BESSEL MODELS FOR REAL UNITARY GROUPS: THE TEMPERED CASE

HANG XUE

ABSTRACT. We prove the local Gan–Gross–Prasad conjecture for tempered L-packets of real unitary groups. The proof is based on theta lifts and is very simple.

CONTENTS

1. INTRODUCTION

The goal of this paper is to prove the local Gan–Gross–Prasad (GGP) conjecture, as stated in [GGP12a, Conjecture 17.3], for all tempered L-packets of *real* unitary groups $\mathrm{U}(n+2t+1)\times\mathrm{U}(n)$. In the simplest case $t=0$, the local GGP conjecture seeks to characterize the nonvanishing of the space $\mathrm{Hom}_{\mathrm{U}(n)}(\pi\widehat{\otimes}\sigma,\mathbb{C})$ where π and σ are irreducible representations of (not necessarily compact) $\mathrm{U}(n+1)$ and $\mathrm{U}(n)$ repsectively. When the unitary groups in question are indeed compact, the characterization is given by Weyl's celebrated branching rule [Wey50, Section 18.2, p. 391]. In this case, irreducible representations of compact unitary groups are all finite dimensional and are parameterized by their highest weights. If π and the dual of σ respectively have highest weights

$$a_1\geq\cdots\geq a_{n+1},\quad b_1\geq\cdots\geq b_n,$$

then Weyl's branching rule states that $\mathrm{Hom}_{\mathrm{U}(W)}(\pi\otimes\sigma,\mathbb{C})$ is at most one dimensional and it is one dimensional precisely when

$$a_1\geq b_1\geq a_2\geq b_2\geq\cdots\geq b_n\geq a_{n+1}.$$

In general when the unitary groups are not necessarily compact, it is also known that this Hom space is at most one dimensional, cf. [SZ12]. The local GGP conjecture rougly states that in each generic Vogan L-packet of $\mathrm{U}(n+1)\times\mathrm{U}(n)$, there is a unique pair (π,σ) such that $\mathrm{Hom}_{\mathrm{U}(W)}(\pi\widehat{\otimes}\sigma,\mathbb{C})\neq 0$, and this pair can be specified by the local root numbers. We prove this conjecture for all tempered L-packets. The precise definitions and the statement of the theorem will be given below in Section 2.

In a subsequent paper [Xue], we will prove this conjecture for real unitary groups in general by reducing it to the tempered case. The techniques used in these two papers are of rather different nature, so we keep them as two separate ones.

The proof of the tempered case in this paper is remarkably simple, both conceptually and technically. Previously, Beuzart-Plessis [BP20], following the strategy of Waldspurger, proved the "multiplicity one in a Vogan packet" part of the local GGP conjecture for all tempered packets of $\mathrm{U}(n+2t+1)\times\mathrm{U}(n)$ (and the full local GGP conjecture for tempered L-packets of p-adic unitary groups). The argument is based on local trace formulae and is long and difficult, but has the advantage of being uniform for both real and p-adic groups. Our proof follows a completely different approach and applies exclusively to real unitary groups. The proof is based on theta lifts. Via theta lifts, He [He17] proved the local GGP conjecture for $\mathrm{U}(n+1)\times\mathrm{U}(n)$ for square integrable representations, and Gan–Ichino [GI16] studied the relation between $\mathrm{U}(n+1)\times\mathrm{U}(n)$ and $\mathrm{U}(n)\times\mathrm{U}(n)$. Part of our argument is inspired by their work. One notable point is that contrary to the usual expectations, our argument is not based on the results in [He17] and it does not reduce the tempered case to the square integrable case. An outline of the proof will be given after the precise statement of the theorem. The reason why the method applies to real unitary groups but not p-adic ones will be clear from the discussion there. Some ideas in the proof have other applications, e.g. studying the local linear forms appearing in the Ichino–Ikeda' conjecture [II10]. We will pursue these ideas in later papers.

Notation and convention. Throughout this paper, we keep the following notation. Let $\psi:\mathbb{R}\to\mathbb{C}^\times$ be the additive character given by $\psi(x)=e^{-2\pi ix}$ and $\psi^{\mathbb{C}}(z)=e^{2\pi(\overline{z}-z)}$. By a character of $\mathbb{C}^\times$ we mean a unitary character. If χ is a character of $\mathbb{C}^\times$, we put $\chi^c(z)=\chi(\overline{z})$. The character χ is called conjugate self-dual of sign $+1$ (resp. -1) if $\chi|_{\mathbb{R}^\times}$ is trivial (resp. the sign character) or equivalently $\chi(z)=(z/\sqrt{z\overline{z}})^m$ for an even (resp. odd) integer m.

We denote by 1_n the $n\times n$ identity matrix, and $\mathrm{diag}(x_1,\cdots,x_n)$ the $n\times n$ diagonal matrix with entries $x_1,\cdots,x_n$.

If V is a hermitian or skew-hermitian space, the hermitian or skew-hermitian form on skew-hermitian form is denote by h_V. If V is hermitian of signature $(n-q,q)$ then we define $\operatorname{disc} V=(-1)^{\frac{n(n-1)}{2}-q}$. If V is skew-hermitian, then we let $-iV$ be the hermitian space with the underline vector space V and hermitian form $-ih_V$. We denote by L_{+1} and L_{-1} the positive and negative hermitian line respectively, i.e. the underline vector spaces are $\mathbb{C}$ and the hermitian forms are given by $h(x,y)=x\overline{y}$ and $h(x,y)=-x\overline{y}$ respectively.

Unless we explicitly mention the contrary, by a representation, we always mean a unitary Casselman–Wallach representation of finite length (Frechet representation of moderate growth), cf. [Wal92, Chapter XII]. The inner product on a representation is denoted by $\langle -,-\rangle$. Let π be a representation. We denote by $\pi^\vee$ the space of continuous linear functionals on π and it is given the strong topology (uniform convergence on bounded subsets). The smooth dual of π, i.e. the subspace of smooth vectors in $\pi^\vee$, is identified with $\overline{\pi}$. By a tempered representation of a reductive group G, we mean a (unitary) representation whose matrix coefficients are in $L^{2+\epsilon}(G/Z)$ for any $\epsilon > 0$ where Z stands for the center of G.

Acknowledgement. I thank Wee Teck Gan, Atsushi Ichino and Yifeng Liu for helpful discussions. I thank Ye Tian and Shou-Wu Zhang for their constant support. I thank the anonymous referees for many suggestions which improve the readability of this paper. This work is partially supported by the NSF grant DMS #1901862. This manuscript is prepared during the coronavirus pandemic when many of us are trapped at home. I thank my family especially my wife for the great support during this difficult and stressful time. She takes care of the kids most of the time so I can escape and do math. This paper can never be finished without her battle with the kids.

2. Restrictions problems

2.1. **Langlands–Vogan packets.** As explained in [GGP12a, Section 8], a tempered L-parameter ϕ for a unitary group in n variables is an n-dimensional conjugate self-dual semisimple continuous representation of $\mathbb{C}^\times$ of sign $(-1)^{n-1}$. As $\mathbb{C}^\times$ is abelian we can write

$$\phi = m_1\phi_1 \oplus \cdots \oplus m_k\phi_k, \tag{2.1}$$

where $m_1, \cdots, m_k$ are integers, $m_1 + \cdots + m_k = n$, and $\phi_1, \cdots, \phi_k$ are distinct characters of $\mathbb{C}^\times$. The parameter ϕ being conjugate self-dual of sign $(-1)^{n-1}$ means the following. If ϕ_i is conjugate self-dual of sign $(-1)^{n-1}$, then m_i is arbitrary. If ϕ_i is conjugate self-dual of sign $(-1)^n$, then m_i is even. If ϕ_i is not conjugate self-dual then there is an i' so that $\phi_i^{c,-1} = \phi_{i'}$ and $m_i = m_{i'}$. In particular the number of conjugate self-dual characters of sign $(-1)^{n-1}$ (counting multiplicity) has the same parity with n.

A component group A_ϕ is defined for each ϕ, cf. [GGP12a, Section 4]. In our case it is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^r$ where r is the number of distinct conjugate self-dual characters of sign $(-1)^{n-1}$ contained in ϕ. By relabeling we may assume that ϕ_i is self-dual of sign $(-1)^{n-1}$ precisely when $1 \leq i \leq r$. We then label elements in A_ϕ as

$$\bigoplus_{i=1}^{r} (\mathbb{Z}/2\mathbb{Z})a_i, \tag{2.2}$$

where a_i is a symbol corresponding to ϕ_i. Without saying the contrary we will follow this convention of labeling characters in the L-parameter.

Let V be a hermitian space of dimension n. To each ϕ, the Langlands correspondence, cf. [GGP12a, Section 9, 10], assigns a finite set Π_ϕ^V of irreducible tempered representations of $\mathrm{U}(V)$, called the L-packet attached to ϕ. The Vogan packet attached to ϕ, denoted by Π_ϕ, is the (disjoint) union of all Π_ϕ^V as V ranges over all (isomorphism classes of) hermitian spaces of dimension n,

$$\Pi_\phi = \bigcup_{V:\ \dim V = n} \Pi_\phi^V.$$

To each representation $\pi \in \Pi_\phi$ there is a character $\eta : A_\phi \to \langle \pm 1 \rangle$ attached to it and this defines a bijection between Π_ϕ and all characters of A_ϕ. This bijection $\Pi_\phi \to \mathrm{Hom}(A_\phi, \langle \pm 1 \rangle)$ depends on the choice of a (equivalence class of) Whittaker datum. When n is odd, we choose it to be the unique Whittaker datum (up to equivalence) of $\mathrm{U}(V)$ where V is of signature $(\frac{n+1}{2}, \frac{n-1}{2})$. When n is even as explained in [GGP12a, Section 10], this is equivalent to choosing a nontrivial additive character of $\mathbb{C}$ which is trivial on $\mathbb{R}$. Throughout this paper, we will take this additive character to be $\overline{\psi^{\mathbb{C}}}$. This choice agrees with the one used in [Ato20], where the author picked the Whittaker datum $\mathfrak{w}_+$ in his notation (cf. the last sentence of the paper), and the definition of $\mathfrak{w}_+$ is in the middle of p. 66 for odd unitary groups, and in the middle of p. 67 for even unitary groups (his notation is μ_+).

Let V' be the skew-hermitian space and $V = -iV'$. The unitary groups $\mathrm{U}(V)$ and $\mathrm{U}(V')$ are physically the same. The use of $-i$ instead of i is compatible with the identifications in [Pau98, Pau00, Ato20]. In [Pau98, Subsection 1.1] and [Ato20, Subsection 3.1], the (skew-hermitian) unitary group preserving skew-hermitian form $i\,\mathrm{diag}(1_p, -1_q)$ is identified with a unitary group of signature (p, q). We have a similar Langlands correspondence for $\mathrm{U}(V')$ as in the case of $\mathrm{U}(V)$. The bijection between Π_ϕ and $\mathrm{Hom}(A_\phi, \langle \pm 1 \rangle)$ again depends on the choice of a Whittaker datum. Following [GGP12a, Section 12], we choose the Whittaker datum so that under the natural identification of $\mathrm{U}(V)$ and $\mathrm{U}(V')$, a representation of $\mathrm{U}(V)$ and hence $\mathrm{U}(V')$ correspond to the same character of A_ϕ.

Properties of the bijection $\Pi_\phi \leftrightarrow \mathrm{Hom}(A_\phi, \langle \pm 1 \rangle)$ have been summarized in [Ato20, Theorem 2.1] and [GGP12a, Section 9, 10]. The most frequently used ones are recalled in the following proposition.

Proposition 2.1. *We have the following assertions.*

(1) [Ato20, Theorem 2.1(5)] *Suppose $V = X \oplus V_0 \oplus Y$ where X, Y are isotropic lines, and $X \oplus Y$ is a two dimensional split hermitian space perpendicular to V_0. Let $P = MN$ be the parabolic subgroup stabilizing X and $M \simeq \mathbb{C}^\times \times \mathrm{U}(V_0)$ be the Levi subgroup. Let π_0 be an irreducible tempered representation of $\mathrm{U}(V_0)$ whose Langlands–Vogan parameter is (ϕ_0, η_0), and ξ a character of $\mathbb{C}^\times$ that is not conjugate self-dual of sign $(-1)^{n-1}$. Then the parabolic induction*

$$\pi = \mathrm{Ind}_P^{\mathrm{U}(V)} \xi \otimes \pi_0$$

is an irreducible tempered representation of $\mathrm{U}(V)$ whose Langlands–Vogan parameter is (ϕ, η) where

$$\phi = \phi_0 \oplus \xi \oplus \xi^{c,-1}, \quad A_\phi = A_{\phi_0}, \quad \eta = \eta_0.$$

(2) [Ato20, Theorem 2.1(8)] *Suppose that π is an irreducible tempered representation of $\mathrm{U}(V)$ with Langlands–Vogan parameter (ϕ, η), then $\pi \otimes \det^k$ has parameter $(\phi \otimes \chi^k, \eta)$, where $\chi_k(z) = (z/\overline{z})^k = (z/\sqrt{z\overline{z}})^{2k}$.*

(3) [Ato20, Theorem 2.1(6)] *If π is an irreducible tempered representation* $\mathrm{U}(V)$ *with Langlands–Vogan parameter* (ϕ, η)*, then the smooth dual of* π*, which has been identified with* $\overline{\pi}$*, has Langlands–Vogan parameter* $(\phi^\vee, \eta^\vee)$ *where*

$$\phi^\vee = m_1\phi_1^{-1} \oplus \cdots \oplus m_k\phi_k^{-1}, \quad A_{\phi^\vee} = A_\phi, \quad \eta^\vee(a_i) = (-1)^{n-1}\eta(a_i).$$

(4) [GGP12a, Section 10] *Let π be an irreducible tempered representation of* $\mathrm{U}(V)$*, where V is of signature* (p, q) *and π has Langlands–Vogan parameter* (ϕ, η)*. Then*

$$\eta(a_1 + \cdots + a_r) = (-1)^{\frac{n(n-1)}{2} - q} = \operatorname{disc} V.$$

Given any nontrivial additive character ψ' of $\mathbb{C}$, one can define the root number $\epsilon(\phi, \psi')$. We only record the following properties which will be used often and refer the readers to [GGP12a, Section 5] for a more detailed discussion. We have

$$\epsilon(\phi, \overline{\psi'}) = \det \phi(-1)\epsilon(\phi, \psi'), \quad \epsilon(\phi, \psi')\epsilon(\phi^\vee, \overline{\psi'}) = 1, \tag{2.3}$$

If ψ' is trivial on $\mathbb{R}$, and ξ is a character of $\mathbb{C}^\times$, then

$$\epsilon(\phi, \psi') = \epsilon(\phi \oplus \xi \oplus \xi^{c,-1}, \psi'). \tag{2.4}$$

We also note that the additive character $\psi^{\mathbb{C}}$ satisfies the condition of being trivial on $\mathbb{R}$.

2.2. **Bessel models.** Let t be a nonnegative integer. Let $W \subset V$ be a pair of hermitian spaces of dimensions n and $n + 2t + 1$ respectively. We say that the pair (W, V) is relevant if $V = W \oplus^\perp Z$ and we can find a basis $z_0, z_{\pm i}, i = 1, \cdots, t$ of Z with

$$h_V(z_i, z_j) = (-1)^n \delta_{i,-j}, \quad i, j = 0, \pm 1, \cdots, \pm t.$$

Note that this definition is slightly more restrictive than [GGP12a, Section 2], where it is only required that Z is split. To study the restriction problem and prove the local GGP conjecture, we however only need to consider the case at hand, because if $h_V(z_0, z_0) = (-1)^{n+1}$, we may consider the restriction problem for the embedding of groups $\mathrm{U}(-W) \subset \mathrm{U}(-V)$ instead.

