhood \( \Omega_{v_3} \) of zero in \( F^n_{v_3} \) so that

\[
F^n \cap \prod \Omega_w = \{0\}.
\]

Let \( U_{v_3} \) be the inverse image of \( \Omega_{v_3} \) in \( G(F_{v_3}) \), then \( U_{v_3} \) contains \( \mathbb{H}(F_{v_3}) \) by definition. We also note that

\[
\left( U_{v_3} \times \prod_{w \neq v_3} \mathbb{H}(F_w)(\text{supp} f_w)\mathbb{H}(F_w) \right) \cap G(F) = \mathbb{H}(F).
\]

Indeed if \( \gamma \) lies in the left hand side, then the reduced characteristic polynomial of \( (s - 1)^2 \in B(F) \) is of the form \( \lambda^n. \) By assumption \( A(F_{v_0}) \) is a central division algebra, which means that \( s = 1 \in A(F_{v_0}) \), hence in \( A(F). \) This is equivalent to that \( \gamma \in \mathbb{H}(F). \) Having all this, we can thus choose an \( f_{v_3} \) supported in \( U_{v_3} \) so that if \( \gamma \in \text{supp} f \cap G(F) \) then \( \gamma \in \mathbb{H}(F). \)

Let us plug this test function in the simple relative trace formula (6.2). It reads

\[
\sum_{\gamma \in \mathbb{H}(F) \backslash G(F)/\mathbb{H}(F)} O(\gamma, f) = \sum_{\sigma} \mathcal{J}_\sigma(f).
\]

By our choice of the test function, the left hand side is reduced to a single term \( \int_{\mathbb{H}(A_F)} f(h)dh, \) which is nonzero. Therefore there is at least one \( \sigma \) on the right hand side so that \( \mathcal{J}_\sigma(f) \neq 0. \) Moreover any \( \sigma \) on the right hand side has the property that \( \mathcal{J}_{\sigma_i}(f_v) \neq 0, \sigma_{v_i} \) is supercuspidal for \( i = 1, 2 \) and in particular \( \mathcal{J}_{\sigma_{v_3}} \) is not identically zero when restricted to the \( \theta \)-elliptic locus. Thus applying Lemma 6.10 concludes the proof of this proposition. \( \square \)

6.4. Epsilon dichotomy. It is now time to reap the fruit of our long labor. Let us go back to the local situation. Assume that \( E/F \) is a quadratic extension of local fields. Again we assume the working hypothesis (FL).
**Theorem 6.12.** Assume (FL). Let π be an irreducible supercuspidal representation of G and π' is its Jacquet–Langlands transfer to GL_{2n}(F). Then π is H(F)-distinguished if and only if the following two conditions hold:

1. π' is H'-distinguished and (H', η)-distinguished.
2. \(\epsilon(π_E)\eta(-1)^n = (-1)^r\), where r is the split rank of G.

Before delving into the proof, we note that if \(G \simeq \text{GL}_{2n}(F)\) (resp. \(G \simeq \text{GL}_n(D)\) with D ramified, resp. \(G \simeq A^\times\) where A is a central division algebra), then \(r = 2n\) (resp. \(r = n\), resp. \(r = 1\)). Thus this theorem covers both Theorem 1.1 and 1.5.

**Proof of Theorem 6.12.** We first assume that π is H(F)-distinguished. By Lemma 6.10, there is a nonzero constant \(\kappa = \kappa(\pi)\) so that

\[ I_{π'}(f') = \kappa I_{π}(f) \]

for any matching test function \(f' \in C_c^\infty(G'(F))\) and \(f \in C_c^\infty(G(F))\). In particular we conclude that π' is H'(F)-distinguished and (H'(F), η)-distinguished \(I_{π'}\) is not identically zero.

