LINEAR INTERTWINING PERIODS AND EPSILON DICHOTOMY FOR LINEAR MODELS

MIYU SUZUKI AND HANG XUE

Contents

1. Introduction

1.1. **Epsilon dichotomy.** Let E/F be a quadratic field extension of local nonarchimedean fields of characteristic zero and η the quadratic character of $F^\times$ attached to E via the local class field theory. Let n be a positive integer. Take a central division algebra D over F of dimension d^2 and suppose that E can be embedded in $\mathrm{Mat}_n(D)$ as F-algebras. Note that this implies that nd is even. Put $G = \mathrm{GL}_n(D)$ and let H be the centralizer of $E^\times$ in G. Note that H is the multiplicative group of a central simple algebra over E. We say that an admissible representation π of G is *H-distinguished* if

$$\mathrm{Hom}_H(\pi, \mathbb{C}) \neq 0.$$

This Hom-space is at most one dimensional if π is irreducible [BM19]. A (special case of a) conjecture of Prasad and Takloo-Bighash [PTB11] predicts when this Hom-space is one dimensional in terms of local root numbers.

Conjecture 1.1. *Let π be an irreducible admissible representation of G with trivial central character. If π is H-distinguished, then*

1. *the Langlands parameter of π takes values in the $\mathrm{Sp}_{nd}(\mathbb{C})$;*
2. *the root number satisfies $\varepsilon(\pi)\varepsilon(\pi \otimes \eta) = (-1)^n \eta_{E/F}(-1)^{nd/2}$.*

Conversely, if π is square integrable and satisfies (1) and (2), then π is H-distinguished.

Date: August 30th 2022.

The original conjecture of Prasad and Takloo-Bighash requires that the Jacquet–Langlands transfer of π to $\mathrm{GL}_{nd}(F)$ is generic. It is explained in [Suz21] that the genericity condition is not necessary. We should note that the converse implication is not expected to hold in the stated form when π is not square integrable.

The case $nd = 2$ recovers the celebrated theorem of Saito and Tunnell. The case $nd = 4$ was proved in [PTB11]. Both cases can be proved by local theta correspondences. There are currently two approaches when $nd > 4$. Using the relative trace formulae proposed by Guo, the second author [Xue21] proved the forward implication of Conjecture 1.1 completely and the converse implication when either π is supercuspidal and $d \leq 2$ or the Jacquet–Langlands transfer of π to $\mathrm{GL}_{nd}(F)$ is supercuspidal. Sécherre [Séc] proved the direct implication using type theory and some intermediate results from [Xue21] and the converse direction when π is supercuspidal and the residue characteristic is odd using type theory.

The goal of this paper is to prove the following theorem. Recall that we always assume F is of characteristic zero.

Theorem 1.2. *Assume that either the residue characteristic of F is odd or $d \leq 2$. Then the converse implication of Conjecture 1.1 holds.*

This completes the proof of Conjecture 1.1 under the stated hypothesis.

1.2. **Distinguished representations.** Let t, k be positive integers and $n = tk$. Let P be the standard parabolic subgroup of G corresponding to the partition $(k, \cdots, k)$ of n and M its standard Levi subgroup isomorphic to $\mathrm{GL}_k(D) \times \cdots \times \mathrm{GL}_k(D)$. Let ρ be an irreducible supercuspidal representation of $\mathrm{GL}_k(D)$. Its Jacquet–Langlands transfer [DKV84] to $\mathrm{GL}_{kd}(F)$ is a square integrable representation. It is well-known by the classification of Zelevinsky [Zel80] that it is the unique irreducible quotient of

$$\rho' \nu^{(1-l_\rho)/2} \times \cdots \times \rho' \nu^{(l_\rho - 1)/2} \tag{1.1}$$

where l_ρ is an integer and ρ' is an irreducible supercuspidal representation of $\mathrm{GL}_{kd/l_\rho}(F)$. Here following the usual convention, the product $\times \cdots \times$ stands for the parabolic induction, and ν stands for the absolute value of the reduced norm of any central simple algebra. The representation

$$\rho\nu^{(1-t)l_\rho/2} \times \rho\nu^{(3-k)l_\rho/2} \times \cdots \times \rho\nu^{(t-1)l_\rho/2}$$

of G has a unique irreducible quotient which is square integrable. By [DKV84, B.2], all irreducible square integrable representations of G are of this form.

Given this description of irreducible square integrable representation, the idea of proving Theorem 1.2 is simple. Assume that π satisfies the conditions in Conjecture 1.1. These two conditions are transformed into conditions on ρ. We thus need to relate the distinguishedness of ρ and π. This is the content of the next theorem. Note that no assumptions on the residue characteristic or d are imposed in this theorem. The L-factor $L(s, \rho', \mathrm{Sym}^2)$ in the theorem is defined either by the local Langlands correspondence or the Langlands–Shahidi method, which agree by [Hen10, CST17].

Theorem 1.3. *Let π and ρ be as above. Assume that ρ is not one dimensional if $k = 1$. We have the following assertions.*

(1) *If t is odd, then $E^\times$ embeds in $\mathrm{GL}_k(D)$ and we denote by H_k the centralizer of $E^\times$. Then π is H-distinguished if and only if ρ is H_k-distinguished.*
(2) *If t is even, then π is H-distinguished if and only if $L(s, \rho', \mathrm{Sym}^2)$ has a pole at $s = 0$.*

Remark 1.4. It is shown in [Yam17] that ρ' is self-dual if $L(s, \rho', \mathrm{Sym}^2)$ has a pole at $s = 0$. See [Yam17, Theorem A and Remark 1.13].

Proof of Theorem 1.2 assuming Theorem 1.3. If $k = 1$ and ρ is a character, then π is a twist of the Steinberg representation of G. This case has been taken care of by [Cho19] so we assume that we are not in this situation and thus Theorem 1.3 applies.

Assume that t is odd first. Since π satisfies the two conditions in Conjecture 1.1, simple computation of the root numbers gives that ρ also satisfies analogous conditions, cf. [Xue21, Section 4.2]. Under the assumption of Theorem 1.2, we see that ρ is H_k-distinguished. Theorem 1.3 then implies that π is H-distinguished.

Now assume that t is even. Again the computation from [Xue21, Section 4.2] shows that condition (2) of Conjecture 1.1 always holds. The Langlands parameter of π takes the form

$$\phi_{\rho'} \boxtimes \mathrm{Sym}^{tl_\rho - 1} \mathbb{C}^2 : W_F \times \mathrm{SL}_2(\mathbb{C}) \to \mathrm{Sp}_{nd}(\mathbb{C}),$$

where $\phi_{\rho'}$ is the Langlands parameter of ρ'. Since t is even, the image of $\mathrm{Sym}^{tl_\rho - 1}(\mathbb{C})$ lies in $\mathrm{Sp}_{tl_\rho}(\mathbb{C})$. Thus condition (1) implies that $\phi_{\rho'}$ lies in the orthogonal group $\mathrm{O}_{kd/l_\rho}(\mathbb{C})$. This is equivalent to that $L(s, \rho', \mathrm{Sym}^2)$ has a pole at $s = 0$. By Theorem 1.3 we conclude that π is H-distinguished. ☐

Remark 1.5. In both cases t being odd or even, we only need the "if" direction to deduce Theorem 1.2. In fact, the "if" direction is one of the main results of this paper, and the "only if" direction can be quickly deduced from previous results. The argument above essentially proves that Conjecture 1.1 holds for all irreducible square integrable representations if it holds for all irreducible supercuspidal representations.

In general, for real numbers $a < b$, we call the unique irreducible quotient

$$\rho\nu^{l_\rho a} \times \cdots \times \rho\nu^{l_\rho b},$$

or the set $\{\rho\nu^{l_\rho a}, \cdots, \rho\nu^{l_\rho b}\}$ a segment. Segments are in one-to-one correspondence with irreducible square integrable representations. Two segments are linked if they do not contain each other and their union is again a segment. Otherwise they are called unlinked. Combining Theorem 1.2 with the classification result from [Suz21] we obtain the following immediate corollary.

Corollary 1.6. *Assume that either the residue characteristic of F is odd or $d \leq 2$. Let $\pi = \Delta_1 \times \cdots \times \Delta_t$ be a parabolic induction of unlinked segments, or equivalently the Jacquet–Langlands transfer of π to $\mathrm{GL}_{nd}(F)$ is generic. Then π is H-distinguished if and only if we can relabel Δ_i's so that*

(1) $\Delta_i \simeq \Delta_{i+1}^\vee$, *for* $i = 1, 3, \cdots, 2a-1$;

(2) Δ_i *satisfies the analogous conditions in Theorem 1.2, for* $i = 2a+1, \cdots, t$.

Remark 1.7. If $D = F$ and $E = F \times F$, the analogues of Theorem 1.3 and Corollary 1.6 have been established by [Mat14, Theorem 6.1], [Mat15, Theorem 3.13] and [Yam17, Theorem 3.18].

1.3. **Intertwining periods.** Let us keep the notation from Theorem 1.3. If π is H-distinguished, it is relatively easy to deduce information about ρ. The other direction is harder. Assume ρ is H_k-distinguished if t is odd and $L(s, \rho', \mathrm{Sym}^2)$ has a pole at $s = 0$ if t is even (as remarked before this implies that ρ' and hence ρ is self-dual). By [Off17, Proposition 7.2], the full induced representation

$$\rho\nu^{(1-t)l_\rho/2} \times \rho\nu^{(3-k)l_\rho/2} \times \cdots \times \rho\nu^{(t-1)l_\rho/2}$$

is H-distinguished and moreover the H-invariant linear form on it is unique up to a scalar. Thus we are reduced to show that this H-invariant linear form factors through its unique irreducible quotient. To proceed let us introduce some notation. For any real number a, the representation

$$\rho\nu^{al_\rho} \times \rho\nu^{(a+1)l_\rho}$$

contains a unique irreducible subrepresentation. We denote this subrepresentation by $Z([a, a+1]_\rho)$. The kernel of the quotient

$$\rho\nu^{(1-t)l_\rho/2} \times \rho\nu^{(3-t)l_\rho/2} \times \cdots \times \rho\nu^{(t-1)l_\rho/2} \to \pi$$

is generated by

$$\rho\nu^{(1-t)l_\rho/2} \times \cdots \times \rho\nu^{(j-3)l_\rho/2} \times Z([(j-1)/2, (j+1)/2]_\rho) \times \rho\nu^{(j+3)l_\rho/2} \times \cdots \times \rho\nu^{(t-1)l_\rho/2}.$$

for $2 - t \leq j \leq t - 2$ and $j \equiv t \pmod 2$. Jacquet module computation implies that these representations are not H-distinguished if $j \neq 0$ or $j = 0$ but $Z([-1/2, 1/2]_\rho)$ as a representation of G_{2k} is not H_{2k}-distinguished. Therefore if t is odd, then Theorem 1.3 follows as j can never be zero. In the case t being even, to prove Theorem 1.3 we are reduced to show that $Z([-1/2, 1/2]_\rho)$ is not H_{2k}-distinguished, or in other words, Theorem 1.3 when $t = 2$.

We now assume that $t = 2$. Following Jacquet, Lapid and Rogawski, one constructs an explicit H-invariant linear form

$$J(\cdot, s) : I(\rho, s) = \rho\nu^s \times \rho\nu^{-s} \to \mathbb{C}.$$

This is the (open) intertwining period alluded in the title of this paper. This linear form is meromorphic in s, holomorphic at $s = -l_\rho/2$, and defines a nonzero H-invariant linear form. Let

$$M(\tau, s) : I(\rho, s) \to I(\rho, -s)$$

be the usual intertwining operator. Then $Z([-1/2, 1/2]_\rho)$ is the cokernel of the intertwining operator $M(\tau, -l_\rho/2)$. Since the space of H-invariant linear forms on $I(\rho, s)$ is at most one dimensional, we conclude that there is a meromorphic function $\alpha(s)$ such that

$$\alpha(s)J(\phi, s) = J(M(\tau, s)\phi, -s). \tag{1.2}$$

The following is the main technical result of this paper.

Proposition 1.8. *Let the notation be as above. We fix a nontrivial additive character ψ of F. Then*

$$\alpha(s) \sim_{\mathbb{C}[q^{\pm s}]^\times} \gamma(-2s, \mathrm{JL}(\rho)^\vee, \wedge^2, \psi)^{-1}\gamma(2s, \mathrm{JL}(\rho), \mathrm{Sym}^2, \psi)^{-1},$$

where $\mathrm{JL}(\rho)$ stands for the Jacquet–Langlands transfer of ρ to $\mathrm{GL}_{kd}(F)$ and the notation $\sim_{\mathbb{C}[q^{\pm s}]^\times}$ means the ratio of both side lies in $\mathbb{C}[q^{\pm s}]^\times$.

With this proposition, elementary computation gives that $\alpha(s)$ has a zero at $s = -l_\rho/2$. The desired Theorem 1.3 in the case $t = 2$ follows quickly from this. The proof of this proposition is neverthelss quite technical. Our proof is inspired by [Mat, Section 10.4] and uses global-to-local arguments. We are not sure if one can prove this proposition using purely local methods because the appearance of the Jacquet–Langlands transfer. The proof uses the global counterpart of the intertwining periods and the global functional equation analogous to (1.2). Global intertwining periods appear as regularized periods of Eisenstein series and the functional equation of intertwining periods is naturally a consequence of that of Eisenstein series. In the computation, we use regularized periods of Eisenstein series introduced in [Zyd] and follow closely the path paved in [JLR99, LR03] by the pioneers. We tailor the computation to precisely what we need and make it very explicit. A thorough study of regularized linear periods of Eisenstein series and general intertwining periods is itself a very interesting subject. We hope to come back to it in our future work.

1.4. **Acknowledgement.** Some initial discussions that led to this paper took place during the RIMS conference "Analytic, geometric and p-adic aspects of automorphic forms and L-function" in January 2020. The authors would like to thank the organizers of the conference for their hospitality. The authors also thank Atsushi Ichino for useful sugesstions, Michał Zydor for answering many questions regarding truncated period integrals and the anonymous referee for careful reading of the manuscript. M. S. is partially supported by the JSPS Research Fellowship for Young Scientists No. 20J00434. H. X. is partially supported by the NSF grant DMS #1901862 and DMS #2154352.

2. Preliminaries

2.1. **General notation.** Without no explicit mention of the contrary, F is always a number field or a local field of characteristic zero. Thus by saying that "F is a local field", we mean that "F is a local field of characteristic zero". If F is a number field we denote by $\mathbb{A}$ or $\mathbb{A}_F$ the ring of adeles. We fix a nontrivial additive character ψ of $\mathbb{A}/F$ (resp. F) if F is a number field (resp. local field).

Suppose that G is a connected algebraic group defined over F. We denote by $X^*(G)$ the group of rational characters of G. Put as usual

$$\mathfrak{a}_G^* = X^*(G) \otimes_{\mathbb{Z}} \mathbb{R}, \quad \mathfrak{a}_G = \mathrm{Hom}_{\mathbb{Z}}(X^*(G), \mathbb{R}),$$

which are real vector spaces dual to each other and $\mathfrak{a}_{G,\mathbb{C}}^*$ and $\mathfrak{a}_{G,\mathbb{C}}$ their complexifications respectively. Let $\mathrm{Lie}(G)$ be the Lie algebra of G.

For any reductive group G over F, we denote by A_G its split center. We fix a minimal parabolic subgroup P_0 of G and a maximal split torus A_0 of G contained in P_0. Put M_0 be the centralizer of A_0 in G and let U_0 denote the unipotnet radical of P_0. Then $P_0 = M_0U_0$ is a Levi decomposition. Parabolic subgroups containing A_0 are called semi-standard. A semi-standard parabolic subgroup has a unique Levi subgroup which contains A_0 and such Levi subgroup is called semi-standard. We consider only semi-standard parabolic subgroups. Thus by "$P = MU$ is a parabolic subgroup" we mean "P is a semi-standard parabolic subgroup, M is its semi-standard Levi subgroup and U is its unipotent radical". For any parabolic subgroup $P = MU$, we denote by $\rho_P \in \mathfrak{a}_M^*$ the usual half sum of all positive roots in $\mathfrak{n} = \mathrm{Lie}(N)$ (counting multiplicity). Let $W_M = N_M(A_0)/M_0$ be the Weyl group of M with respect to A_0, where $N_M(A_0)$ denotes the normalizer of A_0 in M.