Let P be the parabolic subgroup of $\mathrm{U}(V)$ stabilizing the flag of completely isotropic subspaces

$$\langle z_t \rangle \subset \langle z_t, z_{t-1} \rangle \subset \cdots \subset \langle z_t, \cdots, z_1 \rangle.$$

Let N be the unipotent radical of P, and $H = \mathrm{U}(W) \ltimes N$ which is a subgroup of $\mathrm{U}(V)$. We define a character of N as follows. Let $u \in N$, we define a character

$$\nu(u) = \psi\left(\mathrm{Tr}_{\mathbb{C}/\mathbb{R}} \sum_{i=0}^{t-1} h_V(z_{-i-1}, uz_i)\right).$$

As ν is invariant under the conjugation action of $\mathrm{U}(W)$, it admits a unique extension to H which is trivial on $\mathrm{U}(W)$ which we still denote by ν.

Let π and σ be representations of $\mathrm{U}(V)$ and $\mathrm{U}(W)$ respectively. We denote by $\mathrm{Hom}_H(\pi \widehat{\otimes} \sigma, \nu)$ the space of continuous linear forms $\ell : \pi \widehat{\otimes} \sigma \to \mathbb{C}$ with the property that

$$\ell(\pi(h)v \otimes \sigma(h)w) = \nu(h)\ell(v \otimes w), \quad \text{for all } h \in H,\ v \in \pi \text{ and } w \in \sigma.$$

We put $m(\pi, \sigma) = \dim \mathrm{Hom}_H(\pi \widehat{\otimes} \sigma, \nu)$. By [SZ12, JSZ10] if π and σ are irreducible then $m(\pi, \sigma) \leq 1$.

If $t = 0$, then $H = \mathrm{U}(W)$ and ν is trivial. Then $\mathrm{Hom}_{\mathrm{U}(W)}(\pi \widehat{\otimes} \sigma, \mathbb{C})$ is the space of $\mathrm{U}(W)$-invariant continuous linear forms on $\pi \widehat{\otimes} \sigma$.

Assume that π and σ are irreducible and tempered. Let (ϕ_π, η_π) and $(\phi_\sigma, \eta_\sigma)$ be the parameters of π and σ respectively. Write

$$\phi_\pi = m_1\chi_1 \oplus \cdots \oplus m_k\chi_k, \quad \phi_\sigma = n_1\mu_1 \oplus \cdots \oplus n_l\mu_l,$$

and

$$A_{\phi_\pi} = \bigoplus_{i=1}^{r} (\mathbb{Z}/2\mathbb{Z})a_i, \quad A_{\phi_\sigma} = \bigoplus_{j=1}^{s} (\mathbb{Z}/2\mathbb{Z})b_j.$$

The following theorem is the main result of this paper. It confirms the local GGP conjecture for real unitary groups [GGP12a, Conjecture 17.3].

Theorem 2.2. *Assume that π and σ are irreducible and tempered. Then $m(\pi, \sigma) = 1$ if and only if*

$$\eta_\pi(a_i) = \epsilon(\chi_i \otimes \phi_\sigma, \psi^{\mathbb{C}}), \quad \eta_\sigma(b_j) = \epsilon(\phi_\pi \otimes \mu_j, \psi^{\mathbb{C}}),$$

for all $i = 1, \cdots, r$ and $j = 1, \cdots, s$.

2.3. **Tempered intertwining.** Let $W \subset V$ be a relevant pair of hermitian spaces of dimensions n and $n+2t+1$ respectively. Let π and σ be tempered representations of $\mathrm{U}(V)$ and $\mathrm{U}(W)$ respectively. In [BP20, Section 7.2], an $H \times H$ bi-invariant continuous linear form

$$\mathcal{L}_{\pi,\sigma} : \pi \widehat{\otimes} \overline{\pi} \widehat{\otimes} \sigma \widehat{\otimes} \overline{\sigma} \to \mathbb{C}$$

is defined. We remark that it is not required in this definition that π and σ are irreducible. Assume now that $t = 0$. Then when restricted to $\pi \otimes \overline{\pi} \otimes \sigma \otimes \overline{\sigma}$, the linear form $\mathcal{L}_{\pi,\sigma}$ is given by the more familiar integration of matrix coefficients [II10]

$$\mathcal{L}_{\pi,\sigma}(e, e', v, v') = \int_{\mathrm{U}(W)} \langle \pi(h)e, e' \rangle \langle \sigma(h)v, v' \rangle \mathrm{d}h. \tag{2.5}$$

The integral is absolutely convergent [II10]. In general if $t > 0$, a regularization is needed in the definition, cf. [BP20, Section 7.1], but we will not need the expression of $\mathcal{L}_{\pi,\sigma}$ in this case.

Let us recall two results from [BP20, Chapter 7] about $\mathcal{L}_{\pi,\sigma}$. In the local trace formula approach, these results need to be established before one can analyze the spectral side. It does not rely on any local trace formula argument.

In general (any t), it follows from the definition that if $\mathcal{L}_{\pi,\sigma} \neq 0$, then $m(\pi, \sigma) \neq 0$. The following result is [BP20, Theorem 7.2.1], which proves the converse.

Proposition 2.3. *If π and σ are irreducible and tempered, then $m(\pi, \sigma) = 1$ if and only if $\mathcal{L}_{\pi,\sigma} \neq 0$.*

Let $W_1 = V \oplus \langle z_0' \rangle$ with $h_{W_1}(z_0', z_0') = (-1)^{n+1}$. Let X be the isotropic subspace of W_1 spanned by

$$\langle z_0 + z_0', z_1, \cdots, z_t \rangle,$$

and let Q be the parabolic subgroup of $\mathrm{U}(W_1)$ stabilizing X. The Levi subgroup M of Q is isomorphic to $\mathrm{GL}_{t+1}(\mathbb{C}) \times \mathrm{U}(W)$. Let τ be a tempered representation of $\mathrm{GL}_{t+1}(\mathbb{C})$ and we consider the induced representation

$$\sigma_1 = \mathrm{Ind}_Q^{\mathrm{U}(W_1)} \tau \widehat{\otimes} \sigma.$$

We usually denote this induced representation by $\tau \rtimes \sigma$.

The following proposition is [BP20, Proposition 7.4.1]. Proposition 2.3 is deduced from it by an argument using the Plancherel formula.

Proposition 2.4. *For any tempered representation τ, we have $\mathcal{L}_{\pi,\sigma} \neq 0$ if and only if $\mathcal{L}_{\sigma_1,\pi} \neq 0$.*

Proposition 2.3 and Proposition 2.4 reduce Theorem 2.2 to the case $t = 0$.

Corollary 2.5. *Theorem 2.2 in the case $t = 0$ implies all other cases $t > 0$.*

Proof. We keep the notation from Theorem 2.2. Let τ be an irreducible principal series representation of $\mathrm{GL}_{t+1}(\mathbb{C})$, induced from the character $\xi_1 \otimes \cdots \otimes \xi_{t+1}$ of $(\mathbb{C}^\times)^{t+1}$, where $\xi_1, \cdots, \xi_{t+1}$ are not conjugate self-dual. Then by Proposition 2.1(1), σ_1 is irreducible and the parameter of σ_1 is given by

$$\phi_{\sigma_1} = \phi_\sigma \oplus \xi_1 \oplus \xi_1^{c,-1} \oplus \cdots \oplus \xi_{t+1} \oplus \xi_{t+1}^{c,-1}, \quad A_{\phi_{\sigma_1}} = A_{\phi_\sigma}, \quad \eta_{\sigma_1} = \eta_\sigma.$$

Therefore by (2.4) we have $\epsilon(\chi_i \otimes \phi_{\sigma_1}, \psi^{\mathbb{C}}) = \epsilon(\chi_i \otimes \phi_\sigma, \psi^{\mathbb{C}})$. By Proposition 2.3 and Proposition 2.4, Theorem 2.2 for (π, σ) follows from that for (σ_1, π). $\square$

2.4. **Outline of the proof.** We now outline the proof of Theorem 2.2 when $t = 0$. Assume that ϕ_π contains a conjugate self-dual character of sign $(-1)^n$ and ϕ_σ contains b conjugate self-dual character of sign $(-1)^{n-1}$ (counting multiplicity). Note that a has the same parity as $n+1$ and b has the same parity as n. Thus $a + b$ is odd. The trick is to make an induction on $a + b$, not on the size of the unitary group.

The base case is $a + b = 1$. In this case π and σ are both irreducible principal series representations. We have $m(\pi, \sigma) = 1$ and η_π and η_σ are both trivial characters. One in fact checks directly in this case that $\mathcal{L}_{\pi,\sigma}$ is not identically zero. Theorem 2.2 thus holds when $a + b = 1$.

Now we assume $a + b \geq 3$ and proceed with induction. We should have either $a \geq 2$ or $b \geq 2$.

Assume $a \geq 2$ first. The key observation is that, up to twists by characters of the form $\det^k$ for some integer k, π can be constructed by a composition of two theta lifts. More precisely we can find the following data:

- a hermitian space V_0 of dimension $n - 1$ and a skew-hermitian space V' of dimension n;

- an irreducible tempered representation of π_0 of $\mathrm{U}(V_0)$ and an irreducible tempered representation π' of $\mathrm{U}(V')$;

with the property that $\pi' \otimes \det^{k'}$ is the theta lift of π_0 and $\pi \otimes \det^k$ is the theta lift of π' (the integers k and k' depend on π and π'). In this case, via a familiar seesaw argument, Theorem 2.2 for (π, σ) are reduced to that of (σ_0, π_0) where σ_0 (up to twist) is an irreducible tempered representation in the same Vogan packet as σ. The L-parameters of σ_0 contains b conjugate self-dual characters of sign $(-1)^{n-1}$ and the L-parameter of π_0 contains $a-2$ conjugate self-dual characters of sign $(-1)^n$ (see the discussion of the relation between parameters and theta lifts in Section 3). Thus we are done by induction.

Finally assume $b \geq 2$. Let W_1 be the (orthogonal) direct sum of W and a hyperplane and let P be the parabolic subgroup stabilizing an isotropic line in this hyperplane. Then the Levi subgroup is isomorphic to $\mathbb{C}^\times \times \mathrm{U}(W)$. We choose a character χ that is not conjugate self-dual. Consider the induced representation $\sigma_1 = \chi \rtimes \sigma$. Then σ_1 is irreducible and the L-parameter contains b conjugate self-dual characters of sign $(-1)^{n-1}$. We then apply the result in the case $a \geq 2$ to (σ_1, π) and prove Theorem 2.2 for (σ_1, π). Using tempered intertwining, we see that Theorem 2.2 for (σ_1, π) is equivalent to Theorem 2.2 for (π, σ). Then we are done.

The reason why the argument applies exclusively to real unitary groups is now clear. An L-parameter of a real unitary group is a sum of characters and most of the representations (up to twists) can be constructed via theta lifts from smaller unitary groups. Of course this is not true for p-adic unitary groups. For similar reasons, the technique does not seem to work without modification for the local GGP conjecture for orthogonal groups.

3. Theta lifts

3.1. **Weil representations and theta lifts.** Let V and V' be a hermitian space and a skew-hermitian space of dimension n and n' respectively. Then $\mathrm{U}(V) \times \mathrm{U}(V')$ is a dual pair in $\mathrm{Sp}(2nn', \mathbb{R})$ in the sense of Howe. Let ω_ψ be the Weil representation of $\mathrm{Mp}(2nn', \mathbb{R})$ associated to the additive character ψ. The choice of ψ agrees with the choices made in [Pau98, Pau00] by [Ich, Lemma 7.10]. It also agrees with the choice made in [Ato20], cf. [Ato20, p. 32]. In what follows the character ψ is always fixed and will be suppressed from the notation when there is no confusion.

By choosing a conjugate self-dual character χ_V of sign $(-1)^{n'}$ (resp. χ'_V of sign $(-1)^n$) of $\mathbb{C}^\times$, we get a splitting $\mathrm{U}(V) \to \mathrm{Mp}(2nn', \mathbb{R})$ (resp. $\mathrm{U}(V') \to \mathrm{Mp}(2nn', \mathbb{R})$), cf. [Ato20, Section 3.1]. We denote again by ω the restriction of the Weil representation to $\mathrm{U}(V) \times \mathrm{U}(V')$. We remark that it is not a representation by our convention as it is not of finite length. It is a continuous action of $\mathrm{U}(V) \times \mathrm{U}(V')$ on some Schwartz space and it is unitary. In particular the underline space is a nuclear Frechet space. We will describe realizations of the Weil representation on mixed models below. If $V = L_{+1}$, and we use a conjugate self-dual character μ of sign (-1) to split the metaplectic cover over $\mathrm{U}(V')$, we denote the Weil representation of $\mathrm{U}(V')$ by $\omega_{\psi,\mu}$. This is the Weil representation that will appear in the Fourier–Jacobi model below.

Let π be an irreducible representation of $\mathrm{U}(V)$ then $\mathrm{U}(V)$-coinvariance $(\omega\widehat{\otimes}\overline{\pi})_{\mathrm{U}(V)}$ is denoted by $\Theta(\pi)$. It is a (not necessarily unitary) Casselman–Wallach representation of finite length of $\mathrm{U}(V')$ and its maximal semisimple quotient $\theta(\pi)$ of $\Theta(\pi)$ is irreducible, cf. [How89, Theorem 1A]. If $\dim V - \dim V' = 0, \pm 1$ and π is tempered, it is expected that $\Theta(\pi)$ is itself irreducible and hence unitary and $\Theta(\pi) = \theta(\pi)$. The analogous result for p-adic unitary groups holds by the work of Gan and Ichino [GI14, Appendix C]. A rigorous proof for real unitary groups is unfortunately missing in the literature. If we had this, we would be able to prove Theorem 2.2 and Proposition 2.3 simultaneously by combining the seesaw argument in Subsection 3.4 below and the abstract seesaw identity [Ato20, Proposition 3.11], without appealing to the results of [BP20, Chapter 7].

We are going to deal with several theta lifts at the same time, so we add various subscripts to $\theta(\pi)$ to distinguish them. When we need to specify them, for instance, we denote the above $\theta(\pi)$ by $\theta_{V,V',\chi_V,\chi_{V'},\psi}(\pi)$.

We now describe the realization of ω on a mixed model following [GI16, Section 7.4], and use it to deduce an estimate of matrix coefficients of ω. Though [GI16] considers only the nonarchimedean local fields, the formulae for the Weil representation presented there are valid for all local fields of characteristic zero. Let s be the Witt index of V and V_0 be the anisotropic kernel of V, $\dim V_0 = n_0 = n - 2s$. Choose basis $\{v_i, v_i^* \mid i = 1, \cdots, s\}$ of $V_0^\perp$ so that for all $i, j = 1, \cdots, s$,

$$\langle v_i, v_j\rangle = \langle v_i^*, v_j^*\rangle = 0, \quad \langle v_i, v_j^*\rangle = \delta_{ij}.$$

Let $P = MN$ be a minimal parabolic subgroup of $\mathrm{U}(V)$ stabilizing the flag

$$\mathbb{C}\{v_1\} \subset \mathbb{C}\{v_1, v_2\} \subset \cdots \subset \mathbb{C}\{v_1, \cdots, v_s\}.$$

Then $M \simeq (\mathbb{C}^\times)^s \times \mathrm{U}(V_0)$ and we let $A \simeq (\mathbb{R}_{>0})^s$ be the identity component of the maximal split torus in M. Let Δ_P be the roots of A in N and

$$A^+ = \{b \in A \mid |\alpha(b)| \leq 1, \forall \alpha \in \Delta_P\} = \{(b_1, \cdots, b_s) \mid 0 < b_1 \leq \cdots \leq b_s \leq 1\}.$$

Similarly let r be the Witt index of V' and V_0' be the anisotropic kernel of V', $\dim V_0' = n_0' = n' - 2r$, and we have a minimal parabolic subgroup P' and the identity component of a maximal split torus A', and

$$A'^+ = \{a = (a_1, \cdots, a_r) \mid 0 < a_1 \leq \cdots \leq a_r \leq 1\}.$$

To compare the notation with [GI16], the spaces V and W in [GI16] is our V and V'. The dimensions m and n in [GI16] is our n and n' respectively. The Witt indices r and s in [GI16] is our s and r respectively.