It is clear that if \(f\) is supported in a small neighbourhood of identity, we can take \(f'\) to be supported in a small neighbourhood of identity. Thus we have the local character expansion of both sides, i.e.

\[ \sum_{O \subset \mathcal{N}_0} c_O^\prime \widehat{μ}_O^\prime(f'_O) = \kappa \sum_{O \subset \mathcal{N}} c_O \widehat{μ}_O(f_O). \]

As \(f'\) and \(f\) match, do \(f'_O\) and \(f_O\). Moreover it is easy to check that for any \(λ \in F^\times\), \(f'_O^\prime\) and \(f_O\) also match. Thus using homogeneity of the nilpotent orbital integrals on \(\mathfrak{s}'\) and on \(\mathfrak{s}\), cf. Lemma 2.6 and Lemma 4.11, we conclude that

\[ c_{O^\prime}^\prime \widehat{μ}_{O^\prime}^\prime(f'_O) + c_{O}^\prime \widehat{μ}_{O}^\prime(f'_O) = \kappa c_0 \widehat{μ}_0(f_O). \]

By Lemma 5.4, we have

\[ c_{O^\prime}^\prime = \epsilon(π_E)\eta(-1)^n c_{O}^\prime. \]

As \(c_0 \neq 0\) by Lemma 3.5, we conclude that \(c_{O^\prime}^\prime \neq 0\). Thus there is a nonzero constant \(C\) so that

\[ \epsilon(π_E)\eta(-1)^n \widehat{μ}_{O^\prime}^\prime(f'_O) + \widehat{μ}_{O}^\prime(f'_O) = C \widehat{μ}_0(f_O). \]

It is clear that \(\widehat{μ}_0\) is a nonzero constant function on \(\mathfrak{s}\). A little computation also shows that, up to some nonzero constant multiple, \(\widehat{μ}_{O^\prime}^\prime\) are represented by locally constant functions on \(\mathfrak{s}'_{\mathfrak{g}^\prime-\text{reg}}\) and we have

\[ \widehat{μ}_{O^\prime}^\prime \left( \begin{pmatrix} Y & X \\ \end{pmatrix} \right) = η(\det X), \quad \widehat{μ}_{O}^\prime \left( \begin{pmatrix} Y & X \\ \end{pmatrix} \right) = η(\det Y). \]

By Lemma 6.2, \(η(\det XY) = (-1)^r\). Thus we have

\[ \epsilon(π_E)\eta(-1)^n = (-1)^r, \]
for otherwise the left hand side of (6.6) would be identically zero while the right hand side is not. This proves that if \( \pi \) is \( H(F) \)-distinguished, then \( \pi' \) satisfies conditions (1) and (2).

We now prove the other implication. Assume that \( \pi' \) satisfies conditions (1) and (2) in the theorem. By Corollary 5.6, there is an essential matrix coefficient \( f' \) of \( \pi' \) so that

\[
(6.7) \quad \mu'_{\Omega_{\min}^+}(f'_2) + \epsilon(\pi'_E)\eta(-1)^n \mu'_{\Omega_{\min}^-}(f'_2) \neq 0.
\]

By parabolic descent, c.f. (4.2), \( O(\gamma, \eta, f') = 0 \) if \( \gamma \) is \( \theta \)-regular semisimple but not \( \theta \)-elliptic. We now consider the function on \( G'(F)_{\theta-\text{ell}} \) given by

\[
\gamma \mapsto \Omega(\gamma)O(\gamma, \eta, f').
\]

This function is bi-\( H'(F) \)-invariant by definition. We now consider

\[
\gamma \mapsto \Omega(\gamma)O(\gamma, \eta, t f').
\]

On the one hand, we have

\[
\Omega(\gamma)O(\gamma, \eta, t f') = \Omega(\gamma)O(w^{-1} \gamma w^{-1}, \eta, f') = \Omega(w^{-1} \gamma w)O(\gamma, \eta, f'),
\]

since \( w \gamma w^{-1} \) is in the same \( H'(F) \times H'(F) \) double coset as \( \gamma \). On the other hand we have

\[
(6.8) \quad \Omega(\gamma)O(\gamma, \eta, t f') = \epsilon(\pi'_E)\eta(-1)^n \Omega(\gamma)O(\gamma, \eta, f').
\]