2.2. **Automorphic forms.** Let F be a number field. For a parabolic subgroup $P = MU$ and a smooth function ϕ on $P(F)\backslash G(\mathbb{A})$, we define the constant term along P by

$$\phi_P(g) = \int_{U(F)\backslash U(\mathbb{A})} \phi(ug)\mathrm{d}u, \quad g \in G(\mathbb{A}). \tag{2.1}$$

There is a natural function $H_P : M(\mathbb{A}) \to \mathfrak{a}_P$ characterized by

$$\langle \chi, H_P(m)\rangle = \log|\chi(m)|_{\mathbb{A}}$$

for any $\chi \in X^*(M)$ and $m \in M(\mathbb{A})$. Here, for any place v of F we denote by $|\cdot|_v$ the absolute value of F_v normalized in the usual way and by $|\cdot|_{\mathbb{A}} = \prod_v |\cdot|_v$ the absolute value on $\mathbb{A}^\times$. We require that $|\cdot|_{\mathbb{A}^\times}$ takes value one on $F^\times$. The modulus character on $P(\mathbb{A})$ is given by $e^{\langle 2\rho_P, H_P(\cdot)\rangle}$. We fix a maximal compact subgroup K of $G(\mathbb{A})$ so that $G(\mathbb{A}) = P(\mathbb{A})K$ and extend H_P to $G(\mathbb{A})$ by

$$H_P(muk) = H_P(m), \quad m \in M(\mathbb{A}),\ u \in U(\mathbb{A}),\ k \in K.$$

Put $F_\infty = F \otimes \mathbb{R}$. Then $\mathbb{R}$ embeds in F_∞ naturally via $x \mapsto 1 \otimes x$. Choose an isomorphism $A_M \simeq \mathbb{G}_m^l$ for some integer l and let A_M^∞ be the image of $(\mathbb{R}_{>0})^l$ in $A_M(F_\infty)$. Then H_P induces an isomorphism $A_M^\infty \simeq \mathfrak{a}_P$.

We now recall the definition of automorphic forms, Eisenstein series and intertwining operators following the convention in [LR03, Section 5 and 7]. Let $\mathcal{A}(M)$ be the space of automorphic forms on $M(F)\backslash M(\mathbb{A})$. These are smooth moderate growth functions on $M(F)\backslash M(\mathbb{A})$ which are finite under the translation of a maximal compact subgroup of $M(\mathbb{A})$ and the center of the universal enveloping algebra of $\mathrm{Lie}(M(\mathbb{R}))$. Let $\mathcal{A}_P(G)$ denote the set of automorphic forms on $U(\mathbb{A})M(F)\backslash G(\mathbb{A})$. These are smooth, K-finite functions such that for all $k \in K$, the function $m \mapsto \phi(mk)$ belongs to $\mathcal{A}(M)$.

We write $\mathcal{A}_P^1(G)$ for the set of $\phi \in \mathcal{A}_P(G)$ such that the function $m \mapsto e^{-\langle \rho_P, H_P(m)\rangle}\phi(mg)$ on $M(\mathbb{A})$ is A_M^∞-invariant for any $g \in G(\mathbb{A})$ and satisfies

$$\sup_{g\in G(\mathbb{A})} \left| e^{-\langle \rho_P, H_P(g)\rangle}\phi(g)\right| < \infty.$$

For a cuspidal automorphic representation ρ of $M(\mathbb{A})$ with a central character trivial on A_M^∞, let $\mathcal{A}_P^1(G)_\rho$ be the space of functions $\phi \in \mathcal{A}_P^1(G)$ such that for all $k \in K$, the function $m \mapsto \phi(mk)$ belongs to the space of ρ. Set $\mathcal{A}_P^1(G)_c = \sum_\rho \mathcal{A}_P^1(G)_\rho$, where ρ runs through cupidal automorphic representations of $M(\mathbb{A})$ with a central character trivial on A_M^∞.

The Weyl group W_G is $N_G(A_0)/M_0$. Let $P = MU$ and $Q = LV$ be parabolic subgroups of G. We say that P and Q are associate if M and L are conjugate by an element in $N_G(A_0)$. Let $W(M, L)$ be the set of $w \in W_G$ of minimal length in the coset wW_M with $wMw^{-1} = L$. Let $w \in W(M, L)$, and $s \in \mathfrak{a}_{P,\mathbb{C}}^*$, we have the standard intertwining operator

$$M(w, s) : \mathcal{A}_P^1(G) \to \mathcal{A}_Q^1(G),$$

given in the domain of convergence by

$$M(w,s)\phi(g) = e^{\langle -ws, H_Q(g)\rangle}\int_{V(\mathbb{A})\cap wU(\mathbb{A})w^{-1}\backslash V(\mathbb{A})} \phi(w^{-1}ug)e^{\langle s, H_P(w^{-1}ug)\rangle}\mathrm{d}u. \tag{2.2}$$

For $\phi \in \mathcal{A}_P^1(G)_c$ and $s \in \mathfrak{a}_{P,\mathbb{C}}^*$, we have the cuspidal Eisenstein series $E(g, \phi, s)$ which is given, in its domain of absolute convergence, by

$$E(g,\phi,s) = \sum_{\delta\in P(F)\backslash G(F)} \phi(\delta g)e^{\langle s, H_P(\delta g)\rangle}. \tag{2.3}$$

Further properties of cuspidal Eisenstein series and their constant terms will be recalled when we need them.

2.3. **Symmetric pairs.** Let D be a central division algebra over F of dimension d^2. Set $G_n = \mathrm{GL}_n(D)$. We denote by 1_n the identity matrix in G_n and w_n the matrix whose anti-diagonal entries are one and zero elsewhere. We usually suppress the subscript n and write only G when there is no confusion.

Let E/F be a quadratic field extension and τ be an element of $F^\times$ so that $E = F[\sqrt{\tau}]$. Assume that E can be embedded into $\mathrm{Mat}_n(D)$ as an F-subalgebra. If this is the case, nd is even and such an embedding is unique up to an inner automorphism by the Skolem-Noether theorem. Let H be the centralizer of $E^\times$ in G. We fix an explicit embedding of E into $\mathrm{Mat}_n(D)$ as follows. If d is odd and hence n is even we take

$$a + b\sqrt{\tau} \mapsto \begin{pmatrix} a\cdot 1_{n/2} & \tau b w_{n/2} \\ b w_{n/2} & a \cdot 1_{n/2}\end{pmatrix}, \quad a, b \in F.$$

Then $D_E = D \otimes E$ is a central division algebra over E and H is isomorphic to $\mathrm{GL}_{n/2}(D_E)$ and consists of matrices

$$h(a,b) = \begin{pmatrix} a & \tau b w_{n/2} \\ w_{n/2} b & w_{n/2} a w_{n/2} \end{pmatrix}, \quad a, b \in \mathrm{Mat}_{n/2}(D).$$

Let θ be the involution on G defined as the conjugation by $h(0_{n/2}, 1_{n/2})$. Then H is the group of fixed points of θ. If d is even then E embeds into D. We obtain an embedding of E into $\mathrm{Mat}_n(D)$ from the embedding into D. The centralizer H equals $\mathrm{GL}_n(C)$, where C is the centralizer of E in D which is a central division algebra over E. In this case, let θ be the involution on D or on $\mathrm{Mat}_n(D)$ defined as the conjugation by $\sqrt{\tau}$ or $\sqrt{\tau}1_n$. Then H is the group of the fixed points of θ. We also fix an element $\mu \in D^\times$ such that $\theta(\mu) = -\mu$. Such an element exists because θ is an involution on D which is not the identity map.

We will also consider the case $E = F \times F$. This will only play an auxiliary role in the global-to-local argument. We make a further assumption that $d = 1$ which will be sufficient for our purposes. Then $G = \mathrm{GL}_n(F)$ and we take the embedding of $F^\times \times F^\times \to G$ with

$$(a,b) \mapsto \begin{pmatrix} a1_{n/2} & \\ & b1_{n/2} \end{pmatrix}, \quad a, b \in F^\times.$$

The centralizer H of $F^\times \times F^\times$ is isomorphic to $\mathrm{GL}_{n/2}(F) \times \mathrm{GL}_{n/2}(F)$ consisting of diagonal $\frac{n}{2} \times \frac{n}{2}$ blocks. It is the group of fixed points of the involution given by conjugation of the image of $(1, -1)$.

Each (ordered) partition $(n_1, \cdots, n_t)$ of n corresponds to an upper triangular parabolic subgroup $P = MU$. Unless explicitly saying the contrary, by a parabolic subgroup of G we always mean an upper triangular parabolic subgroup. From now on assume that t and k are positive integers such that $n = tk$ and P the parabolic subgroup corresponding to $(k, \cdots, k)$. Explicit coset representatives for $P\backslash G/H$ have been given in [Cho19, Section 2.2 and 3.2]. We do not need it but only the representative of the open double coset. Assume that E is a field. If d is odd we take $\eta = 1_n$. If d is even and t is odd we put

$$\eta_1 = 1_k, \quad \eta_{i+1} = \begin{pmatrix} 1_k & & -\mu 1_k \\ & \eta_i & \\ 1_k & & \mu 1_k \end{pmatrix},$$

and take $\eta = \eta_{(t+1)/2}$. If d and t are even we put

$$\eta_1 = \begin{pmatrix} 1_k & -\mu 1_k \\ 1_k & \mu 1_k \end{pmatrix}, \quad \eta_{i+1} = \begin{pmatrix} 1_k & & -\mu 1_k \\ & \eta_i & \\ 1_k & & \mu 1_k \end{pmatrix},$$

and take $\eta = \eta_{t/2}$. With these choices, the double coset $P\eta H$ is open in G. Now assume that $E = F \times F$ and $d = 1$. Put

$$\eta = \begin{pmatrix} 1_{n/2} & 1_{n/2} \\ 1_{n/2} & -1_{n/2} \end{pmatrix}.$$

Then $P\eta H$ is open in G. For any subset Y of G, we set $Y^\eta = Y \cap \eta H \eta^{-1}$ and $Y(\eta) = \eta^{-1} Y \eta \cap H$. Note that $P(\eta) = M(\eta)U(\eta)$ by [Cho19, Porposition 2.2 and 3.2].

2.4. **Representations.** Suppose that F is a nonarchimedean local field. By a representation we mean an admissible representation. Let $P = MU$ be the standard parabolic subgroup corresponding to the partition $(n_1, \cdots, n_t)$ of n and we fix a maximal compact subgroup K of G such that $G = PK$. As in the global case, there is a natural function $H_P : M \to \mathfrak{a}_P$ characterized by

$$\langle \chi, H_P(m) \rangle = \log |\chi(m)|_F$$

for any $\chi \in X^*(M)$ and $m \in M$, where $|\cdot|_F$ is the normalized absolute value on F. We extend H_P to G by

$$H_P(muk) = H_P(m), \quad m \in M, \ u \in U, \ k \in K.$$

The space $\mathfrak{a}^*_{P,\mathbb{C}}$ is identified with $\mathbb{C}^t$ as usual. Let $s = (s_1, \cdots, s_t) \in \mathbb{C}^t$. Let $\rho_1, \cdots, \rho_t$ be irreducible representations of $G_{n_1}, \cdots, G_{n_t}$ respectively, and $\rho = \rho_1 \boxtimes \cdots \boxtimes \rho_t$ be the representation of M. Let

$$I_P^G(\rho, s) = \mathrm{Ind}_P^G \, \rho_1 \nu^{s_1} \otimes \cdots \otimes \rho_t \nu^{s_t}$$

be the normalized parabolically induced representation, where ν stands for the absolute value of the reduced norm of any central simple algebra. Following usual practice we also write the parabolic induction $I_P^G(\rho, s)$ as

$$\rho_1 \nu^{s_1} \times \cdots \times \rho_t \nu^{s_t}.$$

We stick to the convention that all parabolic inductions in this paper are normalized.

Suppose that $P = MU$ and $Q = LV$ are associate parabolic subgroups, and $w \in W(M, L)$. We have the standard intertwining operator

$$M(w, s) : I_P^G(\rho, s) \to I_Q^G(w\rho, ws)$$

defined in the domain of absolute convergence by

$$M(w, s)\phi(g) = e^{\langle -ws, H_Q(g) \rangle} \int_{V \cap wUw^{-1} \backslash V} \phi(w^{-1}ug) e^{\langle s, H_P(w^{-1}ug) \rangle} \mathrm{d}u. \tag{2.4}$$

Here $w\rho$ is the representation of $L = wMw^{-1}$ given by $w\rho(l) = \rho(w^{-1}lw)$ for all $l \in L$.

We now recall the local Jacquet-Langlands transfer and classification of square integrable representations of G. The local Jacquet-Langlands transfer JL is a bijective map from the set of irreducible square integrable representations of G_k to that of $\mathrm{GL}_{kd}(F)$ [Bad08, Section 2.3]. Let ρ be an irreducible supercuspidal representation of G_k. Since $\mathrm{JL}(\rho)$ is a square integrable representation of $\mathrm{GL}_{kd}(F)$, by the classification of Zelevinsky [Zel80, Theorem 9.3], it is the unique irreducible quotient of

$$\rho' \nu^{(1-l_\rho)/2} \times \cdots \times \rho' \nu^{(l_\rho - 1)/2} \tag{2.5}$$

where l_ρ is an integer and ρ' is an irreducible supercuspidal representation of $\mathrm{GL}_{kd/l_\rho}(F)$. By [DKV84, B.2], the representation

$$\rho\nu^{(1-t)l_\rho/2} \times \rho\nu^{(3-k)l_\rho/2} \times \cdots \times \rho\nu^{(t-1)l_\rho/2}$$

of G has a unique irreducible quotient $\mathrm{St}_t(\rho)$ which is a square integrable representation and all irreducible square integrable representations of G are of this form. Note that $\mathrm{JL}(\mathrm{St}_t(\rho)) = \mathrm{St}_{tl_\rho}(\rho')$.

We make some remarks on the archimedean places. The consideration at the archimedean places only appears in the global-to-local argument. Suppose that $F = \mathbb{R}$ or $\mathbb{C}$. Without explicit mentioning of the contrary, by a representation of a reductive Lie group G, we mean an admissible finitely generated $(\mathfrak{g}, K)$-module, where $\mathfrak{g}$ is the complexified Lie algebra of G, and K is a maximal compact subgroup of G, cf. [Wal92, Section 3.3.2]. One notable exception is Proposition 4.2 where we make use the canonical Casselman–Wallach globalization of an admissible finitely generated $(\mathfrak{g}, K)$-module, cf. [Wal92, Chapter 11], in particular Theorem 11.6.6.

Finally suppose that F is a number field. The global Jacquet-Langlands transfer is an injective map from the set of irreducible discrete series representations of $G_n(\mathbb{A})$ to that of $\mathrm{GL}_{nd}(\mathbb{A})$ [Bad08, Theorem 5.1]. We use JL to denote this map since there is no chance of confusion. Let σ be an irreducible discrete series representation of $G_n(\mathbb{A})$. The representation $\mathrm{JL}(\sigma)$ is characterized by $\mathrm{JL}(\sigma)_v = \sigma_v$ for any place v at which $G_n(F_v)$ is isomorphic to $\mathrm{GL}_{nd}(F_v)$. Note also that if σ_v is a square integrable representation of $G_n(F_v)$, then $\mathrm{JL}(\sigma)_v = \mathrm{JL}(\sigma_v)$.

3. Intertwining periods: global theory

3.1. **Regularized periods.** We recall the theory of relative truncation operators developed by Zydor [Zyd]. We temporarily switch to the setup of reductive symmetric spaces. So G is a reductive group over a number field F, together with an F-rational involution θ on G. In what follows θ acts on various objects and we use $-^\theta$ to denote the fixed point of θ. Let $H = G^\theta$ be a symmetric subgroup. The setup of [Zyd] is more general, but we specialize to this case and translate some terminologies in [Zyd] into more familiar ones of symmetric spaces.