The Weil representation ω is realized on the mixed model. Let us first fix a realization $\mathcal{S}_{00}$ of the Weil representation ω_{V_0',V_0} of $\mathrm{U}(V_0') \times \mathrm{U}(V_0)$. The Weil representation $\omega_{V_0',V}$ of $\mathrm{U}(V_0') \times \mathrm{U}(V)$ is then realized on $\mathcal{S}_0 = \mathcal{S}(V_0'^s)\,\widehat{\otimes}\,\mathcal{S}_{00}$. Finally the Weil representation $\omega_{V',V}$ of $\mathrm{U}(V') \times \mathrm{U}(V)$ is realized on $\mathcal{S} = \mathcal{S}(V^r)\,\widehat{\otimes}\,\mathcal{S}_0$. We view elements in $\mathcal{S}$ as Schwartz functions on $V^r \times V_0'^s$ valued in

$\mathcal{S}_{00}$. We do not need the fully detailed description of the action as in [GI16, Section 7.4], but only the following. Let $a = (a_1, \cdots, a_r) \in A'^+$, $b = (b_1, \cdots, b_s) \in A^+$, and $\phi \in \mathcal{S}$. Then

$$\omega_{V',V}(a,b)\phi(z,w) = \chi_V(a_1 \cdots a_r)\chi_{V'}(b_1 \cdots b_s)(a_1 \cdots a_r)^n (b_1 \cdots b_s)^{n'-2r}\phi(b^{-1}za, wb), \tag{3.1}$$

where $z \in V^r$ (resp. $w \in V_0'^s$) is viewed as a row vector with entries in V (resp. V_0'), and when multiplied from the right, a and b are viewed as diagonal matrices with entries $a_1, \cdots, a_r$ and $b_1, \cdots, b_s$ respectively.

We fix an inner product on $\mathcal{S}_{00}$. Then an inner product on $\mathcal{S}$ is given by

$$\langle \phi, \phi' \rangle = \int_{V^r \times V_0'^s} \langle \phi(z,w), \phi'(z,w) \rangle \mathrm{d}z\mathrm{d}w.$$

This makes sense since ϕ, ϕ' are Schwartz. Define a function on $\mathbb{R}_{>0}$ by

$$\Upsilon(x) = \begin{cases} 1, & x \leq 1 \\ x^{-1}, & x > 1 \end{cases}$$

Lemma 3.1. *There is a continuous seminorm ν on $\mathcal{S}$ such that*

$$|\langle \omega_{V,V'}(b,a)\phi, \phi' \rangle| \leq \prod_{i=1}^r a_i^n \prod_{j=1}^s b_j^{n'-2r} \prod_{i=1}^r \prod_{j=1}^s \Upsilon(a_i b_j^{-1})^2 \nu(\phi)\nu(\phi'), \tag{3.2}$$

where $a = (a_1, \cdots, a_r) \in A'^+$, $b = (b_1, \cdots, b_s) \in A^+$.

Proof. This is a slightly refined estimate of the one given in the proof of [GI14, Lemma D.1]. From the explicit formula (3.1) we see that we only need to prove the following more general result. Let m be an integer and take $\phi, \phi' \in \mathcal{S}(\mathbb{R}^m) \widehat{\otimes} \mathcal{S}_{00}$, viewed as Schwartz functions valued in $\mathcal{S}_{00}$, and $\lambda = (\lambda_1, \cdots, \lambda_m) \in (\mathbb{R}_{>0})^m$. Then there is a seminorm ν on $\mathcal{S}(\mathbb{R}^m) \widehat{\otimes} \mathcal{S}_{00}$ such that

$$\left| \int_{\mathbb{R}^m} \langle \phi(\lambda_1 x_1, \cdots, \lambda_m x_m), \phi(x_1, \cdots, x_m) \rangle \mathrm{d}x_1 \cdots \mathrm{d}x_m \right| \leq \prod_{i=1}^m \Upsilon(\lambda_i^{-1})\nu(\phi)\nu(\phi'). \tag{3.3}$$

By relabeling, we may assume that

$$\lambda_1 \geq \cdots \geq \lambda_l \geq 1 > \lambda_{l+1} \geq \cdots \geq \lambda_m.$$

Make changes of variables $x_i \mapsto \lambda_i^{-1} x_i$, $i = 1, \cdots, l$. The left hand side of (3.3) then becomes $(\lambda_1 \cdots \lambda_l)^{-1}$ times

$$\left| \int_{\mathbb{R}^m} \langle \phi(x_1, \cdots, x_l, \lambda_{l+1}x_{l+1}, \cdots, \lambda_m x_m), \phi'(\lambda_1^{-1}x_1, \cdots, \lambda_l^{-1}x_l, x_{l+1}, \cdots, x_m) \rangle \mathrm{d}x_1 \cdots \mathrm{d}x_m \right|. \tag{3.4}$$

Choose positive polynomials functions $p(x_1, \cdots, x_l)$ and $p'(x_{l+1}, \cdots, x_m)$ such that

$$\int_{\mathbb{R}^m} p(x_1, \cdots, x_l)^{-1} p'(x_{l+1}, \cdots, x_m)^{-1} \mathrm{d}x_1 \cdots \mathrm{d}x_m$$

is convergent. Then we have that (3.4) is bounded by the product of this integral and

$$\begin{aligned} \sup_{x_1, \cdots, x_m} & |\langle p(x_1, \cdots, x_l)\phi(x_1, \cdots, x_l, \lambda_{l+1}x_{l+1}, \cdots, \lambda_m x_m), \\ & p'(x_{l+1}, \cdots, x_m)\phi'(\lambda_1^{-1}x_1, \cdots, \lambda_l^{-1}x_l, x_{l+1}, \cdots, x_m) \rangle| \end{aligned} \tag{3.5}$$

The Cauchy–Schwartz inequality gives that

$$\begin{aligned}(3.5) \leq & \sup_{x_1,\cdots,x_m} \|\langle p(x_1,\cdots,x_l)\phi(x_1,\cdots,x_l,\lambda_{l+1}x_{l+1},\cdots,\lambda_m x_m)\| \\ & \times \sup_{x_1,\cdots,x_m} \|p'(x_{l+1},\cdots,x_m)\phi'(\lambda_1^{-1}x_1,\cdots,\lambda_l^{-1}x_l,x_{l+1},\cdots,x_m)\rangle\|,\end{aligned}$$

where $\|\cdot\|$ is the norm induced by the inner product on $\mathcal{S}_{00}$. Note that p does not contain the variables $x_{l+1},\cdots,x_m$ while p' does not contain the variables $x_1,\cdots,x_l$. Thus both sup terms are independent of the λ_i's. Both sup terms are continuous seminorms on $\mathcal{S}(\mathbb{R}^m)\widehat{\otimes}\mathcal{S}_{00}$ by the definition of the Frechet space structure of $\mathcal{S}(\mathbb{R}^m)\widehat{\otimes}\mathcal{S}_{00}$ and we can find another continuous seminorm ν that dominates them. The desired estimate (3.3) follows, since

$$\prod_{i=1}^m \Upsilon(\lambda_i) = (\lambda_1\cdots\lambda_l)^{-1}$$

by our assumption. □

We now deduce from Lemma 3.1 several convergence lemmas which we need later. First we fix some notation and recall some standard estimates from [BP20, Section 1.5] and [II10, Section 4]. Let G be a real reductive group, we let Ξ^G be the Harish-Chandra Xi function on G [BP20, Section 1.5]. We fix a logarithmic norm ς_G on G [BP20, Section 1.2]. Fix a maximal connected split torus A of G and a minimal parabolic subgroup P_0 of G. Let Δ_{P_0} be the set of roots of A in P_0 and

$$A^+ = \{a \in A \mid |\alpha(a)| \leq 1, \forall \alpha \in \Delta_{P_0}\}.$$

Fix a maximal compact subgroup K of G such that we have the Cartan decomposition $G = KA^+K$. For any $f \in L^1(G)$ we have the following integration formula

$$\int_G f(g)\mathrm{d}g = \int_{A^+} \varphi(a) \iint_{K\times K} f(k_1ak_2)\mathrm{d}k_1\mathrm{d}k_2\mathrm{d}a, \tag{3.6}$$

where $\varphi(a)$ is a nonnegative function on A^+ satisfying

$$\varphi(a) \leq C\delta_{P_0}^{-1}(a) \tag{3.7}$$

where C is a positive constant independent of a and δ_{P_0} stands for the modulus character of P_0. Let π be an irreducible tempered representation of G, and α be a matrix coefficient of π. Then there is a constant A such that $|\alpha(g)| \leq A\Xi^G(g)$ for all $g \in G$. The result of [Sun09] is more precise. There is a seminorm ν on π such that

$$|\langle \pi(g)v, v'\rangle| \leq \Xi^G(g)\nu(v)\nu(v'), \quad v, v' \in \pi. \tag{3.8}$$

Moreover by [BP20, Proposition 1.5.1] there are constants $B, C' > 0$ such that if $a \in A^+$ we have

$$\Xi^G(a) \leq C'\delta_{P_0}^{\frac{1}{2}}(a)\varsigma_G(a)^B.$$

We now come back to our setup of unitary groups $\mathrm{U}(V)$ and $\mathrm{U}(V')$. We keep the notation prior to Lemma 3.1.

Lemma 3.2. *Assume that $n \geq n'$. Let π' be an irreducible tempered representation of $\mathrm{U}(V')$. Take $v, v' \in \pi'$ and $\phi, \phi' \in \mathcal{S}$. Then there are continuous seminorms $\nu_{\pi'}$ on π' and $\nu_{\mathcal{S}}$ on $\mathcal{S}$ such that*

$$\left| \int_{\mathrm{U}(V')} \overline{\langle \pi'(g)v, v' \rangle} \langle \omega(g,1)\phi, \phi' \rangle \mathrm{d}g \right| \leq \nu_{\pi'}(v)\nu_{\pi'}(v')\nu_{\mathcal{S}}(\phi)\nu_{\mathcal{S}}(\phi'). \tag{3.9}$$

Proof. To shorten notation, we write $\omega(g)$, $g \in \mathrm{U}(V')$, instead of $\omega(g,1)$. We write $a = (a_1, \cdots, a_r) \in A^+$, and we have

$$\delta_P(a) = \prod_{i=1}^r a_i^{2n'-4i+2}.$$

Let us fix a maximal compact subgroup K of $\mathrm{U}(V')$ such that we have the Cartan decomposition $\mathrm{U}(V') = KA'^+K$. By the integration formula (3.6), the estimate (3.7) and (3.8) and Lemma 3.1, the left hand side of (3.9) is bounded by

$$\int_{0<a_1\leq \cdots \leq a_r \leq 1} \prod_{i=1}^r a_i^{-\frac{1}{2}(2n'-4i+2)} (1 - \sum_{i=1}^r \log a_i)^B \prod_{i=1}^r a_i^n \mathrm{d}^\times a_1 \cdots \mathrm{d}^\times a_r$$
$$\times \int_K \int_K \widetilde{\nu}_\pi(\pi(k)v)\widetilde{\nu}_\pi(\pi(k'^{-1})v')\widetilde{\nu}_{\mathcal{S}}(\omega(k)\phi)\widetilde{\nu}_{\mathcal{S}}(\omega(k'^{-1})\phi')\mathrm{d}k\mathrm{d}k',$$

where B is a positive constant, $\mathrm{d}^\times a_i = a_i^{-1}\mathrm{d}a_i$ is a multiplicative measure on $\mathbb{R}_{>0}$, and $\widetilde{\nu}_\pi$ (resp. $\widetilde{\nu}_{\mathcal{S}}$) is a continuous seminorm on π (resp. $\mathcal{S}$). The first term simplifies to

$$\int_{0<a_1\leq \cdots \leq a_r \leq 1} \prod_{i=1}^r a_i^{n-n'+2i-1} (1 - \sum_{i=1}^r \log a_i)^B \mathrm{d}^\times a_1 \cdots \mathrm{d}^\times a_r.$$

Since $n \geq n'$, it is absolutely convergent. Since K is compact, the second term is bounded by

$$(\mathrm{vol}\, K)^2 \sup_{k\in K} \widetilde{\nu}_\pi(\pi(k)v) \sup_{k\in K} \widetilde{\nu}_{\mathcal{S}}(\omega(k)\phi) \sup_{k\in K} \widetilde{\nu}_\pi(\pi(k^{-1})v') \sup_{k\in K} \widetilde{\nu}_{\mathcal{S}}(\omega(k^{-1})\phi').$$

Each sup term defines a continuous seminorm on the corresponding space by the uniform boundedness principle [Trè06, Theorem 33.1]. The desired estimate (3.9) then follows. □

Lemma 3.3. *Assume $V = L_{+1}$ is a positive hermitian line. Let π' and σ' be irreducible tempered representations of $\mathrm{U}(V')$, and $v, v' \in \pi'$, $e, e' \in \sigma'$. Let $\phi, \phi' \in \mathcal{S}$. Then there are seminorms $\nu_{\pi'}$, $\nu_{\sigma'}$, and $\nu_{\mathcal{S}}$ on π', σ' and $\mathcal{S}$ respectively, such that*

$$\int_{\mathrm{U}(V')} |\langle \pi'(h)v, v' \rangle \langle \sigma'(h)e, e' \rangle \overline{\langle \omega_{\psi,\mu}(h)\phi, \phi \rangle}| \mathrm{d}h \leq \nu_{\pi'}(v)\nu_{\pi'}(v')\nu_{\sigma'}(e)\nu_{\sigma'}(e')\nu_{\mathcal{S}}(\phi)\nu_{\mathcal{S}}(\phi').$$

Proof. The argument is very similar to the previous lemma, so we will be brief. Using the integration formula (3.6), and applying the estimates (3.7), (3.8), Lemma 3.1, the left hand side of the desired inequality is bounded by the product of

$$\int_{0<a_1\leq \cdots \leq a_r \leq 1} a_1 \cdots a_r (1 - \sum_{i=1}^r \log a_i)^B \mathrm{d}^\times a_1 \cdots \mathrm{d}^\times a_r$$

and

$$\sup_{k\in K}\widetilde{\nu}_{\pi'}(\pi'(k)v)\sup_{k\in K}\widetilde{\nu}_{\pi'}(\pi'(k)v')\sup_{k\in K}\widetilde{\nu}_{\sigma'}(\sigma'(k)e)\sup_{k\in K}\widetilde{\nu}_{\sigma'}(\sigma'(k)e')\sup_{k\in K}\widetilde{\nu}_{\mathcal{S}}(\omega'(k)\phi)\sup_{k\in K}\widetilde{\nu}_{\mathcal{S}}(\omega'(k)\phi'),$$

where B is a positive constant and $\widetilde{\nu}_{\cdot}$'s are various continuous seminorms. The convergence of the first integral is clear. The sup terms are continuous seminorms again by the uniform boundedness principle. □

In the following proposition, if σ is a representation, by a dense subspace of matrix coefficients of σ, we mean a space of functions generated by $\langle\sigma(h)e, e'\rangle$, where $e \in U$ and $e' \in U'$ and U, U' are dense subspaces of σ.

Proposition 3.4. *Assume either $n = n'$ or $n = n' + 1$. Let π' be an irreducible tempered representation of $\mathrm{U}(V')$. Take $v, v' \in \pi'$ and $\phi, \phi' \in \mathcal{S}$.*

(1) *The linear form*

$$(v, v', \phi, \phi') \mapsto \int_{\mathrm{U}(V')} \overline{\langle\pi'(g)v, v'\rangle}\langle\omega(g, 1)\phi, \phi'\rangle \mathrm{d}g$$

continuously extends to a linear form on $\overline{\pi'}\,\widehat{\otimes}\,\pi'\,\widehat{\otimes}\,\omega\,\widehat{\otimes}\,\overline{\omega}$. It is not identically zero if and only if $\theta_{V',V}(\pi') \neq 0$.