This can be seen as follows. Suppose that

\[
\int_{Z_G(F)} f'(zg)dz = \langle \pi'(g)W_1, W_2 \rangle
\]

where \( W_1, W_2 \) are in the Whittaker model \( W \) of \( \pi \). Then by the uniqueness of linear periods [JR96], we can find a constant \( A \) (could be zero), depending on \( \gamma \) and \( \pi' \) but not on \( W_1 \) and \( W_2 \) so that

\[
\Omega(\gamma)O(\gamma, \eta, f') = A l(W_1)\overline{\eta(W_2)}.
\]

Using the result of Lapid and Mao [LM17, Theorem 3.2] again, we get (6.8). We thus conclude that

\[
\Omega(w^{-1} \gamma w) = \Omega(\gamma)\epsilon(\pi'_E)\eta(-1)^n,
\]

if \( O(\gamma, \eta, f') \neq 0 \). By Lemma 6.2, we conclude that \( \gamma \) matches an elliptic element \( \delta \in G(F) \) with \( \epsilon(\pi'_E)\eta(-1)^n = (-1)^r \).

Let \( f \in C^\infty_c(G(F)) \) be the smooth transfer of \( f' \). Direct computation, together with the fact that Fourier transform commutes with smooth transfer, show that

\[
(-1)^r \mu'_{\Omega_{\min}^+}(f'_2) + \mu'_{\Omega_{\min}^-}(f'_2) = B f(1).
\]

We thus conclude that \( f(1) \neq 0 \) from (6.7) and the fact that \( \epsilon(\pi'_E)\eta(-1)^n = (-1)^r \). Note that this is the geometric counterpart of the identity (6.6).

By Proposition 6.11, we can find a \( \sigma \) so that \( J_\sigma(f) \neq 0 \) and again a spherical character identity

\[
I_{JL(\sigma)}(f) = \kappa J_\sigma(f),
\]

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where \( \text{JL}(\sigma) \) stands for the Jacquet–Langlands transfer of \( \sigma \) to \( G'(F) \) and \( \kappa \neq 0 \). We conclude that \( I_{\text{JL}(\sigma)}(f') \neq 0 \). However by our very choice, \( f' \) is essentially a matrix coefficient of \( \pi' \). Thus we conclude that \( \pi' = \text{JL}(\sigma) \). This implies that \( \pi = \sigma \) and is thus \( H(F) \)-distinguished.

This ends the proof of the theorem. \( \square \)
Appendix by Raphaël Beuzart-Plessis

**APPENDIX A. EXISTENCE OF DISTINGUISHED SUPERCUSPIDAL REPRESENTATIONS**

Let $F$ be a $p$-adic field, $G$ be a connected reductive group over $F$ and $\mathbb{H}$ a closed algebraic subgroup of $G$. Set $G = G(F)$ and $H = \mathbb{H}(F)$ and denote by $\mathfrak{g}$ and $\mathfrak{h}$ their Lie algebras. In this appendix we will always make the slight abuse of language of identifying an algebraic subgroup of $G$ with its group of $F$-points. Let $\mathfrak{g}^*$ be the $F$-dual of $\mathfrak{g}$ and $\mathfrak{h}^\perp \subseteq \mathfrak{g}^*$ the orthogonal subspace of $\mathfrak{h}$. We say that an element $X \in \mathfrak{g}^*$ is *elliptic* if its centralizer $G_X$ in $G$ is (the group of $F$-points of) an elliptic torus.

**Remark A.1.** By using a $G$-invariant nondegenerate bilinear form on $\mathfrak{g}$ we could have identified $\mathfrak{g}$ and $\mathfrak{g}^*$ $G$-equivariantly. Then, the above notion of elliptic element would coincide with the usual notion of elliptic element in $\mathfrak{g}$. However, we prefer to work directly with $\mathfrak{g}^*$ since it is more canonical.