We fix a maximal θ-stable F-split torus A_0 of G such that $A_0' = A_0^\theta$ is a maximal F-split torus of H (whose existence is guaranteed by [HW93, Proposition 3.5]). Parabolic subgroups containing A_0 or A_0' will be referred to as semi-standard. All parabolic subgroups that we work with in the regularization process will be semi-standard. We recall from [GO16, Lemma 3.1] that parabolic subgroups P' of H are precisely those of the form $P \cap H$ where P is a θ-stable parabolic subgroup of G (the reference [GO16] works with local nonarchimedean local fields, but this lemma and its proof work over any base field of characteristic not two). We fix a minimal parabolic subgroup $P_0' = M_0'U_0'$ of H and a θ-stable parabolic subgroup $P_0 = M_0U_0$ of G with $P_0' = P_0 \cap H$. In what follows for any θ-stable parabolic subgroup $P = MU$ of G, unless otherwise explained, we always write $P' = M'U' = P \cap H$. To ease notation, $\mathfrak{a}_{P_0}$, $\mathfrak{a}_{P_0'}$ and other related spaces will be abbreviated as $\mathfrak{a}_0$, $\mathfrak{a}_{0'}$ and etc. We will encounter various bilinear pairings and we denote them all by $\langle -, - \rangle$. There should be no confusion which pairing it is.

Let $P = MU$ be a θ-stable parabolic subgroup of G. Let $\Delta_P \subset \mathfrak{a}_P^*$ be the set of simple roots for the action of A_P on $\mathrm{Lie}(U)$. Define the positive chamber

$$\mathfrak{a}_P^+ = \{v \in \mathfrak{a}_P \mid \langle v, \alpha \rangle > 0 \text{ for all } \alpha \in \Delta_P\}.$$

We also define the positive chamber $\mathfrak{a}_{P'}^+$ of $\mathfrak{a}_{P'}$ similarly using the parabolic subgroup P' of H. We write $\mathfrak{a}_{0'}^+$ for $\mathfrak{a}_{P_0'}^+$. The involution θ acts on $\mathfrak{a}_0$. We have $\mathfrak{a}_{0'} = \mathfrak{a}_0^\theta$ and hence for any subset Y of $\mathfrak{a}_0$ we have $Y^\theta = Y \cap \mathfrak{a}_{0'}$. In particular $\mathfrak{a}_P^{+,\theta} = \mathfrak{a}_P^+ \cap \mathfrak{a}_{0'}$ for all θ-stable parabolic subgroups P of G. If Y is a subset of $\mathfrak{a}_{0'}$ we denote by $\overline{Y}$ its closure in $\mathfrak{a}_{0'}$. The relative chambers denoted by $\mathfrak{z}_P$, $\mathfrak{z}_P^+$ and $\overline{\mathfrak{z}_P^+}$ in [Zyd, Section 3] are indeed our $\mathfrak{a}_P^\theta$, $\mathfrak{a}_P^{+,\theta}$ and $\overline{\mathfrak{a}_P^{+,\theta}}$, respectively. We fix an inner product on $\mathfrak{a}_{0'}$ which is invariant under the action of the Weyl group of H.

Let Q' be a parabolic subgroup of H. We set

$$\mathcal{P}^G(Q') = \left\{\theta\text{-stable parabolic subgroups } P \text{ of } G \text{ satisfying } P \cap H = Q'\right\},$$

and

$$\mathcal{F}^G(Q') = \bigcup_{Q' \subset P'} \mathcal{P}^G(P')$$

where the union ranges over all parabolic subgroups P' of H containing Q'. By [Zyd, Example 0.3, Proposition 3.1], $\mathcal{F}^G(P_0')$ is precisely the set of θ-stable parabolic subgroups P of G such that $\mathfrak{a}_P^{+,\theta} \neq \emptyset$. Moreover by [Zyd, (3.1)], we have

$$\mathfrak{a}_{Q'}^+ = \coprod_{P \in \mathcal{P}^G(Q')} \mathfrak{a}_P^{+,\theta}, \quad \overline{\mathfrak{a}_{Q'}^+} = \coprod_{P \in \mathcal{F}^G(Q')} \mathfrak{a}_P^{+,\theta}.$$

For $P, Q \in \mathcal{F}^G(P_0')$ such that $P \subset Q$, let $\mathfrak{a}_P^{Q,\theta}$ be the orthogonal complement of $\mathfrak{a}_Q^\theta$ in $\mathfrak{a}_P^\theta$ with respect to the fixed inner product on $\mathfrak{a}_{0'}$. We write $\mathfrak{a}^{Q,\theta}$ for $\mathfrak{a}_{P_0}^{Q,\theta}$. For $X \in \mathfrak{a}_{0'}$, let X_P (resp. X^Q, resp. X_P^Q) be the orthogonal projection of X onto $\mathfrak{a}_P^\theta$ (resp. $\mathfrak{a}^{Q,\theta}$, resp. $\mathfrak{a}_P^{Q,\theta}$).

For $P, Q \in \mathcal{F}^G(P_0')$ such that $P \subset Q$, let z be any fixed point in $\mathfrak{a}_Q^{+,\theta}$. Following [Zyd, Section 3.2], we put $\hat{\tau}_P^Q$ the characteristic function of the interior of the cone

$$\left\{v \in \mathfrak{a}_{0'} \mid \langle v, x - z \rangle > 0 \text{ for all } x \in \overline{\mathfrak{a}_P^{+,\theta}}\right\},$$

and τ_P^Q the characteristic function of the interior of the dual cone

$$\left\{\lambda(x - z) \in \mathfrak{a}_P^\theta \mid \lambda > 0,\ x \in \overline{\mathfrak{a}_P^{+,\theta}}\right\}.$$

These definitions are independent of the choice of z. We omit the superscript if $Q = G$.

Let us fix an element $\mathcal{T}_{\mathrm{reg}} = \mathcal{T}_{H,\mathrm{reg}} \in \mathfrak{a}_{0'}$ as in [Zyd, Section 3.5]. It is by definition an element so that [Zyd, Lemma 2.7] holds. The definition of this element is part of the reduction theory and is rather technical, but we do not really need the precise form of it. The point is that any element $T \in \mathcal{T}_{\mathrm{reg}} + \mathfrak{a}_{0'}^+$ will be sufficiently regular, i.e. sufficiently away from the walls of the Weyl chambers.

Let Q be a parabolic subgroup in $\mathcal{F}^G(P'_0)$ and ϕ a locally integrable function on $Q(F)\backslash G(\mathbb{A})$. For a parabolic subgroup P contained in Q, recall that the constant term ϕ_P is defined by (2.1). The relative truncation operator is defined as follows, cf. [Zyd, Section 3.7]

$$\Lambda^{T,Q}\phi(x) = \sum_{\substack{P\in\mathcal{F}^G(P'_0)\\ P\subset Q}} (-1)^{\dim \mathfrak{a}_P^{Q,\theta}} \sum_{\delta\in P'(F)\backslash Q'(F)} \widehat{\tau}_P^Q(H_{P'_0}(\delta x)^Q - T^Q)\phi_P(\delta x), \tag{3.1}$$

for all $x \in Q'(F)\backslash H(\mathbb{A})$. The sums in the definition of $\Lambda^{T,Q}$ are all finite, cf. [Zyd, Lemma 2.8]. When $Q = G$, we write $\Lambda^T = \Lambda^{T,G}$. We observe that if ϕ is a locally integrable function on $G(F)\backslash G(\mathbb{A})$ then

$$\Lambda^{T,Q}\phi = \Lambda^{T,Q}\phi_Q.$$

We also have an inversion formula [Zyd, Lemma 3.7]

$$\phi(x) = \sum_{Q\in\mathcal{F}^G(P'_0)} \sum_{\delta\in Q'(F)\backslash H(F)} \tau_Q(H_{P'_0}(\delta x)_Q - T_Q)\Lambda^{T,Q}\phi(\delta x) \tag{3.2}$$

for all ϕ being a locally integrable function on $G(F)\backslash G(\mathbb{A})$ and $x \in H(\mathbb{A})$.

Take a parabolic subgroup $Q = LV$ in $\mathcal{F}^G(P'_0)$. We define

$$L'(\mathbb{A})^{Q,1} = \{x \in L'(\mathbb{A}) \mid H_{Q'}(x)_Q = 0\}.$$

For $T \in \mathcal{T}_{\mathrm{reg}} + \mathfrak{a}_{0'}^+$, the function $h \mapsto \Lambda^T\phi(h)$ is of rapid decay by [Zyd, Theorem 3.9]. Thus we can define a functional $\mathcal{P}^T$ on $\mathcal{A}(G)$ by

$$\mathcal{P}^T(\phi) = \int_{H(F)\backslash H(\mathbb{A})^{G,1}} \Lambda^T\phi(h)\mathrm{d}h.$$

To each $\phi \in \mathcal{A}(G)$ and each $P \in \mathcal{F}^G(P'_0)$, one associates a set of relative exponents $\mathcal{E}_P(\phi)' \subset \mathfrak{a}_{0',\mathbb{C}}$, cf. [Zyd, Section 4.2]. Let $\rho_{\underline{P}}$ be the projection of $\rho_P - 2\rho_{P'}$ to $\mathfrak{a}_{0'}$. Define a subspace of $\mathcal{A}(G)$ by

$$\mathcal{A}(G)^{\mathrm{reg}} = \{\phi \in \mathcal{A}(G) \mid \langle \lambda + \rho_{\underline{P}}, \mathfrak{a}_P^{G,\theta}\rangle \neq 0 \text{ for all } \lambda \in \mathcal{E}_P(\phi)' \text{ and all maximal } P \in \mathcal{F}^G(P'_0)\}.$$

According to [JLR99, Section 9], all cuspidal Eisenstein series defined as in (2.3) whose parameter (s in (2.3)) takes generic values belong to this space. In fact the argument of [JLR99] shows that these Eisenstein series lie in a slightly smaller subspace $\mathcal{A}(G)^{**} \subset \mathcal{A}(G)^{\mathrm{reg}}$. Strictly speaking only the Galois symmetric spaces are considered in [JLR99], but the argument carries over in general without change.

The main assertion of [Zyd, Theorem 4.1] is that the map

$$T \mapsto \mathcal{P}^T(\phi)$$

is a polynomial exponential in T and if moreover $\varphi \in \mathcal{A}(G)^{\mathrm{reg}}$, its purely polynomial part is a constant. This constant is denoted by

$$\mathcal{P}(\varphi) = \int^*_{H(F)\backslash H(\mathbb{A})^{G,1}} \varphi(h)\mathrm{d}h,$$

and is referred to as the regularized period of φ.

3.2. **Global intertwining periods.** In this subsection we return to the setup of the symmetric space introduced in Subsection 2.3. We will consider a very special case, namely $t = 2$ and $k = 1$ so $G = \mathrm{GL}_2(D)$ where D is a central division algebra over a number field F. If d is odd then $H = D_E^\times$, which is anisotropic. If d is even, then E embeds in D and $H = \mathrm{Res}_{E/F}\,\mathrm{GL}_2(C)$ where C is the centralizer of E in D.

Let $P = MU$ be the unique upper triangular parabolic subgroup of G. Then P and its transpose are the only proper parabolic subgroups of G and $\mathfrak{a}_P^{G,*}$ is one dimensional. We fix an identification $\mathfrak{a}_{P,\mathbb{C}}^{G,*} \simeq \mathbb{C}$ which sends the unique simple weight to 1. We also fix an identification $\mathfrak{a}_P^G \simeq \mathbb{R}$ so that the unique simple coroot is sent to 1. Then the pairing between $\mathfrak{a}_{P,\mathbb{C}}^G$ and $\mathfrak{a}_P^{G,*}$ reduces to the usual multiplication of real and complex numbers.

If d is odd, then $G = PH$ and we put $\eta = 1$. If d is even, then there are two double cosets. We have fixed a representative of the open one as $\eta = \begin{pmatrix} 1 & -\mu \\ 1 & \mu \end{pmatrix}$. In this case M^η is isomorphic to $D^\times$, embedded in M as $\begin{pmatrix} a & \\ & \theta(a) \end{pmatrix}$, $a \in D^\times$. The other double coset is PH which is closed and $P \cap H = M \cap H$ is isomorphic to $C^\times \times C^\times$.

Let $\phi \in \mathcal{A}_P^1(G)_c$ and $s \in \mathfrak{a}_{P,\mathbb{C}}^{G,*}$. We define the global (open) intertwining period by

$$J(\phi, s) = \int_{P(\eta)(\mathbb{A})\backslash H(\mathbb{A})} \left(\int_{M^\eta(F)\backslash M^\eta(\mathbb{A})^{P,1}} \phi(m\eta h)\mathrm{d}m \right) e^{\langle s, H_P(\eta h)\rangle}\mathrm{d}h, \tag{3.3}$$

where $M^\eta(\mathbb{A})^{P,1} = M^\eta(\mathbb{A}) \cap M'(\mathbb{A})^{P,1}$. Let us first check the invariance property. Since M^η consists of elements of the form $\begin{pmatrix} a & \\ & \theta(a) \end{pmatrix}$, $a \in D^\times$, the function

$$h \mapsto e^{\langle s, H_P(\eta h)\rangle}$$

is left invariant by $P(\eta)(\mathbb{A})$. Moreover $M^\eta(\mathbb{A})^{P,1} A_\infty^\eta = M^\eta(\mathbb{A})$ where A_∞^η consists of $\begin{pmatrix} a & \\ & a \end{pmatrix}$, $a \in F_\infty^\times$. Since $\phi \in \mathcal{A}_P^1(G)$, we have $\phi(zg) = \phi(g)$ if $z \in A_\infty^\eta$. It follows that the function

$$h \mapsto \int_{M^\eta(F)\backslash M^\eta(\mathbb{A})^{P,1}} \phi(m\eta h)\mathrm{d}m$$

is left invariant by $P(\eta)(\mathbb{A})$. The defining integral of $J(\phi, s)$ thus makes sense.

Theorem 3.1. *Let π be an irreducible cuspidal automorphic representation of $D^\times(\mathbb{A})$, and $\phi \in \mathcal{A}_P^1(G)_{\pi\boxtimes\pi^\vee}$. The integral (3.3) is absolutely convergent when $\mathrm{Re}\, s >> 0$ and has a meromorphic continuation to the whole complex plane. Moreover if π is not self-dual, then we have the functional equation*

$$J(\phi, s) = J(M(w, s)\phi, -s)$$

where $w \in W(M, M)$ is the nontrivial element.

We note that if d is odd, this theorem is essentially trivial. The group H is anisotropic modulo the center so absolute convergence and meromorphic continuation are obvious. Moreover recall that we have the Eisenstein series $E(g, \phi, s)$ on $G(\mathbb{A})$, defined as in (2.3). The usual unfolding argument gives that

$$\int_{H(F)\backslash H(\mathbb{A})^{G,1}} E(h, \phi, s)\mathrm{d}h = J(\phi, s) \tag{3.4}$$

The functional equation then follows from that of the Eisenstein series.

The rest of this subsection will deal with the case d being even. We will prove a relation similar to (3.4), with the period integral on the left hand side replaced by the regularized period. Theorem 3.1 follows immediately.

Let us first check the absolute convergence.

Lemma 3.2. *There is an $s_0 \in \mathbb{R}$ so that if $\mathrm{Re}\, s > s_0$ then the defining integral of $J(\phi, s)$ is absolutely convergent. Moreover for a fixed $s_1 > s_0$, the function*

$$t \mapsto J(\phi, s_1 + \sqrt{-1}t)$$

is bounded.

Proof. The proof is essentially that of [JLR99, Lemma 27]. As D is a division algebra, the inner integral of (3.3) is over a compact region. Since cusp forms are bounded, it is enough to show the integral

$$\int_{P(\eta)(\mathbb{A})\backslash H(\mathbb{A})} e^{\langle s, H_P(\eta h)\rangle}\mathrm{d}h$$

converges absolutely when $\mathrm{Re}\, s$ is sufficiently large. Boundedness in the imaginary part of s also follows from this, since $|e^{\langle s, H_P(\eta h)\rangle}| = e^{\langle \mathrm{Re}\, s, H_P(\eta h)\rangle}$. Explicitly the function $g \mapsto e^{\langle s, H_P(g)\rangle}$ is given by

$$\begin{pmatrix} a & * \\ & b \end{pmatrix} k \mapsto \nu(ab^{-1})^s$$

where $a, b \in D^\times(\mathbb{A})$ and k lies in the maximal compact subgroup K of G used to define H_P.