(2) *Assume $\theta_{V',V}(\pi') \neq 0$. The function*

$$h \mapsto \int_{\mathrm{U}(V')} \overline{\langle\pi'(g)v, v'\rangle}\langle\omega(g, h)\phi, \phi'\rangle \mathrm{d}g$$

defines a (possibly zero) matrix coefficient of $\theta_{V',V}(\pi')$. When v, v' range over a dense subspace of π', and ϕ, ϕ' range over a dense subspace of $\mathcal{S}$, functions of this form generate a dense subspace of matrix coefficients of $\theta_{V',V}(\pi')$.

Proof. The absolute convergence and continuity follow from Lemma 3.2. The nonvanishing is quite subtle. If $n' = n$, it is proved in [LZ98, Proposition 5.2]; if $n' = n + 1$ it is proved in [GQT14, Proposition 11.5]. This proves the first assertion.

For the second assertion, the integral indeed defines some hermitian form on $\Theta_{V',V}(\pi')$. The semi-positivity of this hermitian form follows from [He03, Theorem 1.1]. It then defines an inner product on $\Theta_{V',V}(\pi')/K$ where K is the kernel of it. Therefore $\Theta_{V',V}(\pi')/K$ must be semisimple, and thus coincides with the $\theta_{V',V}(\pi')$. The positivity of this pairing on $\Theta_{V',V}(\pi')/K$ needs some explanation. In fact it follows from the following general fact. If U is a vector space and q is a nonzero semipositive definite hermitian form, and L the kernel of it, then q descends to an inner product on U/L. To see this, if there is an $x \notin L$ such that $q(x, x) = 0$, then take some $y \in U$ with $q(x, y) \neq 0$ and consider $q(ax + y, ax + y)$ for some $a \in \mathbb{C}$, which equals $q(y, y) + 2\mathrm{Re}\, aq(x, y)$. Since a is arbitrary and $q(x, y) \neq 0$, it cannot be nonnegative for all a.

Finally we fix a surjective homomorphism $p : \overline{\pi'}\,\widehat{\otimes}\,\omega \to \theta_{V',V}(\pi')$. The integral in the lemma takes the form $\langle\theta_{V',V}(\pi')p(v, \phi), p(v', \phi')\rangle$, where this $\langle -, -\rangle$ is an inner product on $\theta_{V',V}(\pi')$. The last assertion of the lemma then follows from the surjectivity of p and the density of $\overline{\pi'} \otimes \omega$ in $\overline{\pi'} \widehat{\otimes} \omega$. $\square$

3.2. **Parameters of theta lifts.** Let π be an irreducible tempered representation of $\mathrm{U}(V)$ with the parameter (ϕ, η). In the case $n' - n = 0, 1$, the parameters of theta lifts can be described. We fix χ_V and $\chi_{V'}$ to split the metaplectic cover over $\mathrm{U}(V)$ and $\mathrm{U}(V')$ respectively.

Let us write the parameter of π as

$$\phi = m_1\phi_1 \oplus \cdots \oplus m_k\phi_k, \tag{3.10}$$

where $\phi_1, \cdots, \phi_r$ are self-dual characters of sign $(-1)^{n-1}$ while $\phi_{r+1}, \cdots, \phi_k$ are not. Then $A_\phi = \oplus_{i=1}^r (\mathbb{Z}/2\mathbb{Z})a_i$ as in (2.2).

We first consider the equal rank case $n = n'$. There exists a unique skew-hermitian space V' with $\dim V' = n$ so that $\theta_{V,V',\chi_V,\chi_{V'}}(\pi) \neq 0$. Put

$$\theta(\phi) = \phi \otimes \chi_V^{-1}\chi_{V'}. \tag{3.11}$$

Then $\theta_{V,V',\chi_V,\chi_{V'}}(\pi) \in \Pi_{\theta(\phi)}$ and it corresponds to the character $\theta(\eta) : A_\phi \to \langle \pm 1 \rangle$ given by

$$\theta(\eta)(a_i) = \epsilon(\phi_i \otimes \chi_V^{-1}, \overline{\psi^{\mathbb{C}}})\eta(a_i). \tag{3.12}$$

The discriminants of V and V' are related by

$$\operatorname{disc} V \operatorname{disc}(-iV') = \epsilon(\phi \otimes \chi_V^{-1}, \overline{\psi^{\mathbb{C}}}).$$

These results were proved by Paul and were stated in [Pau98, Theorem 6.1] in a different language. We will make a translation after Lemma 3.5 for readers' convenience.

We now consider the almost equal rank situation. Put

$$\theta(\phi) = (\phi \otimes \chi_V^{-1}\chi_{V'}) \oplus \chi_{V'}. \tag{3.13}$$

There are two cases. If ϕ does not contain χ_V, there are exactly two skew-hermitian spaces V' with $\dim V' = n' = n+1$ such that $\theta_{V,V',\chi_V,\chi_{V'}}(\pi)$ is nonzero. We have

$$\theta_{V,V',\chi_V,\chi_{V'}}(\pi) \in \Pi_{\theta(\phi)}^{V'}, \quad A_{\theta(\phi)} = A_\phi \oplus (\mathbb{Z}/2\mathbb{Z})a_{V'},$$

where the extra copy of $\mathbb{Z}/2\mathbb{Z}$ comes from $\chi_{V'}$. The representation $\theta_{V,V',\chi_V,\chi_{V'}}(\pi)$ corresponds to the character $\theta(\eta) : A_{\theta(\phi)} \to \langle \pm 1 \rangle$ such that

$$\theta(\eta)|_{A_\phi} = \eta, \quad \theta(\eta)(a_{V'}) = \eta(a_1)\cdots\eta(a_r)(-1)^{\frac{n(n+1)}{2}-q}. \tag{3.14}$$

where $-iV'$ is of signature (p, q). In this case theta lift defines a bijection

$$\Pi_\phi \leftrightarrow \bigcup_{\operatorname{disc}(-iV')=\epsilon} \Pi_{\theta(\phi)}^{V'},$$

where $\epsilon = \pm 1$ is a fixed sign and V' ranges over all (isomorphism classes of) skew-hermitian spaces with $\mathrm{disc}(-iV') = \epsilon$. If ϕ contains χ_V, then there is a unique V' such that $\theta_{V,V',\chi_V,\chi_{V'}}(\pi) \neq 0$, and it corresponds to $(\theta(\phi), \theta(\eta))$ where

$$A_{\theta(\phi)} = A_\phi, \quad \theta(\eta) = \eta. \tag{3.15}$$

In this case theta lift defines a bijection

$$\Pi_\phi \leftrightarrow \Pi_{\theta(\phi)}.$$

These results are due to Paul [Pau00]. It is again not stated in this language in [Pau00]. We will make the translation after Lemma 3.5.

One direct consequence of this description of the parameters is the following lemma. It is easy but is crucial to our argument.

Lemma 3.5. *Assume $n' = n+1$. Let π' be an irreducible tempered representation of $\mathrm{U}(V')$ with Langlands–Vogan parameter $(\phi_{\pi'}, \eta_{\pi'})$. Assume that $\phi_{\pi'}$ contains at least one conjugate self-dual character of sign $(-1)^{n'-1}$. Then there is an integer k, a hermitian space V with $\dim V = n$, and an irreducible tempered representation π of $\mathrm{U}(V)$ so that*

$$\theta_{V,V',\chi_V,\chi_{V'}}(\pi) = \pi' \otimes \det{}^k.$$

Proof. By assumption we have $\phi_{\pi'} = \chi' \oplus \phi'_0$ where $\chi'(z) = (z/\sqrt{z\overline{z}})^m$ and m has the same parity with $n(= n'-1)$. Suppose that $\chi_{V'} = (z/\sqrt{z\overline{z}})^\nu$. Then ν has the same parity with m (and with n). Let χ_k be the character $\chi_k(z) = (z/\overline{z})^k$. Then there is an integer k such that $\chi' \otimes \chi_k = \chi_V$, and hence by Proposition 2.1(2) the L-parameter of $\pi' \otimes \det^k$ contains the character $\chi_{V'}$. Put $\phi = \phi'_0 \otimes \chi_k \otimes \chi_V \chi_{V'}^{-1}$. Then ϕ is an L-parameter of unitary groups in n-variables, and A_ϕ is a subgroup of index at most two of $A_{\phi_{\pi'}}$. Let $\eta = \eta_{\pi'}|_{A_\phi}$. Then the representation π with the Langlands–Vogan parameter (ϕ, η) is an irreducible tempered representation of $\mathrm{U}(V)$ with $\dim V = n$, and $\theta_{V,V',\chi_V,\chi_{V'}}(\pi) = \pi' \otimes \det^k$ by (3.13) and (3.14). $\square$

So far we considered only the theta lifts from $\mathrm{U}(V)$ to $\mathrm{U}(V')$. We can also consider the theta lifts from $\mathrm{U}(V')$ to $\mathrm{U}(V)$. As groups we have identifications $\mathrm{U}(V') = \mathrm{U}(-iV')$ and $\mathrm{U}(V) = \mathrm{U}(iV)$. Moreover $V' \otimes V = (-iV') \otimes (iV)$ as skew-hermitian spaces so the Weil representation does not change. Moreover we picked the Whittaker datum so that under the Langlands–Vogan parametrization a representation of $\mathrm{U}(-iV')$ (resp. $\mathrm{U}(iV)$) and hence $\mathrm{U}(V')$ (resp. $\mathrm{U}(V)$) correspond to the same Langlands–Vogan parameter. Therefore the results described above do not change.

The rest of the subsection is devoted to the translation of Paul's results into the language of Langlands–Vogan parametrization. This material will not be used in the proof of the main theorems. Tempered representations are irreducible parabolic inductions from limit of discrete series representations. By the Langlands–Vogan parameters of parabolic inductions, cf. Proposition 2.1(1), and the induction principle of theta lifts, cf. [Pau98, Theorem 4.6.4], the crux of the matter is the case when ϕ is an L-parameter of limit of discrete series representations. Assume from now on that this is the case. Then ϕ_i's are all conjugate self-dual characters of sign $(-1)^{n-1}$, in the expression (3.10).

In [Pau98, Pau00], theta lifts are viewed as correspondences between genuine representations of covers over unitary groups. Let $\widetilde{\mathrm{U}(V)}$ and $\widetilde{\mathrm{U}(V')}$ be the inverse image of $\mathrm{U}(V)$ and $\mathrm{U}(V')$ in $\mathrm{Mp}(2nn',\mathbb{R})$. Then by [Pau98, Section 1.2], we have

$$\widetilde{\mathrm{U}(V)} = \{(g,z) \in \mathrm{U}(V) \times \mathbb{C}^\times \mid (\det g)^{\nu'} = z^2\}$$

where ν' is an integer with the same parity as n'. Different choices of ν' give isomorphic groups, and do not affect the result of theta lifts on the covers. However the choice of ν' does affects how the identifications of representations of $\mathrm{U}(V)$ and $\widetilde{\mathrm{U}(V)}$ are made. There is a genuine character of $\widetilde{\mathrm{U}(V)}$ given by

$$\det{}^{\frac{\nu'}{2}} : \widetilde{\mathrm{U}(V)} \to \mathbb{C}^\times, \quad (g,z) \mapsto z.$$

Thus there is a natural identification of irreducible representations of $\mathrm{U}(V)$ and those of $\widetilde{\mathrm{U}(V)}$ given by $\pi \leftrightarrow \pi \otimes \det^{-\frac{\nu'}{2}}$. This choice is made so that the relation (3.16) below holds. Similarly we have

$$\widetilde{\mathrm{U}(V')} = \{(g,z) \in \mathrm{U}(V') \times \mathbb{C}^\times \mid (\det g)^{\nu} = z^2\}$$

where ν is an integer with the same parity as n. Thus there is a natural identification of irreducible representations of $\mathrm{U}(V')$ and those of $\widetilde{\mathrm{U}(V')}$ given by $\pi' \leftrightarrow \pi' \otimes \det^{-\frac{\nu}{2}}$. The result of [How89] in the setting of covers of unitary groups is the following. If $\widetilde{\pi}$ is an irreducible genuine representation of $\widetilde{\mathrm{U}(V)}$, then the maximal semisimple quotient the $\widetilde{\mathrm{U}(V)}$-coinvariant $(\omega \widehat{\otimes} \widetilde{\pi})_{\widetilde{\mathrm{U}(V)}}$ is a genuine irreducible representation of $\widetilde{\mathrm{U}(V')}$. We denote it by $\widetilde{\theta_{V,V'}}(\widetilde{\pi})$.

Suppose that the splitting characters χ_V and $\chi_{V'}$ are respectively of the form

$$\chi_V(z) = \left(\frac{z}{\sqrt{z\overline{z}}}\right)^{\nu'}, \quad \chi_{V'}(z) = \left(\frac{z}{\sqrt{z\overline{z}}}\right)^{\nu}.$$

Let π be an irreducible representation of $\mathrm{U}(V)$. Then by [Ato20, Proposition 3.5] we have

$$\theta_{V,V',\chi_V,\chi_{V'}}(\pi) \otimes \det{}^{-\frac{\nu}{2}} = \widetilde{\theta_{V,V'}}(\pi \otimes \det{}^{-\frac{\nu'}{2}}). \tag{3.16}$$

Strictly speaking (3.16) is not explicitly stated in [Ato20], but the calculation in the proof gives it.

Limit of discrete series representations of $\mathrm{U}(V)$ are parameterized by the pair (λ,Ψ) in [Pau98, Pau00]. Here

$$\lambda = (\underbrace{\lambda_1,\cdots,\lambda_1}_{p_1},\cdots\underbrace{\lambda_k,\cdots,\lambda_k}_{p_k};\underbrace{\lambda_1,\cdots,\lambda_1}_{q_1},\cdots\underbrace{\lambda_k,\cdots,\lambda_k}_{q_k})$$

with

- $\lambda_i \in \mathbb{Z} + \frac{n-1}{2}$, and $\lambda_1 > \lambda_2 > \cdots > \lambda_k$;
- $p_i, q_i \geq 0$, $(p_i,q_i) \neq (0,0)$, and $|p_i - q_i| \leq 1$;
- (p,q) is the signature of V, where $p = p_1 + \cdots + p_k$, $q = q_1 + \cdots + q_k$,

and Ψ is a set of positive root (of the diagonal Cartan subalgebra) in $\mathfrak{gl}_n(\mathbb{C}) = \mathrm{Lie}(\mathrm{U}(V)) \otimes \mathbb{C}$ satisfying certain properties. The set Ψ can be specified by a real diagonal matrix in $\mathfrak{gl}_n(\mathbb{C})$. The conditions Ψ needs to satisfy are stated explicitly in [Ich, End of Section 3].

Genuine limit of discrete series representations of $\widetilde{\mathrm{U}(V)}$ are parameterized in the same way, except that λ_i's are required to be in $\mathbb{Z} + \frac{1}{2}$ (resp. in $\mathbb{Z}$) if $n' - n$ is even (resp. odd). If π is a limit of discrete series representation of $\mathrm{U}(V)$ with the parameter (λ, Ψ), then the representation $\pi \otimes \det^{-\frac{\nu'}{2}}$ of $\widetilde{\mathrm{U}(V)}$ has the parameter $(\lambda - \frac{\nu'}{2}, \Psi)$ where $\lambda - \frac{\nu'}{2}$ means subtracting $\frac{\nu'}{2}$ from each entry of λ.