**Proposition A.2.** Assume that $\mathfrak{h}^\perp$ contains elliptic elements of $\mathfrak{g}^*$. Then there exists a supercuspidal representation $\pi$ of $G$ which is $H$-distinguished (i.e. admitting a nonzero $H$-invariant linear form).

*Proof.* The proof will follow closely [BP16]. Recall that a function $f \in C^\infty_c(G)$ is a *cusp form* if for every proper parabolic subgroup $P = MN$ of $G$ we have

$$\int_N f(xu)du = 0$$

for every $x \in G$. Let $C^\infty_c(G) \subseteq C^\infty_c(G)$ be the subspace of cusp forms. We will prove the following:

(A.1) There exists $f \in C^\infty_c(G)$ such that $\int_H f(h)dh \neq 0$.

First we explain how (A.1) implies the proposition. Let $f$ be a cusp form as in (A.1) and let $V$ be the subspace of $C^\infty_c(G)$ spanned by the right translates $R(g)f$ of $f$ by elements $g \in G$. Then, $V$ is a smooth representation of $G$ for the action by right translation and by [BP16, (2)] the representation $V$ is supercuspidal. Therefore, by the exactness of Jacquet’s functors, so are all of its subquotients. Let $Z(G)$ be the center of $G$ and set $Z = Z(G)/(Z(G) \cap H)$. Fixing a Haar measure $dz_G$ on $Z(G)$, for every unitary character $\chi$ of $Z$ we set

$$f_\chi(x) = \int_{Z(G)} f(zGx)\chi(z)^{-1}dz_G$$

for every $x \in G$. Then, $f_\chi$ is a locally constant function on $G$ which satisfies $f(zGx) = \chi(zG)f(x)$ for every $(zG, x) \in Z(G) \times G$ and which is compactly supported modulo $Z(G)$. By Fourier inversion on $Z$ applied to the function $z \in Z \mapsto \int_H f(zh)dh$, we see that there exists a unitary character $\chi$ of $Z$ such that

$$\int_{Z(H)\backslash H} f_\chi(h)dh \neq 0.$$
We fix such a character $\chi$ henceforth and let $V_\chi$ be the space of functions on $G$ spanned by the right $G$-translates of $f_\chi$. Then $V_\chi$ is again a smooth representation of $G$ generated by one element and it is a quotient of $V$ hence supercuspidal. Define a linear form $L$ on $V_\chi$ by

$$L(f') = \int_{Z(H)\backslash H} f'(h) dh, \quad f' \in V_\chi.$$ 

By the choice of $\chi$ this linear form is not identically zero and moreover it is obviously $H$-invariant.

Since $Z(G)$ acts on $V_\chi$ via the character $\chi$, by [Ren10, Théorème VI.3.5] $V_\chi$ is a finite direct sum of supercuspidal irreducible representations. The restriction of $L$ to at least one of these irreducible representations is nonzero and this shows the proposition.

We now prove (A.1) and for this we proceed as in [BP16]. Let $C^\infty_c(cusp)(g) \subseteq C^\infty_c(g)$ be the subspace of cusp forms on $g$ i.e. the space of functions $\varphi \in C^\infty_c(g)$ such that for every proper parabolic subgroup $P = MN$ of $G$ we have

$$\int_n \varphi(X + N) dN = 0$$

for every $X \in g$ where $n$ denotes the Lie algebra of $N$. Then, we only need to show

(A.2) There exists $\varphi \in C^\infty_c(cusp)(g)$ such that $\int_H \varphi(X) dX \neq 0$.