As in [JLR99, Lemma 27], we represent the function $g \mapsto e^{\langle s, H_P(g)\rangle}$ by an integral. We consider the integral

$$\int_{D^\times(\mathbb{A})} \phi(t)\nu(t)^s\mathrm{d}t,$$

where ϕ is a Schwartz function on $D(\mathbb{A})$ and $\mathrm{d}t$ is the multiplicative measure on $D^\times(\mathbb{A})$. This integral is a type of Godement–Jacquet zeta integral, which is convergent when $\mathrm{Re}\, s$ is large and is a holomorphic multiple of $\zeta_D(s - \frac{1}{2}(d-1))$, the standard zeta function for D. We can choose ϕ so that it actually equals $\zeta_D(s - \frac{1}{2}(d-1))$. Let Φ be a Schwartz function on $D(\mathbb{A}) \times D(\mathbb{A})$ that is invariant under the right translation of K and $\Phi(0, t) = \phi(t)$. Consider

$$\frac{\nu(g)^s}{\zeta_D(2s - \frac{1}{2}(d-1))} \int_{D^\times(\mathbb{A})} \Phi((0, t)g)\nu(t)^{2s}\mathrm{d}t, \quad g \in G(\mathbb{A}).$$

By our choice this integral equals $e^{\langle s, H_P(g)\rangle}$. We can rewrite this integral as an integral over M^η as

$$e^{\langle s, H_P(g)\rangle} = \frac{1}{\zeta_D(2s - \frac{1}{2}(d-1))} \int_{M^\eta(\mathbb{A})} \Phi((0,1)yg)\nu(yg)^s \mathrm{d}y.$$

After a change of variables we get for $h \in H(\mathbb{A})$

$$e^{\langle s, H_P(\eta h)\rangle} = \frac{1}{\zeta_D(2s - \frac{1}{2}(d-1))} \int_{M(\eta)(\mathbb{A})} \Phi((0,1)\eta y h)\nu(\eta y h)^s \mathrm{d}y.$$

Since $M(\eta) = P(\eta)$ we have

$$\int_{P(\eta)(\mathbb{A})\backslash H(\mathbb{A})} e^{\langle s, H_P(\eta h)\rangle} \mathrm{d}h = \frac{1}{\zeta_D(2s - \frac{1}{2}(d-1))} \int_{H(\mathbb{A})} \Phi((0,1)\eta h)\nu(\eta h)^s \mathrm{d}h. \tag{3.5}$$

For $a, b, c, d \in C$, the function

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \Phi(a - \mu c, b - \mu d)$$

is a Schwartz function on $M_2(C)$. Moreover if

$$h = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(C),$$

we have

$$\Phi((0,1)\eta h) = \Phi(a - \mu c, b - \mu d).$$

It follows that the right hand side of (3.5) is a Godement–Jacquet zeta integral for $\mathrm{GL}_2(C)$. Therefore it converges absolutely for $\operatorname{Re} s$ sufficiently large. This proves the lemma. □

Theorem 3.3. *Let π be an irreducible cuspidal automorphic representation of $D^\times(\mathbb{A})$ and $\phi \in \mathcal{A}^1_P(G)_{\pi\boxtimes\pi^\vee}$. Assume that π is not self-dual. Then if $\Re s$ is sufficiently large then*

$$\int^*_{H(F)\backslash H(\mathbb{A})^{G,1}} E(h, \phi, s)\mathrm{d}h = J(\phi, s). \tag{3.6}$$

Proof. Before we delve into the proof, let us first make various objects in Subsection 3.1 explicit in our context.

The space $\mathfrak{a}_P$ is spanned by two coroots e_1 and e_2, where

$$e_1(a) = \begin{pmatrix} a & \\ & 1 \end{pmatrix}, \quad e_2(a) = \begin{pmatrix} 1 & \\ & a \end{pmatrix}.$$

The involution θ acts trivially on $\mathfrak{a}_P$ and $\mathfrak{a}_P^*$. Thus we have $\mathfrak{a}_P = \mathfrak{a}_{P'}$ and $\mathfrak{a}_P^* = \mathfrak{a}_{P'}^*$. For all $m \in M'(\mathbb{A})$, we have $H_P(m) = H_{P'}(m)$. We also have

$$\rho_P = 2\rho_{P'}$$

as $\dim_F D = 2\dim_F C$.

The subspaces $\mathfrak{a}_G$ and $\mathfrak{a}_P^G$ are spanned by e_1+e_2 and e_1-e_2 respectively. The element e_1-e_2 is the unique simple coroot, and we have made an identification $\mathfrak{a}_P^G \simeq \mathbb{R}$ which sends e_1-e_2 to 1. The function τ_P is the characteristic function of the region

$$\{ae_1+be_2 \mid a>b\}.$$

Therefore when restricted to $\mathfrak{a}_P^G$, it is nothing but the characteristic function of $\mathbb{R}_{>0}$. There is an element $\mathcal{T}_{\mathrm{reg}} \in \mathfrak{a}_P$ and the truncation operator T lies in $\mathcal{T}_{\mathrm{reg}}+\mathfrak{a}_P^+$. In the current situation, this simply means that $T^G \in \mathfrak{a}_P^G$ is a large real number.

There are only two proper parabolic subgroups of G, i.e. P and its transpose. We have

$$\mathcal{F}^G(P')=\{P,G\}.$$

The group $H(\mathbb{A})^{G,1}$ consists of elements h with $\nu(h)=1$. The group $M'(\mathbb{A})^{P,1}$ consists of

$$\begin{pmatrix} a & \\ & b \end{pmatrix}, \quad a,b\in C^\times(\mathbb{A}), \quad \nu(a)=\nu(b)=1.$$

Therefore $M'(F)\backslash M'(\mathbb{A})^{P,1}$ is compact.

Put $A_{M'}^1 = A_{M'}^\infty \cap H(\mathbb{A})^{G,1}$. Then the map $H_{P'}: A_{M'}^1 \to \mathfrak{a}_P^G$ is a bijection and its inverse is denoted by

$$\mathfrak{a}_P^G \to A_{M'}^1, \quad X\mapsto e^X.$$

By the Iwasawa decomposition, cf. [Zyd, Subsection 4.3], we have

$$H(\mathbb{A})^{G,1} = U'(\mathbb{A})A_{M'}^1 M'(\mathbb{A})^{P,1}K \tag{3.7}$$

where K is a maximal compact subgroup of $H(\mathbb{A})$. That is, any element $h\in H(\mathbb{A})^{G,1}$ can be written in the form

$$ue^X mk, \quad u\in U'(\mathbb{A}),\ X\in \mathfrak{a}_P^G,\ m\in M'(\mathbb{A})^{P,1},\ k\in K.$$

We fix a measure on K and on $U'(\mathbb{A})$ such that the volumes of K and $U'(F)\backslash U'(\mathbb{A})$ equal one. The space $\mathfrak{a}_P^G$ is given the usual Lebesgue measure. We fix a measure on $H(\mathbb{A})^{G,1}$ and then there is a unique measure on $M'(\mathbb{A})^{P,1}$ such that

$$\int_{H(\mathbb{A})^{G,1}} f(h)\mathrm{d}h = \int_{U'(\mathbb{A})}\int_{\mathfrak{a}_P^G}\int_{M'(\mathbb{A})^{P,1}}\int_K e^{\langle -2\rho_{P'},X\rangle} f(ue^X mk)\mathrm{d}k\mathrm{d}m\mathrm{d}X\mathrm{d}u, \tag{3.8}$$

for all compactly supported locally integrable function f on $H(\mathbb{A})^{G,1}$.

We have $H_{P'}(ue^X mk)=X$. If $T\in\mathfrak{a}_P^G$, with the identification $\mathfrak{a}_P^G\simeq\mathbb{R}$, then

$$\tau_P(H_{P'}(ue^X mk)-T) = \begin{cases} 1, & X>T \\ 0, & X\le T \end{cases}$$

We now begin the proof of the theorem, following the argument of [LR03, Theorem 9.1.1] closely. Due to our specific situation the computation can be made very explicit.

Let f be a Paley–Wiener function on $\mathfrak{a}_{P,\mathbb{C}}^{G,*}$, cf. [MW95, II.1.2]. In our current setup, with the identification $\mathfrak{a}_{P,\mathbb{C}}^{G,*} \simeq \mathbb{C}$, this means that f is of the form

$$f(s) = \int_{-\infty}^{\infty} \varphi(t) e^{ts} \mathrm{d}t,$$

where φ is a compactly supported function on $\mathbb{R}$. In particular f is holomorphic and is of rapid decay uniformly as $\operatorname{Im} s \to \infty$ in any vertical strip $a \leq \operatorname{Re} s \leq b$.

<u>*Claim*</u>. Assume that f vanishes at 0. Then for some sufficiently large real number s_0 we have

$$\int_{H(F)\backslash H(\mathbb{A})^{G,1}} \int_{\operatorname{Re} s = s_0} E(h,\phi,s) f(s) \mathrm{d}s \mathrm{d}h = \int_{\operatorname{Re} s = s_0} \int^{*}_{H(F)\backslash H(\mathbb{A})^{G,1}} E(h,\phi,s) f(s) \mathrm{d}h \mathrm{d}s, \tag{3.9}$$

both sides being convergent as iterated integrals.

Using the inversion formula (3.2), the left hand side of (3.9) equals

$$\sum_{Q \in \mathcal{F}^G(P')} \int_{Q'(F)\backslash H(\mathbb{A})^{G,1}} \int_{\operatorname{Re} s = s_0} \Lambda^{T,Q} E(h,\phi,s) f(s) \tau_Q(H_{P'}(h)_Q - T_Q) \mathrm{d}s \mathrm{d}h$$

As the central character of $E(h,\phi,s)$ is trivial, we may assume that $T \in \mathfrak{a}_P^G$. We recall that by definition $\Lambda^{T,Q} E(g,\phi,s) = \Lambda^{T,Q} E_Q(g,\phi,s)$ where E_Q stands for the constant term along Q. The sum over Q contains only two terms: $Q = G$ or $Q = P$. If $Q = P$, then the constant term is computed in [JLR99, Section 9]

$$E_P(g,\phi,s) = \phi(g) e^{\langle s, H_P(g) \rangle} + M(w,s)\phi(g) e^{\langle -s, H_P(g) \rangle},$$

where w is the unique nontrivial Weyl group element of G. The left hand side of (3.9) thus equals

$$\begin{aligned} &\int_{H(F)\backslash H(\mathbb{A})^{G,1}} \int_{\operatorname{Re} s = s_0} \Lambda^T E(h,\phi,s) f(s) \mathrm{d}s \mathrm{d}h \\ &+ \int_{P'(F)\backslash H(\mathbb{A})^{G,1}} \int_{\operatorname{Re} s = s_0} \phi(h) e^{\langle s, H_P(h) \rangle} f(s) \tau_P(H_{P'}(h) - T) \mathrm{d}s \mathrm{d}h \\ &+ \int_{P'(F)\backslash H(\mathbb{A})^{G,1}} \int_{\operatorname{Re} s = s_0} M(w,s)\phi(h) e^{\langle -s, H_P(h) \rangle} f(s) \tau_P(H_{P'}(h) - T) \mathrm{d}s \mathrm{d}h. \end{aligned}$$

We denote these terms by $I + II + III$ and compute them separately.

- Since s_0 is large and $\Lambda^T E(h,\phi,s)$ is of rapid decay, the double integral I is absolutely convergent and we can change the order of integration.
- We now compute II. Write $h = ue^X mk$ as in (3.8). Then

  $$II = \int_{M'(F)\backslash M'(\mathbb{A})^{P,1}} \int_T^{\infty} \int_{\operatorname{Re} s = s_0} \phi_K(e^X m) e^{(s - 2\rho_{P'})X} f(s) \mathrm{d}s \mathrm{d}X \mathrm{d}m.$$

  Here $\phi_K(g) = \int_K \phi(gk) \mathrm{d}k$, and we have moved the integral over K inside because K is compact. Since $\phi \in \mathcal{A}_P^1(G)$, we have

  $$\phi(ag) = e^{\langle \rho_P, H_P(a) \rangle} \phi(g)$$

for all $a \in A_M^\infty$ by definition. It follows that

$$\phi_K(e^X m)e^{(s-2\rho_{P'})X} = \phi_K(m)e^{sX}.$$

Thus

$$II = \int_{M'(F)\backslash M'(\mathbb{A})^{P,1}} \int_T^\infty \int_{\mathrm{Re}\, s=s_0} \phi_K(m)e^{sX} f(s)\mathrm{d}s\mathrm{d}X\mathrm{d}m.$$

Let s_0' be a sufficiently negative real number. We can shift the integral over $\mathrm{Re}\, s = s_0$ to $\mathrm{Re}\, s = s_0'$ because f is a Paley–Wiener function. Along the line $\mathrm{Re}\, s = s_0'$ the inner two integrals are absolutely convergent and can be switched. Elementary computation then gives

$$II = \int_{M'(F)\backslash M'(\mathbb{A})^{P,1}} \phi_K(m)\mathrm{d}m \times \int_{\mathrm{Re}\, s=s_0'} -\frac{e^{sT}}{s} f(s)\mathrm{d}s.$$

As $f(s)$ vanishes at $s = 0$, the integrand is holomorphic. Thus we can shift the contour $\mathrm{Re}\, s = s_0'$ back to $\mathrm{Re}\, s = s_0$ and obtain that

$$II = -\int_{M'(F)\backslash M'(\mathbb{A})^{P,1}} \phi_K(m)\mathrm{d}m \times \int_{\mathrm{Re}\, s=s_0} \frac{e^{sT}}{s} f(s)\mathrm{d}s.$$

- The integral III can be computed similarly. Like the integral II, it equals

$$III = \int_{M'(F)\backslash M'(\mathbb{A})^{P,1}} \int_T^\infty \int_{\mathrm{Re}\, s=s_0} (M(w,s)\phi)_K(m)e^{-sX} f(s)\mathrm{d}s\mathrm{d}X\mathrm{d}m.$$

  The domain of the outer integral is compact. As s_0 is a large real number, the intertwining operator $M(w,s)$ is given by the convergent integral (2.2). It follows from (2.2) is bounded by some constant independent of the imaginary part of s. Therefore the integral III is absolutely convergent. We can change of the order of integration, and there is no need of shifting the contour as in the computation of integral II. The result is

$$III = -\int_{\mathrm{Re}\, s=s_0} \frac{e^{-sT}}{-s} f(s) \left(\int_{M'(F)\backslash M'(\mathbb{A})^{P,1}} (M(w,s)\phi)_K(m)\mathrm{d}m \right) \mathrm{d}s.$$

With these computations, the conclusion is that the left hand side of (3.9) equals

$$\begin{aligned} &\int_{\mathrm{Re}\, s=s_0} \int_{H(F)\backslash H(\mathbb{A})^{G,1}} \Lambda^T E(h,\phi,s) f(s)\mathrm{d}s\mathrm{d}h \\ &- \int_{M'(F)\backslash M'(\mathbb{A})^{P,1}} \phi_K(m)\mathrm{d}m \times \int_{\mathrm{Re}\, s=s_0} \frac{e^{sT}}{s} f(s)\mathrm{d}s \\ &- \int_{\mathrm{Re}\, s=s_0} \frac{e^{-sT}}{-s} f(s) \left(\int_{M'(F)\backslash M'(\mathbb{A})^{P,1}} (M(w,s)\phi)_K(m)\mathrm{d}m \right) \mathrm{d}s. \end{aligned} \tag{3.10}$$

The right hand side of (3.9) is computed in [Zyd, Corollary 4.3]. The regularized period of $E(g,\phi,s)$ equals

$$\begin{aligned}(3.11)\qquad &\int_{H(F)\backslash H(\mathbb{A})^{G,1}} \Lambda^T E(h,\phi,s)\mathrm{d}h \\ &-\frac{e^{sT}}{s}\int_{M'(F)\backslash M'(\mathbb{A})^{P,1}} \phi_K(m)\mathrm{d}m - \frac{e^{-sT}}{-s}\int_{M'(F)\backslash M'(\mathbb{A})^{P,1}} (M(w,s)\phi)_K(m)\mathrm{d}m.\end{aligned}$$

The computation in [Zyd] assumes that A_G^∞ is trivial. But as our Eisenstein series has trivial central character, we may view it as an automorphic form on $A_G(\mathbb{A})\backslash G(\mathbb{A})$. In the computation (3.11), this amounts to assuming that $T\in\mathfrak{a}_P^G$, which is what we have done. Thus the result of [Zyd] applies.