For limit of discrete series representations, the Langlands–Vogan parametrization (ϕ, η) and the parametrization (λ, Ψ) are linked as follows. This follows from [MR19, Théorème 1.1] and has been worked out in concrete terms in [Ich, Section 5.3.1]. Write the L-parameter ϕ as $\chi_1 \oplus \chi_2 \oplus \cdots \oplus \chi_n$, where $\chi_i(z) = (z/\sqrt{z\overline{z}})^{\kappa_i + \nu'}$. We may order these characters so that $\kappa_1 \geq \kappa_2 \geq \cdots \geq \kappa_n$. Define an elementary abelian 2-group

$$A_\phi^+ = (\mathbb{Z}/2\mathbb{Z})e_1 \oplus (\mathbb{Z}/2\mathbb{Z})e_2 \oplus \cdots \oplus (\mathbb{Z}/2\mathbb{Z})e_n,$$

where e_i corresponds to κ_i. Then $\mathrm{Hom}(A_\phi, \langle \pm 1 \rangle)$ is identified with a subgroup of $\mathrm{Hom}(A_\phi^+, \langle \pm 1 \rangle)$ consisting of characters η with the property that $\eta(e_i) = \eta(e_j)$ if $\kappa_i = \kappa_j$. Let η be a character of A_ϕ and put

$$(p_i, q_i) = \begin{cases} (1, 0), & \text{if } \eta(e_i) = (-1)^{i-1}; \\ (0, 1), & \text{if } \eta(e_i) = (-1)^i. \end{cases}$$

If π is a representation of $\mathrm{U}(V)$ with Langlands–Vogan parameter (ϕ, η), then its parameter in terms of (λ, Ψ) is given by

$$\lambda = \frac{1}{2}((\underbrace{\kappa_1, \cdots, \kappa_1}_{p_1}, \cdots, \underbrace{\kappa_n, \cdots, \kappa_n}_{p_n}; \underbrace{\kappa_1, \cdots, \kappa_1}_{q_1}, \cdots, \underbrace{\kappa_n, \cdots, \kappa_n}_{q_n}) + (\underbrace{\nu', \cdots, \nu'}_{n})),$$

and the set Ψ is given by the diagonal matrix

$$\mathrm{diag}(\underbrace{x_1, \cdots, x_1}_{p_1}, \cdots, \underbrace{x_n, \cdots, x_n}_{p_n}; \underbrace{x_1, \cdots, x_1}_{q_1}, \cdots, \underbrace{x_n, \cdots, x_n}_{q_n}),$$

with $x_1 > x_2 > \cdots > x_n$.

Finally we can make the translation of the results of Paul. Consider the equal rank case $n = n'$ first. In this case κ_i's are odd integers. Suppose that we have $\kappa_1 \geq \cdots \geq \kappa_l > 0 > \kappa_{l+1} \geq \cdots \geq \kappa_n$. The parameter (λ, Ψ) of π is as described above. The parameter of the representation $\pi \otimes \det^{-\frac{\nu'}{2}}$ of $\widetilde{\mathrm{U}(V')}$ is then $(\widetilde{\lambda}, \Psi)$ where

$$\frac{1}{2}(\underbrace{\kappa_1, \cdots, \kappa_1}_{p_1}, \cdots, \underbrace{\kappa_l, \cdots, \kappa_l}_{p_l}, \underbrace{\kappa_{l+1}, \cdots, \kappa_{l+1}}_{p_{l+1}}, \cdots, \underbrace{\kappa_n, \cdots, \kappa_n}_{p_n};$$
$$\underbrace{\kappa_1, \cdots, \kappa_1}_{q_1}, \cdots, \underbrace{\kappa_l, \cdots, \kappa_l}_{q_l}, \underbrace{\kappa_{l+1}, \cdots, \kappa_{l+1}}_{q_{l+1}}, \cdots, \underbrace{\kappa_n, \cdots, \kappa_n}_{q_n}).$$

By [Pau98, Theorem 6.1], the theta lift $\widetilde{\theta_{V,V'}}(\pi \otimes \det^{-\frac{\nu'}{2}})$ is a limit of discrete series representation of $\widetilde{\mathrm{U}(V')}$ and the parameter is given by $(\Gamma\widetilde{\lambda}, \Gamma\Psi)$, where

$$\Gamma\widetilde{\lambda} = \frac{1}{2}(\underbrace{\kappa_1, \cdots, \kappa_1}_{p_1}, \cdots, \underbrace{\kappa_l, \cdots, \kappa_l}_{p_l}, \underbrace{\kappa_{l+1}, \cdots, \kappa_{l+1}}_{q_{l+1}}, \cdots, \underbrace{\kappa_n, \cdots, \kappa_n}_{q_n};$$
$$\underbrace{\kappa_1, \cdots, \kappa_1}_{q_1}, \cdots, \underbrace{\kappa_l, \cdots, \kappa_l}_{q_l}, \underbrace{\kappa_{l+1}, \cdots, \kappa_{l+1}}_{p_{l+1}}, \cdots, \underbrace{\kappa_n, \cdots, \kappa_n}_{p_n}),$$

and $\Gamma\Psi$ is given by the diagonal matrix

$$\mathrm{diag}(\underbrace{x_1, \cdots, x_1}_{p_1}, \cdots, \underbrace{x_l, \cdots, x_l}_{p_l}, \underbrace{x_{l+1}, \cdots, x_{l+1}}_{q_{l+1}}, \cdots, \underbrace{x_n, \cdots, x_n}_{q_n};$$
$$\underbrace{x_1, \cdots, x_1}_{q_1}, \cdots, \underbrace{x_l, \cdots, x_l}_{q_l}, \underbrace{x_{l+1}, \cdots, x_{l+1}}_{p_{l+1}}, \cdots, \underbrace{x_n, \cdots, x_n}_{p_n}).$$

The definition of $\Gamma\Psi$ in [Pau98, Section 5.2, (5.2.4b)] is given explicitly in terms of the roots, which is equivalent to our description here. Then by the relation (3.16) we conclude that $\theta_{V,V',\chi_V,\chi'_V}(\pi)$ is $(\Gamma\lambda, \Gamma\Psi)$ where

$$\Gamma\lambda = \frac{1}{2}\,(\,(\underbrace{\kappa_1, \cdots, \kappa_1}_{p_1}, \cdots, \underbrace{\kappa_l, \cdots, \kappa_l}_{p_l}, \underbrace{\kappa_{l+1}, \cdots, \kappa_{l+1}}_{q_{l+1}}, \cdots, \underbrace{\kappa_n, \cdots, \kappa_n}_{q_n};$$
$$\underbrace{\kappa_1, \cdots, \kappa_1}_{q_1}, \cdots, \underbrace{\kappa_l, \cdots, \kappa_l}_{q_l}, \underbrace{\kappa_{l+1}, \cdots, \kappa_{l+1}}_{p_{l+1}}, \cdots, \underbrace{\kappa_n, \cdots, \kappa_n}_{p_n}) + (\underbrace{\nu, \cdots, \nu}_{n})\,)\,.$$

Now consider the representation with the Langland–Vogan parameter $(\theta(\phi), \theta(\eta))$. The L-parameter is of the form $\chi'_1 \oplus \chi'_2 \oplus \cdots \oplus \chi'_n$ where $\chi'_i(z) = (z/\sqrt{z\overline{z}})^{\kappa_i+\nu}$. By the relation (3.12), we have

$$\theta(\eta)(e_i) = \epsilon(\chi_i \otimes \chi_V^{-1}, \overline{\psi^{\mathbb{C}}})\eta(e_i).$$

By definition $\chi_i \otimes \chi_V^{-1}$ is the character $z \mapsto (z/\sqrt{z\overline{z}})^{\kappa_i}$. Then by [GGP12b, Proposition 2.1] and (2.3), we have

$$\epsilon(\chi_i \otimes \chi_V^{-1}, \overline{\psi^{\mathbb{C}}}) = \begin{cases} +1, & \kappa_i > 0, \text{ i.e. } 1 \le i \le l; \\ -1, & \kappa_i < 0, \text{ i.e. } l+1 \le i \le n. \end{cases}$$

By the definition of (p_i, q_i), the effect of multiplying $\epsilon(\chi_i \otimes \chi_V^{-1}, \overline{\psi^{\mathbb{C}}})$ to $\eta(e_i)$ is equivalent to keeping (p_i, q_i) intact when $1 \le i \le l$ and swapping (p_i, q_i) when $l+1 \le i \le n$. Comparing this with $(\Gamma\lambda, \Gamma\Psi)$, we conclude that the Langlands–Vogan parameter of $\theta_{V,V',\chi_V,\chi'_V}(\pi)$ is precisely $(\theta(\phi), \theta(\eta))$, which is what we are after.

Let us move to the almost equal rank case $n' = n+1$. In this case κ_i's are all even integers. We assume u, v are integers with $|u - v| \le 1$ and $\kappa_{l+1} = \cdots = \kappa_{l_{u+v}} = 0$ and hence

$$\kappa_1 \ge \cdots \ge \kappa_l \ge 0 = \cdots = 0 \ge \kappa_{l+u+v+1} \ge \cdots \ge \kappa_n.$$

As in the equal rank case, the parameter $(\widetilde{\lambda}, \Psi)$ of the representation $\pi \otimes \det^{-\frac{\nu'}{2}}$ is

$$\widetilde{\lambda} = \frac{1}{2}(\underbrace{\kappa_1, \cdots, \kappa_1}_{p_1}, \cdots, \underbrace{\kappa_l, \cdots, \kappa_l}_{p_l}, \underbrace{0, \cdots, 0}_{u}, \underbrace{\kappa_{l+u+v+1}, \cdots, \kappa_{l+u+v+1}}_{p_{l+u+v+1}}, \cdots, \underbrace{\kappa_n, \cdots, \kappa_n}_{p_n};$$

$$\underbrace{\kappa_1, \cdots, \kappa_1}_{q_1}, \cdots, \underbrace{\kappa_l, \cdots, \kappa_l}_{q_l}, \underbrace{0, \cdots, 0}_{v}, \underbrace{\kappa_{l+u+v+1}, \cdots, \kappa_{l+u+v+1}}_{q_{l+u+v+1}}, \cdots, \underbrace{\kappa_n, \cdots, \kappa_n}_{q_n}),$$

and the set Ψ is again given by the diagonal matrix

$$\mathrm{diag}(\underbrace{x_1, \cdots, x_1}_{p_1}, \cdots, \underbrace{x_n, \cdots, x_n}_{p_n}; \underbrace{x_1, \cdots, x_1}_{q_1}, \cdots, \underbrace{x_n, \cdots, x_n}_{q_n}),$$

with $x_1 > x_2 > \cdots > x_n$. We distinguish two cases.

First assume that $u = v = 0$, i.e. ϕ does not contain χ_V. By [Pau00, Theorem 3.4(c)], there are two hermitian spaces V' such that $\widetilde{\theta_{V,V'}}(\pi \otimes \det^{-\frac{\nu'}{2}}) \neq 0$, and the parameters are given respectively by

$$\Gamma\widetilde{\lambda} = \frac{1}{2}(\underbrace{\kappa_1, \cdots, \kappa_1}_{p_1}, \cdots, \underbrace{\kappa_l, \cdots, \kappa_l}_{p_l}, 0, \underbrace{\kappa_{l+1}, \cdots, \kappa_{l+1}}_{q_{l+1}}, \cdots, \underbrace{\kappa_n, \cdots, \kappa_n}_{q_n};$$

$$\underbrace{\kappa_1, \cdots, \kappa_1}_{q_1}, \cdots, \underbrace{\kappa_l, \cdots, \kappa_l}_{q_l}, \underbrace{\kappa_{l+1}, \cdots, \kappa_{l+1}}_{p_{l+1}}, \cdots, \underbrace{\kappa_n, \cdots, \kappa_n}_{p_n}),$$

$$\Gamma\Psi = \mathrm{diag}(\underbrace{x_1, \cdots, x_1}_{p_1}, \cdots, \underbrace{x_l, \cdots, x_l}_{p_l}, x', \underbrace{x_{l+1}, \cdots, x_{l+1}}_{q_{l+1}} \cdots, \underbrace{x_n, \cdots, x_n}_{q_n};$$

$$\underbrace{x_1, \cdots, x_1}_{q_1}, \cdots, \underbrace{x_l, \cdots, x_l}_{q_l}, \underbrace{x_{l+1}, \cdots, x_{l+1}}_{p_{l+1}} \underbrace{x_n, \cdots, x_n}_{p_n}),$$

and

$$\Gamma\widetilde{\lambda} = \frac{1}{2}(\underbrace{\kappa_1, \cdots, \kappa_1}_{p_1}, \cdots, \underbrace{\kappa_l, \cdots, \kappa_l}_{p_l}, \underbrace{\kappa_{l+1}, \cdots, \kappa_{l+1}}_{q_{l+1}}, \cdots, \underbrace{\kappa_n, \cdots, \kappa_n}_{q_n};$$

$$\underbrace{\kappa_1, \cdots, \kappa_1}_{q_1}, \cdots, \underbrace{\kappa_l, \cdots, \kappa_l}_{q_l}, 0, \underbrace{\kappa_{l+1}, \cdots, \kappa_{l+1}}_{p_{l+1}}, \cdots, \underbrace{\kappa_n, \cdots, \kappa_n}_{p_n}),$$

$$\Gamma\Psi = \mathrm{diag}(\underbrace{x_1, \cdots, x_1}_{p_1}, \cdots, \underbrace{x_l, \cdots, x_l}_{p_l}, \underbrace{x_{l+1}, \cdots, x_{l+1}}_{q_{l+1}} \cdots, \underbrace{x_n, \cdots, x_n}_{q_n};$$

$$\underbrace{x_1, \cdots, x_1}_{q_1}, \cdots, \underbrace{x_l, \cdots, x_l}_{q_l}, x', \underbrace{x_{l+1}, \cdots, x_{l+1}}_{p_{l+1}} \underbrace{x_n, \cdots, x_n}_{p_n}),$$

Here in both cases, we have $x_1 > x_2 > \cdots x_l > x' > x_{l+1} > \cdots > x_n$. Again the description of $\Gamma\Psi$ is explicitly in terms of roots in [Pau00, Definition 3.1], which is equivalent to our description. The representation $\widetilde{\theta}_{V,V'}(\pi \otimes \det^{-\frac{\nu'}{2}}) \otimes \det^{\frac{\nu}{2}}$ of $\mathrm{U}(V')$ then has parameter

$$\Gamma\lambda = \frac{1}{2}((\underbrace{\kappa_1, \cdots, \kappa_1}_{p_1}, \cdots, \underbrace{\kappa_l, \cdots, \kappa_l}_{p_l}, 0, \underbrace{\kappa_{l+1}, \cdots, \kappa_{l+1}}_{q_{l+1}}, \cdots, \underbrace{\kappa_n, \cdots, \kappa_n}_{q_n};$$

$$\underbrace{\kappa_1, \cdots, \kappa_1}_{q_1}, \cdots, \underbrace{\kappa_l, \cdots, \kappa_l}_{q_l}, \underbrace{\kappa_{l+1}, \cdots, \kappa_{l+1}}_{p_{l+1}}, \cdots, \underbrace{\kappa_n, \cdots, \kappa_n}_{p_n}) + \underbrace{(\nu, \cdots, \nu)}_{n+1}),$$

and

$$\Gamma\lambda = \frac{1}{2}((\underbrace{\kappa_1,\cdots,\kappa_1}_{p_1},\cdots,\underbrace{\kappa_l,\cdots,\kappa_l}_{p_l},\underbrace{\kappa_{l+1},\cdots,\kappa_{l+1}}_{q_{l+1}},\cdots,\underbrace{\kappa_n,\cdots,\kappa_n}_{q_n};$$

$$\underbrace{\kappa_1,\cdots,\kappa_1}_{q_1},\cdots,\underbrace{\kappa_l,\cdots,\kappa_l}_{q_l},0,\underbrace{\kappa_{l+1},\cdots,\kappa_{l+1}}_{p_{l+1}},\cdots,\underbrace{\kappa_n,\cdots,\kappa_n}_{p_n})+\underbrace{(\nu,\cdots,\nu)}_{n+1}),$$

respectively and $\Gamma\Psi = \Gamma\widetilde{\Psi}$. Now consider the representation of $\mathrm{U}(V')$ whose Langlands–Vogan parametrization is given by $(\theta(\phi),\theta(\eta))$. We have $\theta(\phi) = \chi'_1 \oplus \cdots \oplus \chi'_{n'}$, where

$$\chi'_i = \begin{cases} (z/\sqrt{z\overline{z}})^{\kappa_i+\nu}, & 1 \le i \le l \\ (z/\sqrt{z\overline{z}})^{\nu}, & i = l+1 \\ (z/\sqrt{z\overline{z}})^{\kappa_{i-1}+\nu}, & l+2 \le i \le n' \end{cases}$$

Therefore the fact that $\theta(\eta)|_{A_{\theta(\phi)}} = \eta$ is equivalent to that we are swapping p_i and q_i when $i \ge l+1$ in the parameters. The fact that $\theta(\eta)(a_{V'}) = \eta(a_1)\cdots\eta(a_r)(-1)^{\frac{n(n+1)}{2}-q}$ is equivalent to $\theta(\eta)(a_1+\cdots+a_r+a_{V'}) = \operatorname{disc} V'$, cf. Proposition 2.1(4) (recall that V' is of signature (p,q) and $\dim V' = n+1$).