Indeed, let $\omega \subseteq g$ be an open neighborhood of 0 on which the exponential map exp is well-defined and realizes an $F$-analytic isomorphism with an open neighborhood $\Omega$ of 1 in $G$. Up to shrinking $\omega$ if necessary, and scaling the Haar measure on $h$, we may also assume that exp induces a measure-preserving isomorphism between $\omega \cap h$ and $\Omega \cap H$. Set $\varphi_\lambda(X) = \varphi(\lambda^{-1}X)$ for every $\lambda \in F^\times$ and $X \in g$. Then if $\varphi$ is as in (A.2) and $\lambda \in F^\times$ is sufficiently small the support of $\varphi_\lambda$ is included in $\omega$ and the function $f$ on $G$ defined by

$$f(g) = \left\{ \begin{array}{ll} \varphi(\exp(X)) & \text{if } g = \exp(X) \text{ for some } X \in \omega, \\ 0 & \text{otherwise}, \end{array} \right.$$ 

is a cusp form by [BP16, (5)]. Moreover, by the assumptions made on $\varphi$ and $\omega$ for such a $\lambda$ we have

$$\int_H f(h) dh = \int_H \varphi_\lambda(X) dX = |\lambda|_F^{\dim(h)} \int_H \varphi(X) dX \neq 0$$

and therefore $f$ satisfies the requirement of (A.1).

Thus, it only remains to show (A.2). Let $\psi$ be a nontrivial additive character of $F$. We define a Fourier transform $\varphi \in C^\infty_c(g^*) \mapsto \hat{\varphi} \in C^\infty_c(g)$ by

$$\hat{\varphi}(X) = \int_{g^*} \varphi(Y) \psi(\langle X,Y \rangle) dY, \quad X \in g,$$

where we have denoted by $\langle ., . \rangle$ the canonical pairing between $g$ and $g^*$. Let $g^*_\text{ell} \subseteq g^*$ be the subset of elliptic elements and $h^\perp_{\text{ell}} = h^\perp \cap g^*_\text{ell}$. Then, $g^*_\text{ell}$ is an open subset of $g^*$ and $h^\perp_{\text{ell}}$ is nonempty by assumption. Let $X \in h_{\text{ell}}$ and $U \subseteq g^*_\text{ell}$ be a compact-open neighborhood of $X$. Set $\varphi = 1_U$ where
1_U denotes the characteristic function of U. By [BP16, (15)], \( \varphi \) is a cusp form and moreover we have
\[
\int_{\mathfrak{h}} \varphi(X) dX = \int_{\mathfrak{h}^\perp} 1_U(Y) dY \neq 0.
\]
Thus, \( \varphi \) satisfies the requirement of (A.2). \( \square \)

Applications:

- Let \( G = \text{GL}_{2n}(F) \) and \( H = \text{GL}_n(F) \times \text{GL}_n(F) \) embedded “diagonally by blocks”. Then, we have \( g = M_{2n}(F) \) and \( \mathfrak{h} \) is the subset of matrices of the form

\[
\begin{pmatrix}
A \\
B
\end{pmatrix}
\]

where \( A, B \in \text{Mat}_n(F) \). We identify \( g \) with its dual through the \( G \)-invariant nondegenerate bilinear form \( B \) given by
\[
B(X,Y) = \text{Tr}(XY), \quad X,Y \in g.
\]
Then \( \mathfrak{h}^\perp \) is the subspace of matrices of the form
\[
\begin{pmatrix}
A \\
B
\end{pmatrix}
\]

where \( A, B \in \text{Mat}_n(F) \). Let \( E_0 \) be a degree \( n \) extension of \( F \). Choose \( x \in E_0 \) which is not a square in \( E_0 \) and generates \( E_0 \) as a \( F \)-algebra (such an element exists since the generators of \( E_0 \) as an \( F \)-algebra form a dense subset of \( E_0^\times \) whereas the set of nonsquare in \( E_0^\times \) is open). Set \( E = E_0[\sqrt{x}] \). Then \( E \) is a degree \( 2n \) extension of \( F \) which is generated by \( \sqrt{x} \) and we have \( E = E_0 \oplus \sqrt{x} E_0 \). Fix an \( F \)-linear isomorphism \( F^{2n} \cong E \) which sends the \( n \) first elements of the canonical basis of \( F^{2n} \) to a basis of \( E_0 \) and the \( n \) last element of this canonical basis to a basis of \( \sqrt{x} E_0 \). This isomorphism induces an identification \( M_{2n}(F) \cong \text{End}_F(E) \) which sends the endomorphism of multiplication by \( \sqrt{x} \) to an elliptic element in \( \mathfrak{h}^\perp \). Thus we see that the pair \( (G, H) \) satisfies the assumption of the proposition and therefore there exists at least one \( H \)-distinguished irreducible supercuspidal representation of \( G \).