Each term of (3.11) is bounded along the line $\mathrm{Re}\, s = s_0$. It follows that the regularized period of $E(g,\phi,s)$ is also bounded along this line. The integration of of (3.11) along $\mathrm{Re}\, s = s_0$ against $f(s)$ is absolutely convergent and it follows from (3.11) that the right hand side of (3.9) equals (3.10). This proves the claim, i.e. the identity (3.9).

Let us now unfold the left hand side of (3.9). The usual unfolding argument gives that the left hand side of (3.9) equals

$$\begin{aligned}(3.12)\qquad &\int_{P(\eta)(F)\backslash H(\mathbb{A})^{G,1}} \phi(\eta h)\left(\int_{\mathrm{Re}\, s=s_0} e^{\langle s, H_P(\eta h)\rangle} f(s)\mathrm{d}s\right)\mathrm{d}h \\ &+\int_{P'(F)\backslash H(\mathbb{A})^{G,1}} \phi(h)\left(\int_{\mathrm{Re}\, s=s_0} e^{\langle s, H_P(h)\rangle} f(s)\mathrm{d}s\right)\mathrm{d}h.\end{aligned}$$

We show that the second term vanishes. Indeed with the choice of the measures (3.8), we can rewrite the second integral as

$$\int_{-\infty}^{\infty}\int_{M'(\mathbb{A})\backslash H(\mathbb{A})}\int_{M'(F)\backslash M'(\mathbb{A})^{P,1}}\int_{\mathrm{Re}\, s=s_0} \phi_K(mh)e^{sX}f(s)\mathrm{d}s\mathrm{d}m\mathrm{d}h\mathrm{d}X.$$

The inner two integrals, i.e. those over $m\in M'(F)\backslash M'(\mathbb{A})^{P,1}$ and over s, can be put in any order, as the integrand containing m and that containing s are separated. By our assumption, π is not self-dual and hence it is not distinguished by $C^\times(\mathbb{A})$. It follows that the integral over $M'(F)\backslash M'(\mathbb{A})^{P,1}$ vanishes. This proves that the second term of (3.12) vanishes.

The first double integral in (3.12) is absolutely convergent by Lemma 3.2. We can switch the order and conclude that the left hand of (3.9) equals

$$\int_{\mathrm{Re}\, s=s_0} J(\phi,s)f(s)\mathrm{d}s.$$

In conclusion, we have proved that

$$\int_{\mathrm{Re}\, s=s_0}\left(\int^*_{H(F)\backslash H(\mathbb{A})^{G,1}} E(h,\phi,s)\mathrm{d}h\right) f(s)\mathrm{d}s = \int_{\mathrm{Re}\, s=s_0} J(\phi,s)f(s)\mathrm{d}s$$

for all Paley–Wiener functions on $\mathfrak{a}_{P,\mathbb{C}}^{G,*}$ which vanish at $s=0$. The desired identity (3.6) then follows from the following calculus fact, which is a very special case of [LR03, Lemma 9.1.2].

Fact. Let α be a bounded countinuous function on $\mathrm{Re}\, s = s_0$. If

$$\int_{\mathrm{Re}\, s=s_0} \alpha(s) f(s) \mathrm{d}s = 0$$

for all Paley–Wiener functions f vanishing at $s = 0$, then $\alpha(s) = 0$. □

Proof of Theorem 3.1. As in the case d being odd, Theorem 3.1 follows from Theorem 3.3 immediately. □

4. Intertwining periods: local theory

4.1. **Local intertwining periods.** We assume that F is a local field and E/F a quadratic étale algebra. Let r, k, d be positive integers, $t = 2r$ and $n = tk$. We keep the setup in Subsection 2.3, i.e. $G = G_n$ with an embedding $E^\times \to G$, H the centralizer of $E^\times$ in G. As in Subsection 2.3, for simplicity we further assume that $d = 1$ if $E = F \times F$. In this section, to ease notation, we usually write G instead of $G(F)$ for the group of F-points. Similar notation applies to other groups.

Assume $t = 2$. Let P be the parabolic subgroup of G corresponding to the partition (k, k) and $P\eta H$ the open double coset in G. We fix a maximal compact subgroup K of G with $G = PK$. Let ρ be an irreducible representation of G_k. Then $\pi = \rho \boxtimes \rho^\vee$ is an irreducible M^η-distinguished representation of M. Indeed, an element of M^η is of the form

$$\begin{pmatrix} m & \\ & \vartheta(m) \end{pmatrix}$$

where $m \in \mathrm{GL}_k(D)$,

$$\vartheta(m) = \begin{cases} w_k m w_k, & E \text{ is a field and } d \text{ is odd,} \\ \theta(m), & E \text{ is a field and } d \text{ is even,} \\ m, & E = F \times F. \end{cases}$$

For any element $h \in G_k$, we define a representation ${}^h\rho$ by ${}^h\rho(g) = \rho(hgh^{-1})$. Put

$${}^\vartheta\rho = \begin{cases} {}^{w_k}\rho, & E \text{ is a field and } d \text{ is odd,} \\ {}^{\sqrt{\tau}}\rho, & E \text{ is a field and } d \text{ is even,} \\ \rho, & E = F \times F. \end{cases}$$

and we fix a nonzero G_k invariant pairing $\langle \cdot, \cdot \rangle$ between ρ and ${}^\vartheta\rho^\vee$. Then

$$\beta(v \otimes v') = \langle v, v' \rangle$$

is a nonzero M^η-invariant linear form on π.

We identify $\mathfrak{a}_{P,\mathbb{C}}^{G,*}$ with $\mathbb{C}$, the unique simple weight being sent to $1 \in \mathbb{C}$. Let $s \in \mathbb{C}$ and ϕ be a section of $I_P^G(\pi, 0)$ and we put

$$\phi_s(g) = \phi(g) e^{\langle s, H_P(g) \rangle} \in I_P^G(\pi, s), \quad \beta(\phi_s(g)) = \beta(\phi(g)) e^{\langle s, H_P(g) \rangle}.$$

These ϕ_s are sections of $I_P^G(\pi, s)$, independent of s when restricted to K and will be referred to as flat sections. We define the local (open) intertwining period

$$J(\phi, s) = \int_{P(\eta)\backslash H} \beta(\phi_s(\eta h))\mathrm{d}h. \tag{4.1}$$

The following result is [BD08, Théorèm 2.8] if F is nonarchimedean and [CD94, Théorème 3] if F is archimedean.

Theorem 4.1. *The integral* (4.1) *converges absolutely when* $\mathrm{Re}\, s$ *is sufficiently large. As a function of* s*,* $J(\phi, s)$ *has a meromorphic continuation to* $\mathbb{C}$*. Moreover for any* s *where* $J(\phi, s)$ *is holomorphic,* $\phi \mapsto J(\phi, s)$ *is an* H*-invariant linear form on* $I_P^G(\pi, s)$*.*

Uniqueness of linear models implies the following functional equation.

Proposition 4.2. *Let* w *be the nontrivial element in* $W(M, M)$*. There is a meromorphic function* $\alpha(s)$ *such that*

$$\alpha(s)J(\phi, s) = J(M(w, s)\phi, -s)$$

for all sections ϕ*.*

Proof. At a generic point s, the induced representation $I_P^G(\pi, s)$ is irreducible, and $J(\phi, s)$ and $J(M(w, s)\phi, -s)$ are holomorphic. For any s we can find some ϕ supported in the open double coset such that the defining integral of $J(\phi, s)$ is convergent and $J(\phi, s) \neq 0$. Both $J(\phi, s)$ and $J(M(w, s)\phi, -s)$ define H-invariant linear forms on $I_P^G(\pi, s)$. The existence of $\alpha(s)$ then follows from the uniqueness of such linear forms (See [JR96, Theorem 1.1] and [AG09, Theorem 8.2.4] when $E = F \times F$ and [BM19, Corollary 5.8 and Theorem 6.7] when E is a field). The archimedean case needs more explanation. We make use of the canonical Casselman–Wallach globalization of an admissible finitely generated $(\mathfrak{g}, K)$-module, cf. [Wal92, Chapter 11]. The statement of [AG09, Theorem 8.2.4] asserts that there is a unique continuous H-invariant linear form on the Casselman–Wallach globalization of $I_P^G(\pi, s)$. By [CD94, Théorème 3], the intertwining period $J(\cdot, s)$ extends continuously to the Cassleman–Wallach globalization of $I_P^G(\pi, s)$. Actually [CD94] showed that when $\mathrm{Re}\, s$ is sufficiently large, the defining integral for $J(\cdot, s)$ is absolutely convergent for all ϕ in the Cassleman–Wallach globalization of $I_P^G(\pi, s)$, and $\phi \mapsto J(\phi, s)$ is a continuous linear form. The existence of $\alpha(s)$ then follows. Finally in any case the function $\alpha(s)$ is meromorphic in s because both $J(\phi, s)$ and $J(M(w, s)\phi, -s)$ are. ☐

4.2. **The local functional equation.** The goal of this subsection is to compute the constant $\alpha(s)$ in Proposition 4.2 when F is nonarchimedean. One can also compute it in the case F being archimedean, but we do not do it as this is not relevant to our purpose. Let us keep the notation from Proposition 4.2. Throughout this subsection, we assume that F is nonarchimedean and ρ is an irreducible supercuspidal representation of G_k. When $k = 1$, we further assume that ρ is not a character. This in particular implies that the Jacquet–Langlands transfer $\mathrm{JL}(\rho)$ is not a twist of the Steinberg representation of $\mathrm{GL}_{kd}(F)$, cf. [Bad08, p. 425].

We use global methods. First we prove a globalization result.

Proposition 4.3. *We can find the following data.*

1. *A quadratic extension of number fields $\mathcal{E}/\mathcal{F}$ which split at all archimedean places, a p-adic place v_0 of $\mathcal{F}$ so that $\mathcal{E}_{v_0}/\mathcal{F}_{v_0} = E/F$, and v_0 is the only place of F above p.*
2. *A central division algebra $\mathcal{D}$ over $\mathcal{F}$ of dimension $(kd)^2$, $\mathcal{D}_{v_0}$ is isomorphic to $M_k(D)$ and $\mathcal{D}_v$ is isomorphic to $\mathrm{Mat}_{kd}(\mathcal{F}_v)$ if v splits in $\mathcal{E}$.*
3. *An irreducible cuspidal automorphic representation σ of $\mathcal{D}^\times(\mathbb{A})$ so that σ is not self-dual, $\sigma_{v_0} \simeq \rho$, $\mathrm{JL}(\sigma)$ is a cuspidal automorphic representation and σ_v is an unramified principal series representation if $v \mid \infty$.*

Proof. We can take $\mathcal{E}/\mathcal{F}$ and v_0 having the property (1) according to [Kab04, Lemma 5] and the proof of [Kab04, Theorem 6]. By Brauer–Hasse–Noether theorem, there is a central division algebra $\mathcal{D}$ with the property (2).

Let $\mathcal{D}_\infty = \prod_{v|\infty} \mathcal{D}_v$. Let $\mathrm{GL}_k(\mathcal{D}_\infty)^1$ be the subgroup of elements $g = (g_v)_v$ in $\mathrm{GL}_k(\mathcal{D}_\infty)$ satisfying $|\det g_v|_v = 1$ for all $v \mid \infty$. Then we have a decomposition $\mathrm{GL}_k(\mathcal{D}_\infty) = \mathrm{GL}_k(\mathcal{D}_\infty)^1 \times Z_\infty$, where Z_∞ is the connected component of the identity of the center of $\mathrm{GL}_k(\mathcal{D}_\infty)$.

Fix a non-empty set $V \subset \widehat{Z_\infty}$, where $\widehat{Z_\infty}$ is the unitary dual of Z_∞. Take a non-empty open set $U^{(1)}$ of irreducible tempered unramified principal series representations of $\mathrm{GL}_k(\mathcal{D}_\infty)^1$ so that if $\sigma \in U^{(1)}$ then it is not self-dual. Note also that $U^{(1)}$ has positive Plancherel measure. Let $U^{(2)} = \{\rho\}$ be a subset of the tempered spectrum of $\mathrm{GL}_k(D)$. Then $U^{(2)}$ is an open subset with positive Plancherel measure. We also take a finite place v' different from v_0 so that $\mathcal{D}_{v'}$ is isomorphic to $\mathrm{Mat}_{kd}(\mathcal{F}_{v'})$ and an irreducible supercuspidal representation ρ' of $\mathcal{D}_{v'}^\times$. Let $U^{(3)} = \{\rho'\}$ be a subset of the tempered spectrum of $\mathcal{D}_{v'}^\times$. Again $U^{(3)}$ is an open subset with positive Plancherel measure. By [Del86], i.e. the limit multiplicity property for $\mathcal{D}^\times$ (the group is anisotropic), there is an irreducible cuspidal automorphic representations σ of $\mathcal{D}^\times(\mathbb{A})$, such that $\sigma_\infty|_{\mathrm{GL}_k(\mathcal{D}_\infty)^1} \in U^{(1)}$, $\sigma_{v_0} = \rho$, $\sigma_{v'} = \rho'$ and the restriction of the central character of σ to Z_∞ is in V. In particular, all archimedean components of σ are unramified principal series. Since $\mathrm{JL}(\sigma)_{v'} = \sigma_{v'}$ is supercuspidal, $\mathrm{JL}(\sigma)$ is a cuspidal automorphic representation of $\mathrm{GL}_{kd}(\mathbb{A})$. This completes the proof. ☐

Remark 4.4. The argument is borrowed from [BP21, Theorem 3.8.1], and it makes use of strong results from [Del86]. What we need to prove essentially is that elements in a dense subset of tempered unramified principal series representations of $\mathrm{GL}_k(\mathcal{D}_\infty)$ can be globalized to cuspidal automorphic representations whose components at v_0 and another split place v' are fixed supercuspidal representations. We have not checked all the details, but a much softer argument like [SV17, Section 16.4] should suffice for this purpose.

We denote by $\mathbb{A}$ the ring of adeles of $\mathcal{F}$. We embed $\mathcal{E}^\times$ in $\mathbf{G} = \mathrm{GL}_2(\mathcal{D})$ as in Subsection 3.2 and let $\mathbf{H}$ be the centralizer of $\mathcal{E}^\times$. Let $\mathbf{P} = \mathbf{MU}$ be the upper triangular parabolic subgroup of $\mathbf{G}$. Let η be the representative of the open double coset as in Subsection 3.2. Let $\phi \in \mathcal{A}^1_{\mathbf{P}}(\mathbf{G})_{\sigma \boxtimes \sigma^\vee}$.

As before we identify $\mathfrak{a}_{P,\mathbb{C}}^{G,*}$ with $\mathbb{C}$. By Theorem 3.1, we have a global intertwining period $J(\phi, s)$ which is given when $\mathrm{Re}\, s >> 0$ by

$$\int_{\mathbf{P}(\eta)(\mathbb{A})\backslash\mathbf{H}(\mathbb{A})}\int_{\mathbf{M}^\eta(F)\backslash\mathbf{M}^\eta(\mathbb{A})^{\mathbf{P},1}} e^{\langle s, H_{\mathbf{P}}(\eta h)\rangle}\phi(m\eta h)\mathrm{d}m\mathrm{d}h.$$

Let us fix a nonzero element

$$\beta_v \in \mathrm{Hom}_{\mathbf{M}^\eta(\mathcal{F}_v)}(\sigma_v \boxtimes \sigma_v^\vee, \mathbb{C})$$

for each place v of $\mathcal{F}$ so that the product of β_v equals the linear form

$$\phi \mapsto \int_{\mathbf{M}^\eta(F)\backslash\mathbf{M}^\eta(\mathbb{A})^{\mathbf{P},1}} \phi(m)\mathrm{d}m.$$

If $\phi = \prod_v \phi_v$ is factorizable, we have

$$J(\phi, s) = \prod_v J_v(\phi_v, s), \tag{4.2}$$

when $\mathrm{Re}\, s >> 0$. The absolute convergence of the right hand side will be proved as a corollary of Proposition 4.5. Note that our proof of the convergence of the defining integral of $J(\phi, s)$, i.e. Lemma 3.2 does not give the absolute convergence of the right hand side. Instead the convergence of the right hand side gives an independent proof of the absolute convergence of the defining integral of $J(\phi, s)$.