Now assume that $u+v \ge 1$, i.e. ϕ contains χ_V. We make use of [Pau00, Theorem 3.4(b)]. In this case there is a unique V' such that $\widetilde{\theta_{V,V'}}(\pi\otimes\det^{-\frac{\nu'}{2}}) \ne 0$. If $u > v$ or if $u = v$ and $(p_{l+1},q_{l+1}) = (0,1)$, the parameter of $\widetilde{\theta_{V,V'}}(\pi\otimes\det^{-\frac{\nu'}{2}})$ is

$$\Gamma\widetilde{\lambda} = \frac{1}{2}(\underbrace{\kappa_1,\cdots,\kappa_1}_{p_1},\cdots,\underbrace{\kappa_l,\cdots,\kappa_l}_{p_l},\underbrace{0,\cdots,0}_{u},\underbrace{\kappa_{l+u+v+1},\cdots,\kappa_{l+u+v+1}}_{q_{l+u+v+1}},\cdots,\underbrace{\kappa_n,\cdots,\kappa_n}_{q_n};$$

$$\underbrace{\kappa_1,\cdots,\kappa_1}_{q_1},\cdots,\underbrace{\kappa_l,\cdots,\kappa_l}_{q_l},\underbrace{0,\cdots,0}_{v+1},\underbrace{\kappa_{l+u+v+1},\cdots,\kappa_{l+u+v+1}}_{p_{l+u+v+1}},\cdots,\underbrace{\kappa_n,\cdots,\kappa_n}_{p_n}),$$

$$\Gamma\Psi = \mathrm{diag}(\underbrace{x_1,\cdots,x_1}_{p_1},\cdots,\underbrace{x_{l+u+v},\cdots,x_{l+u+v}}_{p_{l+u+v}},\underbrace{x_{l+u+v+1},\cdots,x_{l+u+v+1}}_{q_{l+u+v+1}}\cdots,\underbrace{x_n,\cdots,x_n}_{q_n};$$

$$\underbrace{x_1,\cdots,x_1}_{q_1},\cdots,\underbrace{x_{l+u+v},\cdots,x_{l+u+v}}_{q_{l+u+v}},x',\underbrace{x_{l+u+v+1},\cdots,x_{l+u+v+1}}_{p_{l+u+v+1}}\cdots,\underbrace{x_n,\cdots,x_n}_{p_n}).$$

If $u < v$ or if $u = v$ and $(p_{l+1},q_{l+1}) = (1,0)$, the parameter of $\widetilde{\theta_{V,V'}}(\pi\otimes\det^{-\frac{\nu'}{2}})$ is

$$\Gamma\widetilde{\lambda} = \frac{1}{2}(\underbrace{\kappa_1,\cdots,\kappa_1}_{p_1},\cdots,\underbrace{\kappa_l,\cdots,\kappa_l}_{p_l},\underbrace{0,\cdots,0}_{u+1},\underbrace{\kappa_{l+u+v+1},\cdots,\kappa_{l+u+v+1}}_{q_{l+u+v+1}},\cdots,\underbrace{\kappa_n,\cdots,\kappa_n}_{q_n};$$

$$\underbrace{\kappa_1,\cdots,\kappa_1}_{q_1},\cdots,\underbrace{\kappa_l,\cdots,\kappa_l}_{q_l},\underbrace{0,\cdots,0}_{v},\underbrace{\kappa_{l+u+v+1},\cdots,\kappa_{l+u+v+1}}_{p_{l+u+v+1}},\cdots,\underbrace{\kappa_n,\cdots,\kappa_n}_{p_n}),$$

$$\Gamma\Psi = \mathrm{diag}(\underbrace{x_1,\cdots,x_1}_{p_1},\cdots,\underbrace{x_{l+u+v},\cdots,x_{l+u+v}}_{p_{l+u+v}},x',\underbrace{x_{l+u+v+1},\cdots,x_{l+u+v+1}}_{q_{l+u+v+1}}\cdots,\underbrace{x_n,\cdots,x_n}_{q_n};$$

$$\underbrace{x_1,\cdots,x_1}_{q_1},\cdots,\underbrace{x_{l+u+v},\cdots,x_{l+u+v}}_{q_{l+u+v}},\underbrace{x_{l+u+v+1},\cdots,x_{l+u+v+1}}_{p_{l+u+v+1}},\cdots,\underbrace{x_n,\cdots,x_n}_{p_n}),$$

In both cases we have $x_1 > x_2 > \cdots > x_{l+u+v} > x' > x_{l+u+v+1} > \cdots > x_n$. As in the previous case, translating these results from $\widetilde{\mathrm{U}(V')}$ to $\mathrm{U}(V')$ gives the desired Langlands–Vogan parameter.

3.3. **Fourier–Jacobi models.** We will need an auxiliary model in our proof of Theorem 2.2. Let V' be a skew-hermitian space of dimension n. Let π' and σ' be representations of $\mathrm{U}(V')$. Let μ be a conjugate self-dual character of $\mathbb{C}^\times$ of sign (-1), e.g. we may take $\mu(z) = z(z\overline{z})^{-\frac{1}{2}}$. Let $\omega_{\psi,\mu}$ be the Weil representation of $\mathrm{U}(V')$ (see Subsection 3.1). We denote by $\mathrm{Hom}_{\mathrm{U}(V')}(\pi' \otimes \sigma' \otimes \overline{\omega_{\psi,\mu}}, \mathbb{C})$ the space of continuous $\mathrm{U}(V')$-invariant linear forms on $\pi' \widehat{\otimes} \sigma' \widehat{\otimes} \overline{\omega_{\psi,\mu}}$ and by $m(\pi', \sigma')$ its dimension. If π' and σ' are irreducible, then $m(\pi', \sigma') \leq 1$ by [SZ12].

Define a linear form

$$
\begin{aligned}
&\mathcal{L}_{\pi',\sigma',\psi,\mu} : \pi' \widehat{\otimes} \overline{\pi'} \widehat{\otimes} \sigma' \widehat{\otimes} \overline{\sigma'} \widehat{\otimes} \overline{\omega_{\psi,\mu}} \widehat{\otimes} \omega_{\psi,\mu} \to \mathbb{C}, \\
&(v, v', e, e', \phi, \phi') \mapsto \int_{\mathrm{U}(V')} \langle \pi'(h)v, v' \rangle \langle \sigma'(h)e, e' \rangle \overline{\langle \omega_{\psi,\mu}(h)\phi, \phi \rangle} \mathrm{d}h.
\end{aligned} \tag{3.17}
$$

By Lemma 3.3 the integral is absolutely convergent and the linear form indeed extends continuously to the completed tensor product.

Assume that π' and σ' are both irreducible and tempered. Let $(\phi_{\pi'}, \eta_{\pi'})$ and $(\phi_{\sigma'}, \eta_{\sigma'})$ be the parameters of π' and σ' respectively. Let us write

$$
\phi_{\pi'} = m_1\chi_1 \oplus \cdots \oplus m_k\chi_k, \quad \phi_{\sigma'} = n_1\mu_1 \oplus \cdots \oplus n_l\mu_l,
$$

and

$$
A_{\phi'_\pi} = \bigoplus_{i=1}^{r} (\mathbb{Z}/2\mathbb{Z})a_i, \quad A_{\phi'_\sigma} = \bigoplus_{j=1}^{s} (\mathbb{Z}/2\mathbb{Z})b_j,
$$

following the convention in (2.2).

Proposition 3.6. *Assume that π' and σ' are irreducible and tempered. Then $\mathcal{L}_{\pi',\sigma',\psi,\mu} \neq 0$ if and only if*

$$
\eta_{\pi'}(a_i) = \epsilon(\chi_i \otimes \phi_{\sigma'} \otimes \mu^{-1}, \overline{\psi^{\mathbb{C}}}), \quad \eta_{\sigma'}(b_j) = \epsilon(\phi_{\pi'} \otimes \mu_j \otimes \mu^{-1}, \overline{\psi^{\mathbb{C}}}).
$$

This proposition will be proved along with Theorem 2.2 later. The local GGP conjecture for $\mathrm{U}(n) \times \mathrm{U}(n)$ [GGP12a, Conjecture 17.3] is a similar statement with $\mathcal{L}_{\pi',\sigma'} \neq 0$ replaced by $m(\pi', \sigma') = 1$. Since $\mathcal{L}_{\pi',\sigma'} \neq 0$ clearly implies $m(\pi', \sigma') \neq 0$, Proposition 3.6 proves half of this conjecture. There are two possible ways to prove $m(\pi', \sigma') = 1$ if and only if $\mathcal{L}_{\pi',\sigma'} \neq 0$. One is to mimic the argument of [BP20, Chapter 7]. The other is to use theta lift. If we had $\Theta(\pi) = \theta(\pi)$ for irreducible tempered representations in the almost equal rank situation, we would have an explicit relation between $m(\pi', \sigma')$ and $m(\pi, \sigma)$ when π' (resp. σ') and π (resp. σ) are connected via theta lifts, cf. [Ato20, Proposition 3.11]. In this way we can prove $\mathcal{L}_{\pi',\sigma',\psi,\mu} \neq 0$ if and only if $m(\pi', \sigma') = 1$ along with Theorem 2.2 and Proposition 3.6.

3.4. **Seesaw.** We now explain the relation between the tempered intertwining maps (2.5) and (3.17) via theta lifts. We consider a hermitian space V and a skew-hermitian space V' and the Weil representation of $\mathrm{U}(V) \times \mathrm{U}(V')$. We fix the following choice of characters.

- We fix the additive character ψ in the definition of theta lifts. We let $\mu(z) = z(z\overline{z})^{-\frac{1}{2}}$ be a conjugate self-dual character of $\mathbb{C}^\times$ of sign (-1).
- If $\dim V = \dim V' = n$, then we use
$$\chi_V = \mu^n, \quad \chi_{V'} = \mu^n.$$
- If $\dim V = \dim V' \pm 1 = n \pm 1$, then we use
$$\chi_V = \mu^n, \quad \chi_{V'} = \mu^{n+(-1)^n}.$$
- If $V = L_{(-1)^n}$, $\dim V' = n$, then we use
$$\chi_V = \mu^n, \quad \chi_{V'} = \mu^{(-1)^n}.$$

With these choices of the characters, the Weil representation of $\mathrm{U}(V) \times \mathrm{U}(V')$ is denoted by $\omega_{V,V'}$. Let π' be an irreducible representation of $\mathrm{U}(V')$. We denote by $\theta_{V',V}(\pi')$ the theta lift of π' to $\mathrm{U}(V)$. Recall also that we have the Weil representation $\omega_{\psi,\mu}$ of $\mathrm{U}(V')$.

Let us consider the following setup. Let $W \subset V$ be a relevant pair of hermitian spaces of dimensions n and $n+1$ respectively. Let $L = W^\perp = L_{(-1)^n}$. Let V' be a skew-hermitian space of dimension n. Let π' be an irreducible tempered representation of $\mathrm{U}(V')$ and $\pi = \theta_{V',V}(\pi')$. Let σ be an irreducible tempered representation of $\mathrm{U}(W)$. The next two lemmas exploit the following seesaw diagram.

$$\begin{tikzcd}
\mathrm{U}(V') \times \mathrm{U}(V') \arrow[d, dash] \arrow[dr, dash] & \mathrm{U}(V) \arrow[d, dash] \arrow[dl, dash] \\
\mathrm{U}(V') & \mathrm{U}(W) \times \mathrm{U}(L)
\end{tikzcd}.$$

Lemma 3.7. *Let α' be a matrix coefficient of π', β be a matrix coefficient of σ, and $\phi, \phi' \in \mathcal{S}$. Then the integral*
$$\int_{\mathrm{U}(W)} \int_{\mathrm{U}(V)} |\alpha'(g)\beta(h)\langle \omega(g,h)\phi, \phi' \rangle| \mathrm{d}h \mathrm{d}g$$
is absolutely convergent.

Proof. Let us first record a simple calculus fact from which the lemma will eventually be deduced.

Fact. Let $r_1, \cdots, r_N$ and B be real numbers, and $r_1 + \cdots + r_i > 0$ for any $1 \leq i \leq N$. The integral
$$\int_{0 < x_1 \leq \cdots \leq x_N \leq 1} x_1^{r_1} \cdots x_N^{r_N} \left(1 - \sum_{i=1}^N \log x_i\right)^B \mathrm{d}^\times x_1 \cdots \mathrm{d}^\times x_N$$
is convergent.

Now back to the proof of the lemma. Fix minimal parabolic subgroups $P \subset Q$ of $\mathrm{U}(W)$ and $\mathrm{U}(V)$ respectively. Let s be the split rank of $\mathrm{U}(W)$ and we fix maximal connected split tori

$A_W \simeq (\mathbb{R}_{>0})^s \subset A_V$ of $\mathrm{U}(W)$ and $\mathrm{U}(V)$ contained in P and Q respectively. As the notation before Lemma 3.1, we have

$$A_W^+ = \{(b_1, \cdots, b_s) \in A_W \mid 0 < b_1 \leq \cdots \leq b_s \leq 1\}.$$

Fix a maximal compact subgroup K of $\mathrm{U}(W)$ such that we have the Cartan decomposition $\mathrm{U}(W) = KA_W^+K$. Similarly assume the split rank of $\mathrm{U}(V')$ is r, we have a minimal parabolic subgroup P' of $\mathrm{U}(V')$, a maximal connected split torus $A' \simeq (\mathbb{R}_{>0})^r$ of $\mathrm{U}(V')$ contained in P',

$$A'^+ = \{(a_1, \cdots, a_r) \in A' \mid 0 < a_1 \leq \cdots \leq a_r \leq 1\}.$$

Fix a maximal compact subgroup K' of $\mathrm{U}(V')$ such that we have the Cartan decomposition $\mathrm{U}(V') = K'A'^+K'$.