- This example can be generalized to the setting of the paper as follows. Let \( A \) be a central simple algebra over \( F \) of degree \( 2d \) for some integer \( d \geq 1 \). Let \( E/F \) be a quadratic extension and choose a \( F \)-embedding \( E \hookrightarrow A \). Let \( A_E = \text{Cent}_A(E) \) be the centralizer of \( E \) in \( A \) and set \( G = A^\times, \ H = A_E^\times \). Then, we have natural identifications \( g = A \) and \( \mathfrak{h} = A_E \). Let \( A_E^- \) be the subspace of elements \( a \in A \) such that \( ax = \overline{x}a \) for every \( x \in E \) where we have denoted by \( x \mapsto \overline{x} \) the unique nontrivial \( F \)-automorphism of \( E \). By the Skolem-Noether theorem we have
\[
A = A_E \oplus A_E^-\]
and moreover \( A_E^- \cap A^\times \) is nonempty. We identify \( g \) with its dual through the \( G \)-invariant nondegenerate bilinear form \( B \) given by

\[
B(X, Y) = \text{Tr}_{A/F}(XY), \quad X, Y \in g,
\]

where \( \text{Tr}_{A/F} \) stands for the reduced trace. Then, \( h^\perp \) gets identified with the subspace \( A_E^- \) of \( g \). Let \( F_0/F \) be a degree \( d \) extension such that \( F_E = F_0 \otimes_F E \) is a field (such an extension is easy to construct: if \( d \) is odd choose any degree \( d \) extension, if \( d \) is even we can assume by induction that we already constructed a degree \( d/2 \) extension \( F'_0/F \) with \( F'_E = F'_0 \otimes_F E \) a field and then take for \( F_0 \) any quadratic extension of \( F'_0 \) disjoint from \( F'_E \)). Notice that \( A_E \) is a central simple algebra of degree \( d \) over \( E \). Therefore, as \( E \) is \( p \)-adic, there exists an \( E \)-embedding \( F_E \hookrightarrow A_E \). Let \( A_{F_0} \) be the centralizer of \( F_0 \) in \( A \). It is a degree 2 central simple algebra over \( F_0 \) with \( A_{F_0} \cap A_E = F_E \) and setting \( F_E^- = A_{F_0} \cap A_E^- \) we have

\[
A_{F_0} = F_E \oplus F_E^-.
\]

Moreover the set \( \{ x^2 \mid x \in F_E^- \} \) is a coset for \( N_{F_E/F_0}(F_E^\times) \) in \( F_0^\times \) (where we have denoted by \( N_{F_E/F_0} \) the norm map from \( F_E \) to \( F_0 \)). Since the intersection of this coset with the set of non-squares in \( F_0^\times \) is open and the set of \( y \in F_0^\times \) that generates \( F_0 \) over \( F \) is dense, we can find \( x \in F_E^- \subseteq A_E^- \) such that \( x^2 \) generates \( F_0 \) over \( F \) and is not a square in \( F_0^\times \). Then, \( x \) is of degree \( 2d \) over \( F \) and is thus an elliptic element of \( g \). Therefore we see that the pair \( (G, H) \) satisfies the assumption of the proposition and consequently there exists at least one \( H \)-distinguished irreducible supercuspidal representation of \( G \).
References


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