Our next goal is to compute $J_v(\phi_v, s)$ for almost all v. Let us first remark that the embedding $\mathcal{E}_v \to \mathcal{D}_v$ at the place v might not be the same the one fixed in Subsection 2.3, but this does not affect the calculation as all such embeddings are conjugate and hence different embeddings give the same function $\alpha(s)$. We will take the embedding of $\mathcal{E}_v \to \mathcal{D}_v$ as given in Subsection 2.3.

Let us assume the following.

- The place v is either unramified or split in $\mathcal{E}$. If v is nonarchimedean then the residue characteristic is odd.
- The division algebra $\mathcal{D}$ splits at v.
- The representation σ is unramified at v and $\phi_v \in \sigma_v$ is spherical.
- The linear form β_v is chosen so that $\beta_v(\phi_v(1)) = 1$.

Of course, these conditions are satisfied at almost all places.

Proposition 4.5. *With the above assumptions, we have*

$$J_v(\phi_v, s) = \frac{L(2s, \sigma_v, \wedge^2)L(s+\frac{1}{2}, \sigma_v)L(s+\frac{1}{2}, \sigma_v \otimes \eta_v)}{L(2s+1, \sigma_v, \mathrm{Sym}^2)}.$$

Proof. This has been established in [Off04] (v nonsplit) and [LO18] (v split), and we just need to transport their results to our situation. First we observe that we have

$$L(2s, \sigma_v, \wedge^2) = L(0, \sigma_v|\cdot|^s, \wedge^2)$$

and similar equalities hold for other L-factors appearing in the proposition. Therefore by suitably twisting σ_v's we only need to prove the proposition when $s = 0$.

As σ_v is unramified, we assume that σ_v is a subrepresentations of

$$\chi_1 \times \chi_2 \times \cdots \times \chi_{kd},$$

where $\chi_1, \cdots, \chi_{kd}$ are unramified characters of $F^\times$. A pairing between σ_v and $\sigma_v^\vee$ is given by

$$\langle f, f' \rangle = \int_{P_{kd}(\mathcal{F}_v)\backslash \mathrm{GL}_{kd}(\mathcal{F}_v)} f(g) f'(g w_{kd}) \mathrm{d}g, \quad f \in \sigma_v, \quad f' \in \sigma_v^\vee.$$

where P_{kd} is the upper triangular Borel subgroup of $\mathrm{GL}_{kd}(\mathcal{F}_v)$ and the measure is taken so that the volume of $P_{kd}(\mathcal{F}_v)\backslash \mathrm{GL}_{kd}(\mathcal{F}_v)$ equals one. Recall that $\mathbf{M}^\eta(\mathcal{F}_v)$ consists of matrices of the form

$$\begin{pmatrix} m & \\ & w_{kd} m w_{kd} \end{pmatrix}, \quad m \in \mathrm{GL}_{kd}(\mathcal{F}_v).$$

The $\mathbf{M}^\eta(\mathcal{F}_v)$-invariant linear form β_v on $\sigma_v \boxtimes \sigma_v^\vee$ is then given by

$$\beta_v(f \otimes f') = \langle f, f' \rangle.$$

Let f°, $f^{\circ,\vee}$ be spherical sections of σ_v and $\sigma_v^\vee$ respectively, normalized so that $f^\circ(1) = f^{\circ,\vee}(1) = 1$. Let ϕ be the spherical section of $\mathrm{Ind}_{\mathbf{P}(\mathcal{F}_v)}^{\mathbf{G}(\mathcal{F}_v)} \sigma_v \boxtimes \sigma_v^\vee$ satisfying

$$\phi(1) = f^\circ \otimes f^{\circ,\vee}.$$

It is also viewed as a spherical sections of the unramified principal series

$$\chi_1 \times \cdots \times \chi_{kd} \times \chi_1^{-1} \times \cdots \times \chi_{kd}^{-1}.$$

Let $\phi_{\underline{\chi}}$ be the spherical section of

$$\chi_1 \times \cdots \times \chi_{kd} \times \chi_{kd}^{-1} \times \cdots \times \chi_1^{-1}$$

normalized so that $\phi_{\underline{\chi}}(1) = 1$. There is a standard intertwining operator

$$M(w') : \chi_1 \times \cdots \times \chi_{kd} \times \chi_{kd}^{-1} \times \cdots \times \chi_1^{-1} \to \chi_1 \times \cdots \times \chi_{kd} \times \chi_1^{-1} \times \cdots \times \chi_{kd}^{-1},$$

given by the Weyl group element

$$w' = \begin{pmatrix} 1_{kd} & \\ & w_{kd} \end{pmatrix}.$$

By the Gindikin–Karpelevich formula we have

$$M(w')\phi_{\underline{\chi}} = c(\underline{\chi}) \cdot \phi \tag{4.3}$$

where

$$c(\underline{\chi}) = \prod_{1 \le a < b \le kd} \frac{L(0, \chi_a \chi_b^{-1})}{L(1, \chi_a \chi_b^{-1})}.$$

We write $\chi_i = |\cdot|^{s_i}$ for some $s_i \in \mathbb{C}$. Let us assume that

$$\mathrm{Re}\, s_1 >> \mathrm{Re}\, s_2 >> \cdots >> \mathrm{Re}\, s_{kd} >> 1.$$

which guarantees that the integrals under consideration (including the defining integral of $M(w')$) are absolutely convergent. It is enough to prove the proposition under these assumptions as both sides are meromorphic with respect to $s_1, \cdots, s_{kd}$. Write the Levi decomposition of P_{kd} as $P_{kd} = T_{kd}U_{kd}$, where U_{kd} is the unipotent radical of P_{kd} and T_{kd} is the diagonal torus. Note that the defining integral (2.4) of the standard intertwining operator becomes in this case

$$M(w')\phi_{\underline{\chi}}(g) = \int_{U_{kd}(\mathcal{F}_v)} \phi_{\underline{\chi}}\left(w'\begin{pmatrix}1_{kd} & \\ & u\end{pmatrix} g\right) \mathrm{d}u.$$

Let $\mathbf{P}_0$ be the standard upper triangular Borel subgroup of $\mathbf{G}$. Since $\mathbf{H}(\mathcal{F}_v) \cap \mathbf{P}_0(\mathcal{F}_v)$ consists of matrices of the form

$$\begin{pmatrix}t & \\ & w_{kd}tw_{kd}\end{pmatrix}, \quad t \in T_{kd}(\mathcal{F}_v)$$

and $\mathbf{H}(\mathcal{F}_v) \cap \mathbf{P}(\mathcal{F}_v)$ coincides with $\mathbf{M}^\eta(\mathcal{F}_v)$, we have

$$\begin{aligned}
\beta_v(M(w')\phi_{\underline{\chi}}(h)) &= \int_{P_{kd}(\mathcal{F}_v)\backslash \mathrm{GL}_{kd}(\mathcal{F}_v)} M(w')\phi_{\underline{\chi}}\left(\begin{pmatrix}g & \\ & gw_{kd}\end{pmatrix} h\right) \mathrm{d}g \\
&= \int_{P_{kd}(\mathcal{F}_v)\backslash \mathrm{GL}_{kd}(\mathcal{F}_v)} \int_{U_{kd}(\mathcal{F}_v)} \phi_{\underline{\chi}}\left(w'\begin{pmatrix}1_{kd} & \\ & u\end{pmatrix}\begin{pmatrix}g & \\ & gw_{kd}\end{pmatrix} h\right) \mathrm{d}g\mathrm{d}u \\
&= \int_{P_{kd}(\mathcal{F}_v)\backslash \mathrm{GL}_{kd}(\mathcal{F}_v)} \int_{U_{kd}(\mathcal{F}_v)} \phi_{\underline{\chi}}\left(\begin{pmatrix}u & \\ & w_{kd}uw_{kd}\end{pmatrix}\begin{pmatrix}g & \\ & w_{kd}gw_{kd}\end{pmatrix} h\right) \mathrm{d}g\mathrm{d}u \\
&= \int_{\mathbf{H}(\mathcal{F}_v)\cap\mathbf{P}_0(\mathcal{F}_v)\backslash\mathbf{H}(\mathcal{F}_v)\cap\mathbf{P}(\mathcal{F}_v)} \phi_{\underline{\chi}}(mh)\mathrm{d}g.
\end{aligned}$$

Hence we obtain

$$\begin{aligned}
\int_{\mathbf{H}(\mathcal{F}_v)\cap\mathbf{P}_0(\mathcal{F}_v)\backslash\mathbf{H}(\mathcal{F}_v)} \phi_{\underline{\chi}}(h)\mathrm{d}h &= \int_{\mathbf{H}(\mathcal{F}_v)\cap\mathbf{P}_0(\mathcal{F}_v)\backslash\mathbf{H}(\mathcal{F}_v)\cap\mathbf{P}(\mathcal{F}_v)} \int_{\mathbf{H}(\mathcal{F}_v)\cap\mathbf{P}(\mathcal{F}_v)\backslash\mathbf{H}(\mathcal{F}_v)} \phi_{\underline{\chi}}(mh)\mathrm{d}m\mathrm{d}h \\
&= \int_{\mathbf{H}(\mathcal{F}_v)\cap\mathbf{P}(\mathcal{F}_v)\backslash\mathbf{H}(\mathcal{F}_v)} \beta_v(M(w')\phi_{\underline{\chi}}(h))\mathrm{d}h \\
&= c(\underline{\chi}) \cdot J_v(\phi, 0).
\end{aligned}$$

For the last equality, we used (4.3). By [Off04, Lemma 5.6 and (89)] and [LO18, Section 4.5], the first integral equals

$$\prod_{1\le i<j\le kd} \frac{L(0,\chi_i\chi_j)L(0,\chi_i\chi_j^{-1})}{L(1,\chi_i\chi_j)L(1,\chi_i\chi_j^{-1})} \prod_{i=1}^{kd} \frac{L(1/2,\chi_i)L(1/2,\chi_i\eta_{E/F})}{L(1,\chi_i^2)}.$$

A little computation gives that this simplifies to

$$c(\underline{\chi}) \times \frac{L(0,\sigma_v,\wedge^2)L(1/2,\sigma_v)L(1/2,\sigma_v\otimes\eta_v)}{L(1,\sigma_v,\mathrm{Sym}^2)}.$$

The proposition then follows. ∎

Corollary 4.6. *The right hand side of* (4.2) *is absolutely convergence when* $\mathrm{Re}\, s >> 0$.

Proof. Let S be a finite subset of places of F such that the conditions before Proposition 4.5 are satisfied if $v \notin S$. By Proposition 4.5, the right hand side of (4.2) equals

$$\prod_{v \notin S} \frac{L(2s, \sigma_v, \wedge^2) L(s+\frac{1}{2}, \sigma_v) L(s+\frac{1}{2}, \sigma_v \otimes \eta_v)}{L(2s+1, \sigma_v, \mathrm{Sym}^2)} \times \prod_{v \in S} J(\phi_v, s).$$

The product over $v \notin S$ is absolutely convergent because of the convergence of the partial L-functions. The product over $v \in S$ is finite. □

For convenience we introduce the following notation. Let R be a commutative ring, R' be a subring and $a, b \in R$. We write $a \sim_{R'^\times} b$ if there is some $c \in R'^\times$ so that $a = cb$.

Let us fix a nontrivial additive character $\psi = \otimes \psi_v$ of $\mathcal{F} \backslash \mathbb{A}$. The gamma factors in the next proposition are those defined by the Langlands–Shahidi method which coincide with those defined using local Langlands correspondences [CST17, Theorem 2.1].

Proposition 4.7. *Let the notation be as in Proposition 4.2. Then*

$$\alpha(s) \sim_{\mathbb{C}[q^{\pm s}]^\times} \gamma(-2s, \mathrm{JL}(\rho)^\vee, \wedge^2, \psi_{v_0})^{-1} \gamma(2s, \mathrm{JL}(\rho), \mathrm{Sym}^2, \psi_{v_0})^{-1}.$$

Proof. Let S be the finite set of place such that if $v \notin S$ then the assumptions before Proposition 4.5 are satisfied. By Proposition 4.5, we obtain

$$J(\phi, s) = \frac{L^S(2s, \sigma, \wedge^2) L^S(s+\frac{1}{2}, \sigma) L^S(s+\frac{1}{2}, \sigma \otimes \eta_{\mathcal{E}/\mathcal{F}})}{L^S(2s+1, \sigma, \mathrm{Sym}^2)} \times \prod_{v \in S} J_v(\phi_v, s).$$

This equality, a priori holds for $\mathrm{Re}\, s >> 0$, in fact holds for all s as both side have meromorphic continuation. Moreover for any $v \notin S$, by the Gindikin–Karpelevich formula, we have

$$M(w, s)\phi_v = \frac{L(2s, \sigma_v \times \sigma_v)}{L(2s+1, \sigma_v \times \sigma_v)} {}^w\phi_v,$$

where ${}^w\phi_v$ is the spherical section of $\mathrm{Ind}_{\mathbf{P}(\mathcal{F}_v)}^{\mathbf{G}(\mathcal{F}_v)} \sigma_v^\vee \boxtimes \sigma_v$, normalized so that ${}^w\phi_v(1) = 1$. This, combined with the global functional equation $J(\phi, s) = J(M(w, s)\phi, -s)$, gives that

$$\prod_{v \in S} \frac{J_v(M(w, s)\phi_v, -s)}{J_v(\phi_v, s)}$$

equals

$$\frac{L^S(2s+1, \sigma, \wedge^2)}{L^S(-2s, \sigma^\vee, \wedge^2)} \frac{L^S(s+\frac{1}{2}, \sigma)}{L^S(\frac{1}{2}-s, \sigma^\vee)} \frac{L^S(s+\frac{1}{2}, \sigma \otimes \eta_{\mathcal{E}/\mathcal{F}})}{L^S(\frac{1}{2}-s, \sigma^\vee \otimes \eta_{\mathcal{E}/\mathcal{F}})} \frac{L^S(1-2s, \sigma^\vee, \mathrm{Sym}^2)}{L^S(2s, \sigma, \mathrm{Sym}^2)}.$$

We have the functional equations

$$L^S(s, \sigma, \wedge^2) = \prod_v \gamma_v(s, \mathrm{JL}(\sigma_v), \wedge^2, \psi_v) L^S(1-s, \sigma^\vee, \wedge^2) \tag{4.4}$$

and similar functional equations for all partial L-functions above. This needs some explanation. Recall that $\mathrm{JL}(\sigma)$ is a cuspidal automorphic representation of $\mathrm{GL}_{kd}(\mathbb{A})$ with $\mathrm{JL}(\sigma)_v = \sigma_v$ if $v \notin S$ (we have identified $\mathrm{GL}_k(\mathcal{D}_v)$ with $\mathrm{GL}_{kd}(\mathcal{F}_v)$). Then by definition we have

$$L^S(s, \sigma, \wedge^2) = L^S(s, \mathrm{JL}(\sigma), \wedge^2),$$

and similar equalities for all other partial L-functions. Thus (4.4) is nothing but the global functional equation of $L^S(s, \mathrm{JL}(\sigma), \wedge^2)$.