Using the integration formula (3.6), the estimates (3.7), (3.8) and Lemma 3.1, we are reduced to prove that for any $B > 0$, the integral

$$\begin{aligned}\iint_{\substack{0<a_1\leq\cdots\leq a_r\leq 1\\ 0<b_1\leq\cdots\leq b_s\leq 1}} \prod_{i=1}^r a_i^{-(n+1-2i)}(1-\sum_{i=1}^r \log a_i)^B \prod_{j=1}^s b_j^{-(n+1-2j)}(1-\sum_{j=1}^s \log b_j)^B \\ \prod_{i=1}^r a_i^{n+1} \cdot \prod_{j=1}^s b_j^{n-2r} \cdot \prod_{i=1}^r\prod_{j=1}^s \Upsilon(a_ib_j^{-1})^2 \, \mathrm{d}^\times a_1 \cdots \mathrm{d}^\times a_r \mathrm{d}^\times b_1 \cdots \mathrm{d}^\times b_s\end{aligned}$$

is convergent. This simplifies to

$$\begin{aligned}\iint_{\substack{0<a_1\leq\cdots\leq a_r\leq 1\\ 0<b_1\leq\cdots\leq b_s\leq 1}} \prod_{i=1}^r a_i^{2i}(1-\sum_{i=1}^r \log a_i)^B \prod_{j=1}^s b_j^{2j-1-2r}(1-\sum_{j=1}^s \log b_j)^B \\ \prod_{i=1}^r\prod_{j=1}^s \Upsilon(a_ib_j^{-1})^2 \mathrm{d}^\times a_1 \cdots \mathrm{d}^\times a_r \mathrm{d}^\times b_1 \cdots \mathrm{d}^\times b_s.\end{aligned} \tag{3.18}$$

Let $(p_1, \cdots, p_{s+1})$ be a $(s+1)$-tuple of non-negative integers with $p_1 + \cdots + p_{s+1} = r$. Define $S_{p_1,\cdots,p_{s+1}}$ to be the region in $A'^+ \times A_W^+$ defined by the inequalities

$$\begin{aligned}&0 < a_1 \leq \cdots \leq a_{p_1} \leq b_1 \leq a_{p_1+1} \leq \cdots \leq a_{p_1+p_2} \leq b_2 \\ &\leq \cdots \leq a_{p_1+\cdots+p_s} \leq b_s \leq a_{p_1+\cdots+p_s+1} \leq \cdots \leq a_{p_1+\cdots+p_{s+1}} \leq 1.\end{aligned}$$

The domain of integration is a finite union of the regions of the form $S_{p_1,\cdots,p_{s+1}}$. It is thus enough to show that the integral (3.18), with the domain of integration replaced by $S_{p_1,\cdots,p_{s+1}}$, is convergent. Elementary computation gives that when a_i's and b_j's are in this region, we have

$$\begin{aligned}\prod_{\substack{i=1,\cdots,r\\ j=1,\cdots,s}} \Upsilon\left(a_ib_j^{-1}\right) =&(a_{p_1+1}\cdots a_{p_1+p_2})^{-1} \cdot (a_{p_1+p_2+1}\cdots a_{p_1+p_2+p_3})^{-2} \\ &\cdots (a_{p_1+\cdots+p_s+1}\cdots a_r)^{-s} \cdot b_1^{r-p_1} \cdot b_2^{r-p_1-p_2} \cdots b_s^{r-p_1-\cdots-p_s}.\end{aligned}$$

We now check that the integration over $S_{p_1,\cdots,p_{s+1}}$ satisfies the condition of the fact that we have recorded at the beginning of the proof. The sum of the exponents of the terms up to $a_{p_1+\cdots+p_l+t}$

$(0 \le t \le p_{l+1})$ is

$$2+4+\cdots+2(p_1+\cdots+p_l+t)-2p_2-4p_3-\cdots-2(l-1)p_l-2lt$$
$$+l^2-2rl+2(r-p_1)+\cdots+2(r-p_1-\cdots-p_l).$$

Here the first line is the sum of the exponents of a_i and the second is the sum of the exponents of b_j. It simplifies to

$$(p_1+\cdots+p_l+t-l)^2+(p_1+\cdots+p_l+t)>0.$$

Similarly the sum of the exponents of the terms up to b_l $(l \ge 1)$ is

$$2+4+\cdots+2(p_1+\cdots+p_l)-2p_2-4p_3-\cdots-2(l-1)p_l$$
$$+l^2-2rl+2(r-p_1)+\cdots+2(r-p_1-\cdots-p_l).$$

Here again the first line is the sum of the exponents of a_i and the second is the sum of the exponents of b_j. It simplifies to

$$(p_1+\cdots+p_l-l)^2+(p_1+\cdots+p_l)>0.$$

The convergence of the integral (3.18) is thus proved. $\square$

Lemma 3.8. *We have $\mathcal{L}_{\pi,\sigma} \neq 0$ if and only if $\sigma' = \theta_{W,V'}(\overline{\sigma}) \neq 0$ and $\mathcal{L}_{\pi',\overline{\sigma'},\psi^{(-1)^n},\mu^{(-1)^n}} \neq 0$.*

Proof. Consider the Weil representation $\omega_{V,V'}$ of $\mathrm{U}(V) \times \mathrm{U}(V')$. By [Pau98, Lemma 2.8] we have

$$\omega_{V,V'}|_{\mathrm{U}(W)\times\mathrm{U}(V')} \simeq \omega_{W,V'} \widehat{\otimes} \omega_{L,V'}.$$

Let $\phi, \phi' \in \omega_{W,V'}$ and $\varphi, \varphi' \in \omega_{L,V'}$. Then

$$\langle \omega_{V,V'}(g,h)\phi \otimes \varphi, \phi' \otimes \varphi' \rangle = \langle \omega_{W,V'}(g,h)\phi, \phi' \rangle \langle \omega_{L,V'}(h)\varphi, \varphi' \rangle$$

if $g \in \mathrm{U}(W)$ and $h \in \mathrm{U}(V')$.

Let $\alpha'(h)$ be a matrix coefficient of π'. Then by Proposition 3.4 the function

$$g \mapsto \int_{\mathrm{U}(V')} \overline{\alpha'(h)} \langle \omega_{V,V'}(g,h)\phi \otimes \varphi, \phi' \otimes \varphi' \rangle \mathrm{d}h, \quad g \in \mathrm{U}(V)$$

is a matrix coefficient of π, and when α' varies over all matrix coefficients of π' and ϕ, ϕ', φ, φ' vary over all elements in $\omega_{W,V'}$ and $\omega_{L,V'}$ respectively, the matrix coefficients of this form generate a dense subspace of all matrix coefficients of π. It follows the continuity of $\mathcal{L}_{\pi,\sigma}$ that if $\mathcal{L}_{\pi,\sigma} \neq 0$, then we can choose α', ϕ, ϕ', φ, φ', and a matrix coefficient β of σ such that

$$\int_{\mathrm{U}(W)} \beta(g) \left(\int_{\mathrm{U}(V')} \overline{\alpha'(h)} \langle \omega_{W,V'}(g,h)\phi, \phi' \rangle \langle \omega_{L,V'}(h)\varphi, \varphi' \rangle \mathrm{d}h \right) \mathrm{d}g \neq 0.$$

As π' and σ are both tempered, the double is absolutely convergent by Lemma 3.7.

We can then switch the order of integration and integrate over $\mathrm{U}(W)$ first. Then by Proposition 3.4 again the integral

$$h \mapsto \int_{\mathrm{U}(W)} \beta(g) \langle \omega_{W,V'}(g,h)\phi, \phi' \rangle \mathrm{d}g$$

defines a nonzero matrix coefficient of $\sigma' = \theta(\overline{\sigma})$. In particular $\sigma' \neq 0$. Denote this matrix coefficient by $\gamma(h)$. We conclude that

$$\int_{U(V')} \overline{\alpha'}(h)\gamma(h)\langle \omega_{L,V'}(h)\varphi, \varphi'\rangle \mathrm{d}h \neq 0.$$

Note that as a representation of $\mathrm{U}(V')$, $\omega_{L,V'}$ is isomorphic to $\omega_{\psi,\mu}$ if n is even and to $\overline{\omega_{\psi,\mu}} = \omega_{\psi^{-1},\mu^{-1}}$ if n is odd (note that the splitting characters are different in these two cases). Therefore $\mathcal{L}_{\pi',\overline{\sigma'},\psi^{(-1)^n},\mu^{(-1)^n}} \neq 0$.

Reversing the argument gives the converse implication. ☐

Now let V' be a skew-hermitian space of dimension n, and let $V_0 \subset W_0$ be a relevant pair of hermitian spaces of dimension $n-1$ and n respectively. Let π' and σ' be irreducible tempered representations of $\mathrm{U}(V')$. Assume that there is an irreducible tempered representation π_0 of $\mathrm{U}(V_0)$ so that $\pi' = \theta_{V_0,V'}(\pi_0)$. Let $\sigma_0 = \theta_{V',W_0}(\overline{\sigma'})$.

Lemma 3.9. *We have $\mathcal{L}_{\pi',\sigma',\psi^{(-1)^n},\mu^{(-1)^n}} \neq 0$ if and only if $\sigma_0 \neq 0$ and $\mathcal{L}_{\overline{\sigma_0},\pi_0} \neq 0$.*

We omit the proof of this lemma, which exploits the following seesaw diagram and is exactly the same as that of Lemma 3.8.

$$\begin{tikzcd}
\mathrm{U}(V') \times \mathrm{U}(V') \arrow[d, dash] \arrow[dr, dash] & \mathrm{U}(W_0) \arrow[d, dash] \arrow[dl, dash] \\
\mathrm{U}(V') & \mathrm{U}(V_0) \times \mathrm{U}(V_0^{\perp})
\end{tikzcd} .$$

4. Proof of the main theorem

4.1. **Setup.** We prove Theorem 2.2 in the case $t = 0$ and Proposition 3.6 in this section. Let us recall the setup.

<u>$\mathrm{U}(n+1) \times \mathrm{U}(n)$</u>.

- We have a relavant pair of hermitian spaces $W \subset V$ of dimensions n and $n+1$ respectively.
- Let π and σ be irreducible tempered representations of $\mathrm{U}(V)$ and $\mathrm{U}(W)$ respectively. The parameters of π and σ are (ϕ_π, η_π) and $(\phi_\sigma, \eta_\sigma)$ respectively.

<u>$\mathrm{U}(n) \times \mathrm{U}(n)$</u>.

- We have a skew-hermitian space V' of dimension n.
- We fix the character $\mu(z) = z(z\overline{z})^{-\frac{1}{2}}$ of $\mathbb{C}^\times$. We have a Weil representation $\omega_{\psi,\mu}$ of $\mathrm{U}(V')$.
- Let π' and σ' be irreducible tempered representations of $\mathrm{U}(V')$. The parameters of π' and σ' are $(\phi_{\pi'}, \eta_{\pi'})$ and $(\phi_{\sigma'}, \eta_{\sigma'})$ respectively.

Assume that ϕ_π contains a conjugate self-dual characters of sign $(-1)^n$ and ϕ_σ contains b conjugate self-dual characters of sign $(-1)^{n-1}$ (both counting multiplicity). We prove by induction on $a + b = N$. Note that a (resp. b) has the same parity as $n+1$ (resp. n), and thus N is odd.

The base case is $a+b=1$. In this case π and σ are both irreducible (tempered) principal series representations. More precisely assume that $W \subset V$ are split relevant hermitian spaces, i.e. the signature of them are either $(p,p),(p+1,p)$ or $(p,p-1),(p,p)$ respectively. Let $B_2 = T_2 U_2$ be a Borel subgroup of $\mathrm{U}(V)$ with U_2 the unipotent radical and T_2 a maximal torus. Let $B_1 = T_1 U_1$ be a Borel subgroup of $\mathrm{U}(W)$ with U_1 the unipotent radical and T_1 a maximal torus. We fix the following data.

- Assume that $\dim V = 2p+1$. Let $\chi_1, \cdots, \chi_p$ be characters of $\mathbb{C}^\times$, χ_0 be a character of $\mathbb{C}^1$ and $\underline{\chi}$ be a character of $T_2 \simeq (\mathbb{C}^\times)^p \times \mathbb{C}^1$ given by $\chi_1 \otimes \cdots \otimes \chi_p \otimes \chi_0$. Let $\mu_1, \cdots, \mu_p$ be characters of $\mathbb{C}^\times$ and $\underline{\mu}$ be a character of $T_1 \simeq (\mathbb{C}^\times)^p$ given by $\mu_1 \otimes \cdots \otimes \mu_p$.
- Assume that $\dim V = 2p$. Let $\chi_1, \cdots, \chi_p$ be characters of $\mathbb{C}^\times$ and $\underline{\chi}$ be a character of $T_2 \simeq (\mathbb{C}^\times)^p$ given by $\chi_1 \otimes \cdots \otimes \chi_p$. Let $\mu_1, \cdots, \mu_{p-1}$ be characters of $\mathbb{C}^\times$, μ_0 be a character of $\mathbb{C}^1$, and $\underline{\mu}$ be a character of T_1 given by $\mu_1 \otimes \cdots \otimes \mu_{p-1} \otimes \mu_0$.

In both cases let $\pi = \mathrm{Ind}_{B_2}^{\mathrm{U}(V)} \underline{\chi}$ and $\sigma = \mathrm{Ind}_{B_1}^{\mathrm{U}(W)} \underline{\mu}$ be principal series representations of $\mathrm{U}(V)$ and $\mathrm{U}(W)$ respectively.

Lemma 4.1. *We have $\mathcal{L}_{\pi,\sigma} \neq 0$. In particular, if π and σ are both irreducible, then Theorem 2.2 holds for (π,σ).*

Proof. We prove $\mathcal{L}_{\pi,\sigma} \neq 0$ by induction on $\dim V = n+1$. The base case $n=0$, $\dim V = 1$ is trivial. Assume now that $\dim V \geq 2$. Let $V_0 \subset W$ be relevant hermitian spaces, $\dim V_0 = n-1$, then $\mathrm{U}(V_0)$ is quasi-split. Let π_0 be the principal series representation of $\mathrm{U}(V_0)$ defined as follows. By conjugating the Borel subgroup B_2 suitably, we may assume that $B_0 = B_2 \cap \mathrm{U}(V_0)$ is a Borel subgroup of $\mathrm{U}(V_0)$ and $T_0 = T \cap \mathrm{U}(V_0)$ a maximal torus. Let $\underline{\chi_0}$ be the character of T_0 given by $\chi_1 \otimes \cdots \otimes \chi_{p-1} \otimes \chi_0$ if $\dim V = 2p+1$ and by $\chi_1 \otimes \cdots \otimes \chi_{p-1}$ if $\dim V = 2p$. Let $\pi_0 = \mathrm{Ind}_{B_0}^{\mathrm{U}(V_0)} \underline{\chi_0}$ be a principal series representation of $\mathrm{U}(V_0)$. Then $\pi = \chi_p \rtimes \pi_0$. By the induction hypothesis we have $\mathcal{L}_{\sigma,\pi_0} \neq 0$. Then Proposition 2.4 implies that $\mathcal{L}_{\pi,\sigma} \neq 0$.

Assume that π and σ are irreducible, i.e. none of χ_i's, $i \neq 0$, are conjugate self-dual of sign $(-1)^{\dim V - 1}$ and none of μ_j's, $j \neq 0$ are conjugate self-dual of sign $(-1)^{\dim W - 1}$. If $\dim V = 2p+1$, then $A_{\phi_\pi} = \mathbb{Z}/2\mathbb{Z}$ and $A_{\phi_\sigma} = \{1\}$. If $\dim V = 2p$, then $A_{\phi_\pi} = \{1\}$ and $A_{\phi_\sigma} = \mathbb{Z}/2\mathbb{Z}$. In both cases η_π and η_σ are trivial characters. We thus have $m(\pi,\sigma) = 1$ and η_π and η_σ are both trivial characters. This follows from our choice of the Whittaker datum in the Langlands–Vogan parametrization, as we picked the Whittaker data for $\mathrm{U}(\frac{n+1}{2}, \frac{n-1}{2})$ if n is odd. Therefore Theorem 2.2 holds. $\square$

From now on let us assume that $a+b = N \geq 3$ and Theorem 2.2 has been proved for all (π,σ) with $a+b < N$. This induction hypothesis is in effect throughout this section.

4.2. **The case $a \geq 2$.** We first look at the Fourier-Jacobi model. Assume that $\phi_{\pi'}$ (resp. $\phi_{\sigma'}$) contains a' (resp. b') conjugate self-dual characters of sign $(-1)^{n-1}$ (again both counting multiplicity).

Lemma 4.2. *Assume that $a'+b' < N$ and $a' \geq 1$. Then Proposition 3.6 holds for (π',σ').*

Proof. We will be dealing with various theta lifts. We fix the choice of characters to split the metaplectic cover as in Subsection 3.4.

Note first that $\mathcal{L}_{\pi',\sigma',\psi,\mu}$ and the local root numbers in Proposition 3.6 will not change if π' and σ' are replaced by $\pi' \otimes \det^k$ and $\sigma' \otimes \det^{-k}$ respectively for all integers k. It follows that Proposition 3.6 holds for (π', σ') if and only if it holds for $(\pi' \otimes \det^k, \sigma' \otimes \det^{-k})$. Because of the assumption $a' \geq 1$, by Lemma 3.5, we may assume that there is a hermitian space V_0 of dimension $n-1$ and an irreducible tempered representation π_0 of $\mathrm{U}(V_0)$ so that $\pi' = \theta_{V_0,V'}(\pi_0)$. The cases n being even or odd differ slightly in notation. We will treat the case n being even and leave the case n being odd to the interested reader.