We thus conclude that

$$\prod_{v\in S}\alpha_v(s) = \prod_{v\in S}\frac{\gamma(s+\frac{1}{2}, \mathrm{JL}(\sigma_v), \psi_v)\gamma(s+\frac{1}{2}, \mathrm{JL}(\sigma_v\otimes\eta_v), \psi_v)}{\gamma(-2s, \mathrm{JL}(\sigma_v)^\vee, \wedge^2, \psi_v)\gamma(2s, \mathrm{JL}(\sigma_v), \mathrm{Sym}^2, \psi_v)}.$$

The point is that S contains only nonarchimedean places and by [Mat, Lemma 9.3] the factors with different residue characteristics are algebraically independent. By Proposition 4.3, the place v_0 is p-adic and it is the only place above p in S. It follows that there is a nonzero constant C so that

$$\alpha_{v_0}(s) = C\times\frac{\gamma(s+\frac{1}{2}, \mathrm{JL}(\rho), \psi_{v_0})\gamma(s+\frac{1}{2}, \mathrm{JL}(\rho\otimes\eta_{v_0}), \psi_{v_0})}{\gamma(-2s, \mathrm{JL}(\rho)^\vee, \wedge^2, \psi_{v_0})\gamma(2s, \mathrm{JL}(\rho), \mathrm{Sym}^2, \psi_{v_0})}.$$

The representation $\mathrm{JL}(\rho)$ is a square integrable representation of $\mathrm{GL}_{kd}(F)$. It is the unique irreducible quotient of

$$\rho'\nu^{(1-l_\rho)/2}\times\cdots\times\rho'\nu^{(l_\rho-1)/2}$$

where l_ρ is a positive integer, ρ' be an irreducible supercuspidal representation of $\mathrm{GL}_{kd/l_\rho}(F)$. Thus by [CPS17, Theorem 2.3] we have

$$L(s, \mathrm{JL}(\rho)) = L\left(s+\frac{l_\rho-1}{2}, \rho'\right).$$

By our assumption, $\mathrm{JL}(\rho)$ is not a twist of the Steinberg representation. Thus $nd/l_\rho\neq 1$. Therefore by [CPS17, Section 2.6.1], $L(s, \mathrm{JL}(\rho)) = L\left(s+\frac{l_\rho-1}{2}, \rho'\right)$ is the constant 1. Similarly $L(s, \mathrm{JL}(\rho\otimes\eta_{v_0}))$ is also the constant 1. It follows that $\gamma(s+\frac{1}{2}, \mathrm{JL}(\rho), \psi_{v_0})\gamma(s+\frac{1}{2}, \mathrm{JL}(\rho\otimes\eta_{v_0}), \psi_{v_0})$ equals the epsilon factor $\epsilon(s+\frac{1}{2}, \mathrm{JL}(\rho), \psi_{v_0})\epsilon(s+\frac{1}{2}, \mathrm{JL}(\rho\otimes\eta_{v_0}), \psi_{v_0})$, which is an element in $\mathbb{C}[q^{\pm s}]^\times$. The proposition follows. □

4.3. **Applications to distinction.** The goal of this subsection is to prove Theorem 1.3. The setup is the same as Theorem 1.3 which we now recall. The following notation will be kept in this subsection. We have $n = kt$ and P the parabolic subgroup of $G = \mathrm{GL}_n(D)$ corresponding to the partition $(k, k, \cdots, k)$. Note that unlike the previous two subsections, here t can be an arbitrary integer. Recall that η is the fixed representative of the open double coset $P\eta H$ in G and $M^\eta = M\cap\eta^{-1}H\eta$. Let ρ be an irreducible self-dual supercuspidal representation of G_k with $\mathrm{JL}(\rho)$ being the unique irreducible quotient of

$$\rho'\nu^{(1-l_\rho)/2}\times\cdots\times\rho'\nu^{(l_\rho-1)/2}$$

where ρ' be an irreducible self-dual supercuspidal representation of $\mathrm{GL}_{kd/l_\rho}(F)$. We assume that if $k = 1$, then ρ is not a character. We consider distinction of π, where π is the unique irreducible quotient of

$$\rho\nu^{(1-t)l_\rho/2}\times\rho\nu^{(3-t)l_\rho/2}\times\cdots\times\rho\nu^{(t-1)l_\rho/2}.$$

First let us link the local factors of ρ to those of ρ'.

Lemma 4.8. *We have*

$$\gamma(-s,\mathrm{JL}(\rho),\wedge^2,\psi)^{-1}\gamma(s,\mathrm{JL}(\rho),\mathrm{Sym}^2,\psi)^{-1}$$
$$\sim_{\mathbb{C}[q^{\pm s}]^\times}\begin{cases}\dfrac{L(-s,\rho',\mathrm{Sym}^2)}{L(-s+l_\rho,\rho',\mathrm{Sym}^2)}\dfrac{L(s,\rho',\wedge^2)}{L(s+l_\rho,\rho',\wedge^2)} & \textit{if } l_\rho \textit{ is even,}\\ \dfrac{L(-s,\rho',\wedge^2)}{L(-s+l_\rho,\rho',\mathrm{Sym}^2)}\dfrac{L(s,\rho',\mathrm{Sym}^2)}{L(s+l_\rho,\rho',\wedge^2)} & \textit{if } l_\rho \textit{ is odd.}\end{cases}$$

Proof. To simplify notation, we set

$$L_{+1}(s,\mathrm{JL}(\rho))=L(s,\mathrm{JL}(\rho),\wedge^2),\qquad L_{-1}(s,\mathrm{JL}(\rho))=L(s,\mathrm{JL}(\rho),\mathrm{Sym}^2)$$
$$\varepsilon_{+1}(s,\mathrm{JL}(\rho),\psi)=\varepsilon(s,\mathrm{JL}(\rho),\wedge^2,\psi),\qquad \varepsilon_{-1}(s,\mathrm{JL}(\rho),\psi)=\varepsilon(s,\mathrm{JL}(\rho),\mathrm{Sym}^2,\psi)$$
$$\gamma_{+1}(s,\mathrm{JL}(\rho),\psi)=\gamma(s,\mathrm{JL}(\rho),\wedge^2,\psi),\qquad \gamma_{-1}(s,\mathrm{JL}(\rho),\psi)=\gamma(s,\mathrm{JL}(\rho),\mathrm{Sym}^2,\psi).$$

Since we have the relation

$$L_{+1}(s,\mathrm{JL}(\rho))=\prod_{i=1}^{l_\rho}L_{(-1)^{i-1}}(s+l_\rho-i,\rho'),\quad L_{-1}(s,\mathrm{JL}(\rho))=\prod_{i=1}^{l_\rho}L_{(-1)^{i}}(s+l_\rho-i,\rho'),\tag{4.5}$$

we get

$$\frac{L_{-1}(s,\mathrm{JL}(\rho))}{L_{+1}(1+s,\mathrm{JL}(\rho))}=\prod_{i=1}^{l_\rho}\frac{L_{(-1)^i}(s+l_\rho-i,\rho')}{L_{(-1)^{i-1}}(s+l_\rho-i+1,\rho')}=\frac{L_{(-1)^{l_\rho}}(s,\rho')}{L_{+1}(s+l_\rho,\rho')}.$$

Similarly we have

$$\frac{L_{+1}(-s,\mathrm{JL}(\rho))}{L_{-1}(1-s,\mathrm{JL}(\rho))}=\prod_{i=1}^{l_\rho}\frac{L_{(-1)^{i-1}}(-s+l_\rho-i,\rho')}{L_{(-1)^i}(-s+l_\rho-i+1,\rho')}=\frac{L_{(-1)^{l_\rho-1}}(-s,\rho')}{L_{-1}(-s+l_\rho,\rho')}.$$

Because ρ is self-dual we conclude that there is some $a(s)\in\mathbb{C}[q^{\pm s}]^\times$

$$\begin{aligned}\gamma(-s,\mathrm{JL}(\rho),\wedge^2,\psi)^{-1}\gamma(s,\mathrm{JL}(\rho),\mathrm{Sym}^2,\psi)^{-1}&=a(s)\frac{L_{+1}(-s,\mathrm{JL}(\rho))}{L_{-1}(1-s,\mathrm{JL}(\rho))}\frac{L_{-1}(s,\mathrm{JL}(\rho))}{L_{+1}(1+s,\mathrm{JL}(\rho))}\\&=a(s)\frac{L_{(-1)^{l_\rho-1}}(-s,\rho')}{L_{-1}(-s+l_\rho,\rho')}\frac{L_{(-1)^{l_\rho}}(s,\rho')}{L_{+1}(s+l_\rho,\rho')}.\end{aligned}$$

This proves the lemma. □

We now begin the proof of Theorem 1.3. The strategy is to treat $t=2$ first and then reduce the general case to it.

Lemma 4.9. *The space of H-invariant linear forms on $\rho\nu^{-sl_\rho}\times\rho\nu^{sl_\rho}$ is one dimensional for all s.*

Proof. First we assume that $s\neq\frac{1}{2}$. In this case, the induced representation $\rho\nu^{-sl_\rho}\times\rho\nu^{sl_\rho}$ is irreducible and the lemma follows from [BM19, Corollary 5.8].

Next we consider the case $s=\frac{1}{2}$. In the proof of [BM19, Proposition 5.6], it is observed that the only double coset which contributes to the space of H-invariant linear forms is $P\eta H$. Denote by

$$\left(\rho\nu^{-\frac{1}{2}l_\rho}\times\rho\nu^{\frac{1}{2}l_\rho}\right)^\circ$$

the H-invariant subspace of sections supported in this open coset. We have

$$\mathrm{Hom}_H\left(\rho\nu^{-\frac{1}{2}l_\rho}\times\rho\nu^{\frac{1}{2}l_\rho},\mathbb{C}\right)=\mathrm{Hom}_H\left(\left(\rho\nu^{-\frac{1}{2}l_\rho}\times\rho\nu^{\frac{1}{2}l_\rho}\right)^\circ,\mathbb{C}\right), \tag{4.6}$$

the map being given by restriction. By Frobenius reciprocity the right hand side of the above equality is isomorphic to

$$\mathrm{Hom}_{M^\eta}(\rho\nu^{-\frac{1}{2}l_\rho}\boxtimes\rho\nu^{\frac{1}{2}l_\rho},\mathbb{C})$$

Recall that M^η consists of elements of the form $\begin{pmatrix}a & \\ & \theta(a)\end{pmatrix}$, $a\in\mathrm{GL}_k(D)$. Therefore this space is one dimensional. □

Lemma 4.10. *The local intertwining period $J(\cdot,s)$ is holomorphic at the point $s=-\frac{1}{2}l_\rho$ and defines a nonzero H-invariant linear form on $\rho\nu^{-\frac{1}{2}l_\rho}\times\rho\nu^{\frac{1}{2}l_\rho}$.*

Proof. Suppose that the local intertwining period is not holomorphic at $-\frac{1}{2}l_\rho$, then for some positive integer a

$$(s+\tfrac{1}{2}l_\rho)^a J(\cdot,s)$$

defines a nonzero H-invariant linear form on $\rho\nu^{-\frac{1}{2}l_\rho}\times\rho\nu^{\frac{1}{2}l_\rho}$. However for any section supported in the open double coset the defining integral of the local intertwining period is absolutely convergent and therefore $(s+\frac{1}{2}l_\rho)^a J(\cdot,s)$ is zero when restricted to $\left(\rho\nu^{-\frac{1}{2}l_\rho}\times\rho\nu^{\frac{1}{2}l_\rho}\right)^\circ$. This is contradictory to (4.6). Thus the local intertwining period is holomorphic. It is also clear that it is nonzero because we can choose some ϕ supported in the open cell such that $J(\phi,-\frac{1}{2}l_\rho)\neq 0$. □

Let us introduce the following notation. For a real number a, the representation

$$\rho\nu^{l_\rho a}\times\rho\nu^{l_\rho(a+1)}$$

has a unique irreducible quotient. Its kernel is denoted by $Z([a,a+1]_\rho)$. It is also the unique irreducible quotient of

$$\rho\nu^{l_\rho(a+1)}\times\rho\nu^{l_\rho a}.$$

Recall that ρ and hence ρ' are self-dual. It follows that precisely one of $L(s,\rho',\mathrm{Sym}^2)$ and $L(s,\rho',\wedge^2)$ has a pole at $s=0$. The next proposition proves Theorem 1.3 in the case $t=2$.

Proposition 4.11. *Let the notation be as above. The following are equivalent.*

1. *$Z([-1/2,1/2]_\rho)$ is H-distinguished.*
2. *$L(s,\rho',\wedge^2)$ has a pole at $s=0$.*

Proof. Let $\sigma=\rho\boxtimes\rho$ be a representation of M and take a non-zero element β of $\mathrm{Hom}_{M^\eta}(\sigma,\mathbb{C})$. Let w be the nontrivial element in $W(M,M)$. The intertwining operator

$$M(w,-\tfrac{1}{2}l_\rho):\rho\nu^{-\frac{1}{2}l_\rho}\times\rho\nu^{\frac{1}{2}l_\rho}\to\rho\nu^{\frac{1}{2}l_\rho}\times\rho\nu^{-\frac{1}{2}l_\rho}.$$

is holomorphic and nonzero according to [Mat, Proposition 7.1]. Simple Jacquet module computation shows that all intertwining maps $\rho\nu^{-\frac{1}{2}l_\rho}\times\rho\nu^{\frac{1}{2}l_\rho}\to\rho\nu^{\frac{1}{2}l_\rho}\times\rho\nu^{-\frac{1}{2}l_\rho}$ are multiples of the intertwining operator $M(w,-\frac{1}{2}l_\rho)$. It follows that the image of $M(w,-\frac{1}{2}l_\rho)$ is the unique irreducible submodule of $\rho\nu^{\frac{1}{2}l_\rho}\times\rho\nu^{-\frac{1}{2}l_\rho}$. By Lemma 4.10 the space of H-invariant linear forms on $\rho\nu^{\frac{1}{2}l_\rho}\times\rho\nu^{-\frac{1}{2}l_\rho}$ is one dimensional and is given by multiples of the local intertwining period. It follows that $Z([-1/2,1/2]_\rho)$ is H-distinguished if and only if $J(\cdot,\frac{1}{2}l_\rho)$ vanishes on the image of $M(w,-\frac{1}{2}l_\rho)$.

Now assume (1). Let ϕ_s be a flat section of $I_P^G(\rho,s)$. By Proposition 4.7 and Lemma 4.8, $J(M(w,s)\phi,-s)$ equals $J(\phi,s)$ multiplied by

$$\frac{L(-2s,\rho',\mathrm{Sym}^2)L(2s,\rho',\wedge^2)}{L(-2s+l_\rho,\rho',\mathrm{Sym}^2)L(2s+l_\rho,\rho',\wedge^2)} \tag{4.7}$$

and an element in $\mathbb{C}[q^{\pm s}]^\times$. Thus (4.7) has a zero at $s=-\frac{l_\rho}{2}$. All factors in (4.7) are holomorphic and nonzero at $s=-\frac{l_\rho}{2}$ except for $L(2s+l_\rho,\rho',\wedge^2)$ which could have a pole. Therefore we see that (1) implies (2).

The converse direction follows from reversing the argument. ☐

We return to the situation of an arbitrary t. For each $2-t\le j\le t-2$ with $j\equiv t \mod 2$, we define a subrepresentation $\pi_j(\rho)$ of $\rho\nu^{(1-t)l_\rho/2}\times\cdots\times\rho\nu^{(t-1)l_\rho/2}$ by

$$\pi_j(\rho)=\rho\nu^{(1-t)l_\rho/2}\times\cdots\times\rho\nu^{(j-3)l_\rho/2}\times Z([(j-1)/2,(j+1)/2]_\rho)\times\rho\nu^{(j+3)l_\rho/2}\times\cdots\times\rho\nu^{(t-1)l_\rho/2}.$$

Set

$$\mathcal{I}(\pi)=\sum_{\substack{2-t\le j\le t-2\\ j\equiv t \mod 2}}\pi_j(\rho).$$

It follows from the proof of [Tad90, Proposition 2.7] that this is the maximal (proper) subrepresentation of $\rho\nu^{(1-t)l_\rho/2}\times\cdots\times\rho\nu^{(t-1)l_\rho/2}$ and is the kernel of the natural projection

$$\rho\nu^{(1-t)l_\rho/2}\times\cdots\times\rho\nu^{(t-1)l_\rho/2}\to\pi.$$

Lemma 4.12. *For $2-t\le j\le t-2$ which satisfies $j\equiv t \mod 2$, suppose that the representation $\pi_j(\rho)$ is H-distinguished. Then we have $j=0$ and $L(s,\rho',\wedge^2)$ has a pole at $s=0$. In particular, if t is odd, any $\pi_j(\rho)$ is not H-distinguished.*

Proof. Assume to the contrary that either $j\neq 0$ or $j=0$ but $L(s,\rho',\wedge^2)$ does not have a pole at $s=0$. Let us prove that $\pi_j(\rho)$ is not H-distinguished. Let $P_j=M_jU_j$ be the parabolic subgroup of G corresponding to the partition $(k,\dots,k,2k,k,\dots,k)$ of n with $2k$ in position $\frac{t+j}{2}$. Let

$$\sigma=\rho\nu^{(1-t)l_\rho/2}\boxtimes\cdots\boxtimes\rho\nu^{(j-3)l_\rho/2}\boxtimes Z([(j-1)/2,(j+1)/2]_\rho)\boxtimes\rho\nu^{(j+3)l_\rho/2}\boxtimes\cdots\boxtimes\rho\nu^{(t-1)l_\rho/2}$$

be a representation of M_j. Then we have $\pi_j(\rho)=\mathrm{Ind}_{P_j}^G\sigma$. In [Cho19, Section 2.2 and 3.2], representatives of $P_j\backslash G/H$ have been analyzed and to each double coset representative λ there is an associated parabolic subgroup $P_{j,\lambda}$. We write r^λ for the Jacquet module functor along the unipotent radical of $P_{j,\lambda}\cap M_j$. Mackey theory implies that $\mathrm{Hom}_H(\pi_j(\rho),\mathbb{C})$ is embedded in

$$\bigoplus_\lambda \mathrm{Hom}_{M_j^\lambda}(r^\lambda(\sigma),\mathbb{C}),$$

where the sum runs over all double coset representatives of $P_j\backslash G/H$. Let us prove that all summands are zero. Assume that $r^\lambda(\sigma)$ admits an M_j^λ-invariant linear form. Note also that ρ is supercuspidal and the only nonzero Jacquet module of $Z([(j-1)/2,(j+1)/2]_\rho)$ is from the parabolic subgroup corresponding to the partition (k,k), cf. [Tad90, Proposition 2.7]. Hence there are only two possibilities: we either have $P_{j,\lambda}=P_j$ or $P_{j,\lambda}=P$.

First suppose that $P_{j,\lambda}=P$, then $r^\lambda(\sigma)$ is isomorphic to $\rho_1\boxtimes\cdots\boxtimes\rho_t$, where $\rho_i=\rho\nu^{(1-t+2(i-1))l_\rho/2}$. The description of the double coset representative in [Cho19, Section 2.2 and 3.2] indicates that λ corresponds to a symmetric matrix $S=(s_{i,j})$ of size $t-1$ with non-negative integer entries, which satisfies the following.

(a) The sum of i-th row equals $2k$ if $i=\frac{t+j}{2}$ and k otherwise.

(b) Let $\lambda'=(\lambda'_1,\ldots,\lambda'_t)$ be the partition $(s_{1,1},s_{1,2},\ldots,s_{1,t-1},s_{2,1},\ldots,s_{t-1,t-1})$ of n, where 0-entries are ignored. Then $\lambda'=(k,k,k,\ldots,k)$.

Let ι be the involution on the set of idices $\{1,2,\ldots,t\}$ induced from transpose $s_{i,j}\mapsto s_{j,i}$. By [Cho19, Proposition 2.2 and 3.2], M_j^λ-distinction implies the following.

- If $\iota(i)\neq i$, then $\rho_i\simeq\rho_{\iota(i)}^\vee$.
- If $\iota(i)=i$, then ρ_i is H_k-distinguished, where H_k is the centralizer of $E^\times$ in $\mathrm{GL}_k(D)$.

In any case we have $\rho_i\simeq\rho_{\iota(i)}^\vee$ and taking the central characters into consideration, the only possibility is $\iota(i)=t-i+1$ for all i. Therefore the matrix S should be anti-diagonal, but this is not possible because of the conditions (a) and (b).

Now we suppose that $P_{j,\lambda}=P_j$, then $r^\lambda(\sigma)=\sigma$ and $Z([(j-1)/2,(j+1)/2]_\rho)$ as a representation of $\mathrm{GL}_{2k}(D)$ is $\mathrm{GL}_{2k}(C)$-distinguished if d is even and $\mathrm{GL}_k(D_E)$-distinguished if d is odd. But this is not possible if $j\neq 0$ since the central character of $Z([(j-1)/2,(j+1)/2]_\rho)$ is not trivial and is also not possible if $j=0$ by Proposition 4.11. □

Finally we prove Theorem 1.3. For readers' convenience we restate the result as follows. We do not assume that ρ is self-dual.

Theorem 4.13. (1) *Suppose t is odd and hence $E^\times$ embeds in $\mathrm{GL}_k(D)$ and we let H_k be the centralizer of $E^\times$. Then π is H-distinguished if and only if ρ is H_k-distinguished.*

(2) *Suppose t is even. Then π is H-distinguished if and only if $L(s,\rho',\mathrm{Sym}^2)$ has a pole at $s=0$.*

Proof. Assume that t is odd first. By (the proof of) [BM19, Proposition 5.6] π being H-distinguished implies that ρ is H_k-distinguished.

Conversely, suppose that ρ is H_k-distinguished. Then it is self-dual. By [Off17, Proposition 7.2], $\rho\nu^{(1-t)l_\rho/2}\times\cdots\times\rho\nu^{(t-1)l_\rho/2}$ is H-distinguished. From Lemma 4.12, we see that $\mathcal{I}(\pi)$ is not H-distinguished. Therefore any non-zero H-invariant linear form on $\rho\nu^{l_\rho(1-t)/2}\times\cdots\times\rho\nu^{l_\rho(t-1)/2}$ factors through π. This proves the first assertion.

Now assume that $t = 2r$ is even. Suppose that π is H-distinguished. The Langlands parameter of π takes the form

$$\phi_{\rho'} \boxtimes \mathrm{Sym}^{tl_\rho - 1} \mathbb{C}^2 : W_F \times \mathrm{SL}_2(\mathbb{C}) \to \mathrm{GL}_{nd}(\mathbb{C}),$$

where $\phi_{\rho'}$ is the Langlands parameter of ρ'. By [Xue21, Theorem 1.1], the Langlands parameter of π takes value in $\mathrm{Sp}_{nd}(\mathbb{C})$. Since t is even, the image of $\mathrm{Sym}^{tl_\rho-1}(\mathbb{C})$ lies in $\mathrm{Sp}_{tl_\rho}(\mathbb{C})$. This implies that $\phi_{\rho'}$ lies in the orthogonal group $\mathrm{O}_{kd/l_\rho}(\mathbb{C})$. This is equivalent to that $L(s, \rho', \mathrm{Sym}^2)$ has a pole at $s = 0$.

Conversely, suppose that $L(s, \rho', \mathrm{Sym}^2)$ has a pole at $s = 0$. Then ρ' is self-dual by [Yam17, Theorem 3.18]. Therefore ρ is self-dual, and $L(s, \rho', \wedge^2)$ is holomorphic at $s = 0$. By [Off17, Proposition 7.2] again, $\rho\nu^{(1-t)l_\rho/2} \times \cdots \times \rho\nu^{(t-1)l_\rho/2}$ is H-distinguished. It follows from Lemma 4.12 that $\mathcal{I}(\pi)$ is not H-distinguished. Therefore the non-zero H-invariant linear form on $\rho\nu^{(1-t)l_\rho/2} \times \cdots \times \rho\nu^{(t-1)l_\rho/2}$ factors through π. □

Data availability: Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

Conflict of interest statement: On behalf of all authors, Hang Xue states that there is no conflict of interest.

References

[AG09] A. Aizenbud and D. Gourevitch, *Generalized Harish-Chandra descent, Gelfand pairs, and an Archimedean analog of Jacquet-Rallis's theorem*, Duke Math. J. **149** (2009), no. 3, 509–567, DOI 10.1215/00127094-2009-044. With an appendix by the authors and Eitan Sayag. MR2553879 ↑21

[Bad08] A. I. Badulescu, *Global Jacquet-Langlands correspondence, multiplicity one and classification of automorphic representations*, Invent. Math. **172** (2008), no. 2, 383–438, DOI 10.1007/s00222-007-0104-8. With an appendix by Neven Grbac. MR2390289 ↑9, 10, 21

[BD08] P. Blanc and P. Delorme, *Vecteurs distributions H-invariants de représentations induites, pour un espace symétrique réductif p-adique G/H*, Ann. Inst. Fourier (Grenoble) **58** (2008), no. 1, 213–261 (French, with English and French summaries). MR2401221 ↑21

[BM19] P. Broussous and N. Matringe, *Multiplicity one for pairs of Prasad–Takloo-Bighash type*, Int. Math. Res. Not. IMRN, in press, posted on 2019, DOI 10.1093/imrn/rnz254. ↑1, 21, 28, 31

[BP21] R. Beuzart-Plessis, *Archimedean theory and ϵ-factors for the Asai Rankin-Selberg integrals*, Relative trace formulas, Simons Symposia, Springer, Cham, 2021, pp. 1–50, DOI 10.1007/978-3-030-68506-5_1. ↑22

[CD94] J. Carmona and P. Delorme, *Base méromorphe de vecteurs distributions H-invariants pour les séries principales généralisées d'espaces symétriques réductifs: equation fonctionnelle*, J. Funct. Anal. **122** (1994), no. 1, 152–221, DOI 10.1006/jfan.1994.1065 (French, with English summary). MR1274587 ↑21

[Cho19] M. Chommaux, *Distinction of the Steinberg representation and a conjecture of Prasad and Takloo-Bighash*, J. Number Theory **202** (2019), 200–219, DOI 10.1016/j.jnt.2019.01.009. MR3958071 ↑3, 8, 9, 30, 31

[CPS17] J. W. Cogdell and I. I. Piatetski-Shapiro, *Derivatives and L-functions for GL_n*, Representation theory, number theory, and invariant theory, Progr. Math., vol. 323, Birkhäuser/Springer, Cham, 2017, pp. 115–173. MR3753910 ↑27

[CST17] J. W. Cogdell, F. Shahidi, and T.-L. Tsai, *Local Langlands correspondence for* GL_n *and the exterior and symmetric square* ε*-factors*, Duke Math. J. **166** (2017), no. 11, 2053–2132, DOI 10.1215/00127094-2017-0001. MR3694565 ↑3, 26

[DKV84] P. Deligne, D. Kazhdan, and M.-F. Vignéras, *Représentations des algèbres centrales simples p-adiques*, Representations of reductive groups over a local field, Travaux en Cours, Hermann, Paris, 1984, pp. 33–117 (French). MR771672 ↑2, 10

[Del86] P. Delorme, *Formules limites et formules asymptotiques pour les multiplicités dans* $L^2(G/\Gamma)$, Duke Math. J. **53** (1986), no. 3, 691–731, DOI 10.1215/S0012-7094-86-05338-X (French). MR860667 ↑22

[GO16] M. Gurevich and O. Offen, *A criterion for integrability of matrix coefficients with respect to a symmetric space*, J. Funct. Anal. **270** (2016), no. 12, 4478–4512, DOI 10.1016/j.jfa.2016.02.008. MR3490774 ↑10

[HW93] A. G. Helminck and S. P. Wang, *On rationality properties of involutions of reductive groups*, Adv. Math. **99** (1993), no. 1, 26–96, DOI 10.1006/aima.1993.1019. MR1215304 ↑10

[Hen10] G. Henniart, *Correspondance de Langlands et fonctions L des carrés extérieur et symétrique*, Int. Math. Res. Not. IMRN **4** (2010), 633–673, DOI 10.1093/imrn/rnp150 (French). MR2595008 ↑3

[JLR99] H. Jacquet, E. Lapid, and J. Rogawski, *Periods of automorphic forms*, J. Amer. Math. Soc. **12** (1999), no. 1, 173–240, DOI 10.1090/S0894-0347-99-00279-9. MR1625060 ↑5, 12, 14, 17

[JR96] H. Jacquet and S. Rallis, *Uniqueness of linear periods*, Compositio Math. **102** (1996), no. 1, 65–123. MR1394521 ↑21

[Kab04] A. C. Kable, *Asai L-functions and Jacquet's conjecture*, Amer. J. Math. **126** (2004), no. 4, 789–820. MR2075482 ↑22

[LO18] E. M. Lapid and O. Offen, *On the distinguished spectrum of* Sp_{2n} *with respect to* $\mathrm{Sp}_n \times \mathrm{Sp}_n$, Kyoto J. Math. **58** (2018), no. 1, 101–171, DOI 10.1215/21562261-2017-0019. MR3776281 ↑23, 25

[LR03] E. M. Lapid and J. D. Rogawski, *Periods of Eisenstein series: the Galois case*, Duke Math. J. **120** (2003), no. 1, 153–226, DOI 10.1215/S0012-7094-03-12016-5. MR2010737 ↑5, 6, 16, 19

[Mat14] N. Matringe, *Linear and Shalika local periods for the mirabolic group, and some consequences*, J. Number Theory **138** (2014), 1–19, DOI 10.1016/j.jnt.2013.11.012. MR3168918 ↑4

[Mat15] ———, *On the local Bump-Friedberg L-function*, J. reine angew. Math. **2015** (2015), no. 709, 119-170, DOI 10.1515/crelle-2013-0083. MR3430877 ↑4

[Mat] ———, *Gamma factors of intertwining periods and distinction for inner forms of* $\mathrm{GL}(n)$, available at arXiv:1709.00987. to appear in J. Funct. Anal. ↑5, 27, 29

[MW95] C. Mœglin and J.-L. Waldspurger, *Spectral decomposition and Eisenstein series*, Cambridge Tracts in Mathematics, vol. 113, Cambridge University Press, Cambridge, 1995. Une paraphrase de l'Écriture [A paraphrase of Scripture]. MR1361168 ↑17

[Off04] O. Offen, *Relative spherical functions on* $\wp$*-adic symmetric spaces (three cases)*, Pacific J. Math. **215** (2004), no. 1, 97–149, DOI 10.2140/pjm.2004.215.97. MR2060496 ↑23, 25

[Off17] ———, *On parabolic induction associated with a p-adic symmetric space*, J. Number Theory **170** (2017), 211–227, DOI 10.1016/j.jnt.2016.06.014. MR3541705 ↑4, 31, 32

[PTB11] D. Prasad and R. Takloo-Bighash, *Bessel models for* $\mathrm{GSp}(4)$, J. Reine Angew. Math. **655** (2011), 189–243, DOI 10.1515/CRELLE.2011.045. MR2806111 ↑1, 2

[SV17] Y. Sakellaridis and A. Venkatesh, *Periods and harmonic analysis on spherical varieties*, Astérisque **396** (2017), viii+360 (English, with English and French summaries). MR3764130 ↑22

[Séc] V. Sécherre, *Représentations cuspidales de* $\mathrm{GL}_r(D)$ *distinguées par une involution intérieure.*, available at arXiv:2005.05615. ↑2

[Suz21] M. Suzuki, *Classification of standard modules with linear periods*, J. Number Theory **218** (2021), 302–310, DOI 10.1016/j.jnt.2020.07.005. MR4157701 ↑2, 3

[Tad90] M. Tadić, *Induced representations of* $\mathrm{GL}(n, A)$ *for* p*-adic division algebras* A, J. Reine Angew. Math. **405** (1990), 48–77, DOI 10.1515/crll.1990.405.48. MR1040995 ↑30, 31

[Wal92] N. R. Wallach, *Real reductive groups. II*, Pure and Applied Mathematics, vol. 132, Academic Press, Inc., Boston, MA, 1992. MR1170566 ↑10, 21

[Xue21] H. Xue, *Epsilon dichotomy for linear models*, Algebra Number Theory **15** (2021), no. 1, 173–215, DOI 10.2140/ant.2021.15.173. MR4226986 ↑2, 3, 32

[Yam17] S. Yamana, *Local symmetric square* L*-factors of representations of general linear groups*, Pacific J. Math. **286** (2017), no. 1, 215–256, DOI 10.2140/pjm.2017.286.215. MR3582406 ↑3, 4, 32

[Zel80] A. V. Zelevinsky, *Induced representations of reductive* $\mathfrak{p}$*-adic groups. II. On irreducible representations of* $\mathrm{GL}(n)$, Ann. Sci. École Norm. Sup. (4) **13** (1980), no. 2, 165–210. MR584084 ↑2, 9

[Zyd] M. Zydor, *Periods of automorphic forms over reductive subgroups.* to appear in Annales Scientifiques de l'École Normale Supérieure. ↑5, 10, 11, 12, 16, 19

Miyu Suzuki, Faculty of Mathematics and Physics, Institute of Science and Engineering, Kanazawa University, Kakumamachi, Kanazawa, Ishikawa, 920-1192

E-mail address: `miyu-suzuki@staff.kanazawa-u.ac.jp`

Hang Xue, Department of Mathematics, The University of Arizona, Tucson, AZ 85721

E-mail address: `xuehang@math.arizona.edu`