Assume that n is even. Put $W_0 = V_0 \oplus L_{-1}$ and $\sigma_0 = \theta_{V',W_0}(\overline{\sigma'})$. By Lemma 3.9 we have $\mathcal{L}_{\pi',\sigma',\psi,\mu} \neq 0$ if and only if $\sigma_0 \neq 0$ and $\mathcal{L}_{\overline{\sigma_0},\pi_0} \neq 0$.

Let us now compute the parameters. Since $\pi' = \theta_{V_0,V'}(\pi_0)$, by (3.13) and our choice of the splitting characters in Subsection 3.4, the parameter $\phi_{\pi'}$ contains μ^{n+1}. We write

$$\phi_{\pi'} = m_1\chi_1 \oplus \cdots \oplus m_k\chi_k, \quad \phi_{\sigma'} = n_1\mu_1 \oplus \cdots \oplus n_l\mu_l,$$

where

- $\chi_i = \mu^{\alpha_i}$, $i = 1, \cdots, r$, with α_i distinct odd integers, $\alpha_r = n+1$, and χ_i, $i > r$, are not conjugate self-dual of sign $(-1)^{n-1}$;
- $\mu_j = \mu^{\beta_j}$, $i = 1, \cdots, s$, with β_j distinct odd integers, and μ_j, $j > s$, are not conjugate self-dual of sign $(-1)^{n-1}$;

Then

$$A_{\phi_{\pi'}} = \bigoplus_{i=1}^{r} (\mathbb{Z}/2\mathbb{Z})a_i, \quad A_{\phi_{\sigma'}} = \bigoplus_{j=1}^{s} (\mathbb{Z}/2\mathbb{Z})b_j.$$

By (3.13), we have

$$\phi_{\pi_0} = m_1\mu^{\alpha_1 - 1} \oplus \cdots \oplus m_{r-1}\mu^{\alpha_{r-1}-1} \oplus (m_r - 1)\mu^{\alpha_r - 1} \oplus *$$

and

$$\phi_{\sigma_0} = n_1\mu^{-\beta_1} \oplus \cdots \oplus n_s\mu^{-\beta_s} \oplus *.$$

where $*$ consists of characters that are not conjugate self-dual of sign $(-1)^n$ in ϕ_{π_0} and of sign $(-1)^{n-1}$ in ϕ_{σ_0} respectively.

Assume $\mathcal{L}_{\pi',\sigma',\psi,\mu} \neq 0$. Then $\sigma_0 \neq 0$ and $\mathcal{L}_{\overline{\sigma_0},\pi_0} \neq 0$ and hence $\mathcal{L}_{\sigma_0,\overline{\pi_0}} \neq 0$. We distinguish two cases: $m_r > 1$ and $m_r = 1$.

Assume first that $m_r > 1$. Then $A_{\phi_{\pi'}} = A_{\phi_{\pi_0}}$ and $A_{\phi_{\sigma'}} = A_{\phi_{\sigma_0}}$. By the induction hypothesis we have

$$\eta_{\pi_0}(a_i) = \eta_{\overline{\pi_0}}(a_i) = \epsilon(\phi_{\sigma_0} \otimes \mu^{-\alpha_i+1}, \psi^{\mathbb{C}}), \quad \eta_{\sigma_0}(b_j) = \epsilon(\phi_{\overline{\pi_0}} \otimes \mu^{-\beta_j}, \psi^{\mathbb{C}}).$$

Here the first equality $\eta_{\pi_0} = \eta_{\overline{\pi_0}}$ follows from Proposition 2.1(3). Therefore again by (3.15) we have

$$\eta_{\pi'}(a_i) = \eta_{\pi_0}(a_i) = \epsilon(\phi_{\sigma_0} \otimes \mu^{-\alpha_i+1}, \psi^{\mathbb{C}}) = \epsilon(\phi_{\sigma'} \otimes \mu^{\alpha_i - 1}, \overline{\psi^{\mathbb{C}}}), \quad i = 1, \cdots, r$$

and

$$\eta_{\sigma'}(b_j) = -\eta_{\sigma_0}(b_j)\epsilon(\mu^{-\beta_j-n}, \overline{\psi^{\mathbb{C}}}) = -\epsilon(\phi_{\pi_0} \otimes \mu^{\beta_j}, \overline{\psi^{\mathbb{C}}})\epsilon(\mu^{-\beta_j-n}, \overline{\psi^{\mathbb{C}}}) = \epsilon(\phi_{\pi'} \otimes \mu^{\beta_j-1}, \overline{\psi^{\mathbb{C}}}).$$

Here the minus sign is a result of $\eta_{\overline{\sigma_0}}(b_j) = -\eta_{\sigma_0}(b_j)$ which again follows from Proposition 2.1(3). The last equality follows from the fact that $\phi_{\pi'} = (\phi_{\pi_0} \otimes \mu) \oplus \mu^{n+1}$ by (3.13) and the identity

$$\epsilon(\mu^{\beta_j+n}, \overline{\psi^{\mathbb{C}}})\epsilon(\mu^{-\beta_j-n}, \overline{\psi^{\mathbb{C}}}) = \mu^{\beta_j+n}(-1) = -1,$$

where the first equality follows from (2.3) and the second follows from the fact that β_j is odd and n is even. This proves that the parameters of π' and σ' are as described in Proposition 3.6 when $m_r > 1$.

Now assume that $m_r = 1$. Then

$$A_{\phi_{\pi_0}} = \bigoplus_{i=1}^{r-1} (\mathbb{Z}/2\mathbb{Z})a_i, \quad A_{\phi_{\sigma_0}} = A_{\phi_{\sigma'}} = \bigoplus_{j=1}^{s} (\mathbb{Z}/2\mathbb{Z})b_j.$$

The values $\eta_{\pi'}(a_i)$, $i = 1, \cdots, r-1$ and $\eta_{\sigma'}(b_j)$, $j = 1, \cdots, s$ can be computed in the same way as in the case $m_r > 1$. It remains to compute $\eta_{\pi'}(a_r)$. Let us temporarily assume that the signature of W_0 is (p, q) and the signature of $-iV'$ is (p', q'). Then $\operatorname{disc}(-iV') = (-1)^{\frac{n(n-1)}{2}-q'}$ and $\operatorname{disc} W_0 = (-1)^{\frac{n(n-1)}{2}-q}$. By (3.14) and Proposition 2.1(4), we have

$$\epsilon(\phi_{\overline{\sigma'}} \otimes \mu^{-n}, \psi^{\mathbb{C}}) = \operatorname{disc} W_0 \operatorname{disc}(-iV') = (-1)^{-q'-q}.$$

We also have

$$\eta_{\pi'}(a_r) = \eta_{\pi'}(a_1 + \cdots + a_{r-1} + a_r)\eta_{\pi_0}(a_1 + \cdots + a_{r-1}) = (-1)^{\frac{n}{2}-q'}(-1)^{(\frac{n}{2}-1)-(q-1)} = (-1)^{-q'-q}.$$

It follows that

$$\eta_{\pi'}(a_r) = \epsilon(\phi_{\overline{\sigma'}} \otimes \mu^{-n}, \psi^{\mathbb{C}}) = \epsilon(\phi_{\sigma'} \otimes \mu^{n}, \overline{\psi^{\mathbb{C}}}).$$

This proves the parameters of π' and σ' are as described in Proposition 3.6 if $m_r = 1$.

The converse implication can be proved by reversing this argument. $\square$

Lemma 4.3. *Theorem 2.2 holds for (π, σ) if $a \geq 2$.*

Proof. We again note first that Theorem 2.2 holds for (π, σ) if and only if it holds for $(\pi \otimes \det^k, \sigma \otimes \det^{-k})$ for any k. Thus we may assume that there is a skew-hermitian space V' of dimension n and an irreducible tempered representation π' such that $\pi = \theta_{V',V}(\pi')$. By Proposition 2.3, $m(\pi, \sigma) = 1$ is equivalent to $\mathcal{L}_{\pi,\sigma} \neq 0$. Thus we are reduced to prove Theorem 2.2 with the condition $m(\pi, \sigma) = 1$ replaced $\mathcal{L}_{\pi,\sigma} \neq 0$. Let $\sigma' = \theta_{W,V'}(\overline{\sigma})$. By Lemma 3.8, $\mathcal{L}_{\pi,\sigma} \neq 0$ if and only if $\sigma' = \theta_{W,V'}(\overline{\sigma}) \neq 0$ and $\mathcal{L}_{\pi',\overline{\sigma'},\psi^{(-1)^n},\mu^{(-1)^n}} \neq 0$. Assume that $\phi_{\pi'}$ and $\phi_{\sigma'}$ contains a' and b' conjugate self-dual characters of sign $(-1)^{n-1}$ respectively. Then by (3.10) and (3.13), we have $a' = a - 1 \geq 1$ and $b' = b$, $a' + b' = N - 1 < N$. Therefore Proposition 3.6 holds for (π', σ') by Lemma 4.2. The rest of the argument is to deduce Theorem 2.2 for (π, σ) from Proposition 3.6 for (π', σ'). This does not differ much from Lemma 4.2 and we omit the details. $\square$

4.3. The case $b \geq 2$.

Lemma 4.4. *Theorem 2.2 holds for (π, σ) if $b \geq 2$.*

Proof. Let χ be any character of $\mathbb{C}^\times$ that is *not* conjugate self-dual of sign $(-1)^{n-1}$. Let $W_1 = W \oplus^\perp \langle z_1, z_{-1} \rangle$ where $z_{\pm 1}$ are isotropic vectors with $h_{W_1}(z_1, z_{-1}) = 1$. Let $\sigma_1 = \chi \rtimes \sigma$ be the induced representation of $\mathrm{U}(W_1)$. Then σ_1 is irreducible and tempered, and we have $\phi_{\sigma_1} = \phi_\sigma \oplus \chi \oplus \chi^{c,-1}$, $A_{\phi_{\sigma_1}} = A_{\phi_\sigma}$ and $\eta_{\sigma_1} = \eta_\sigma$. Note that ϕ_{σ_1} contains b conjugate self-dual characters of sign $(-1)^{n+1}$, ϕ_π contains a conjugate self-dual characters of sign $(-1)^n$.

Assume that $m(\pi, \sigma) = 1$. Then $m(\sigma_1, \pi) = 1$ by Proposition 2.4. Now apply Lemma 4.3 to (σ_1, π), we conclude that the parameters of σ_1 and π are as specified in Theorem 2.2. It then follows that the parameters of π and σ are as specified in Theorem 2.2.

Let us now assume the parameters of π and σ are given as Theorem 2.2. Then by our construction the parameters of σ_1 and π are given as in Theorem 2.2. It follows from Lemma 4.3 that $m(\sigma_1, \pi) = 1$. Then $m(\pi, \sigma) = 1$ by Proposition 2.3 and 2.4. $\square$

4.4. **End of the proof.** We now prove Theorem 2.2 and Proposition 3.6. As $a + b \geq 3$, we have either $a \geq 2$ or $b \geq 2$. So we conclude by either Lemma 4.3 or 4.4. This completes the induction and proves Theorem 2.2. Proposition 3.6 then follows from Theorem 2.2 by a similar argument of Lemma 4.2.

References

[Ato20] H. Atobe, *On the non-vanishing of theta liftings of tempered representations of* $\mathrm{U}(p,q)$, Adv. Math. **363** (2020), 106984, 76, DOI 10.1016/j.aim.2020.106984. MR4054051 ↑4, 5, 8, 9, 16, 21

[BP20] R. Beuzart-Plessis, *A local trace formula for the Gan-Gross-Prasad conjecture for unitary groups: the Archimedean case*, Astérisque **418** (2020), viii + 299, DOI 10.24033/ast (English, with English and French summaries). MR4146145 ↑2, 6, 7, 9, 11, 21

[GGP12a] W. T. Gan, B. H. Gross, and D. Prasad, *Symplectic local root numbers, central critical L values, and restriction problems in the representation theory of classical groups*, Astérisque **346** (2012), 1–109 (English, with English and French summaries). Sur les conjectures de Gross et Prasad. I. MR3202556 ↑1, 3, 4, 5, 6, 21

[GGP12b] ———, *Restrictions of representations of classical groups: examples*, Astérisque **346** (2012), 111–170 (English, with English and French summaries). Sur les conjectures de Gross et Prasad. I. MR3202557 ↑18

[GI14] W. T. Gan and A. Ichino, *Formal degrees and local theta correspondence*, Invent. Math. **195** (2014), no. 3, 509–672, DOI 10.1007/s00222-013-0460-5. MR3166215 ↑9, 10

[GI16] ———, *The Gross-Prasad conjecture and local theta correspondence*, Invent. Math. **206** (2016), no. 3, 705–799, DOI 10.1007/s00222-016-0662-8. MR3573972 ↑2, 9, 10

[GQT14] W. T. Gan, Y. Qiu, and S. Takeda, *The regularized Siegel-Weil formula (the second term identity) and the Rallis inner product formula*, Invent. Math. **198** (2014), no. 3, 739–831, DOI 10.1007/s00222-014-0509-0. MR3279536 ↑13

[He03] H. He, *Unitary representations and theta correspondence for type I classical groups*, J. Funct. Anal. **199** (2003), no. 1, 92–121, DOI 10.1016/S0022-1236(02)00170-2. MR1966824 ↑13

[He17] ———, *On the Gan-Gross-Prasad conjecture for $U(p, q)$*, Invent. Math. **209** (2017), no. 3, 837–884, DOI 10.1007/s00222-017-0720-x. MR3681395 ↑2

[How89] R. Howe, *Transcending classical invariant theory*, J. Amer. Math. Soc. **2** (1989), no. 3, 535–552, DOI 10.2307/1990942. MR985172 ↑9, 16

[Ich] A. Ichino, *Theta lifting for tempered representations of real unitary groups*, available at `https://arxiv.org/abs/2008.06174`. ↑8, 17

[II10] A. Ichino and T. Ikeda, *On the periods of automorphic forms on special orthogonal groups and the Gross-Prasad conjecture*, Geom. Funct. Anal. **19** (2010), no. 5, 1378–1425, DOI 10.1007/s00039-009-0040-4. MR2585578 ↑2, 6, 11

[JSZ10] D. Jiang, B. Sun, and C.-B. Zhu, *Uniqueness of Bessel models: the Archimedean case*, Geom. Funct. Anal. **20** (2010), no. 3, 690–709, DOI 10.1007/s00039-010-0077-4. MR2720228 ↑6

[LZ98] S. T. Lee and C.-B. Zhu, *Degenerate principal series and local theta correspondence*, Trans. Amer. Math. Soc. **350** (1998), no. 12, 5017–5046, DOI 10.1090/S0002-9947-98-02036-4. MR1443883 ↑13

[MR19] C. Mœglin and D. Renard, *Sur les paquets d'Arthur des groupes unitaires et quelques conséquences pour les groupes classiques*, Pacific J. Math. **299** (2019), no. 1, 53–88, DOI 10.2140/pjm.2019.299.53 (French, with English and French summaries). MR3947270 ↑17

[Pau98] A. Paul, *Howe correspondence for real unitary groups*, J. Funct. Anal. **159** (1998), no. 2, 384–431, DOI 10.1006/jfan.1998.3330. MR1658091 ↑4, 8, 14, 15, 16, 18, 24

[Pau00] ———, *Howe correspondence for real unitary groups. II*, Proc. Amer. Math. Soc. **128** (2000), no. 10, 3129–3136, DOI 10.1090/S0002-9939-00-05359-4. MR1664375 ↑4, 8, 15, 16, 19, 20

[Sun09] B. Sun, *Bounding matrix coefficients for smooth vectors of tempered representations*, Proc. Amer. Math. Soc. **137** (2009), no. 1, 353–357, DOI 10.1090/S0002-9939-08-09598-1. MR2439460 ↑11

[SZ12] B. Sun and C.-B. Zhu, *Multiplicity one theorems: the Archimedean case*, Ann. of Math. (2) **175** (2012), no. 1, 23–44, DOI 10.4007/annals.2012.175.1.2. MR2874638 ↑1, 6, 21

[Trè06] F. Trèves, *Topological vector spaces, distributions and kernels*, Dover Publications, Inc., Mineola, NY, 2006. Unabridged republication of the 1967 original. MR2296978 ↑12

[Wal92] N. R. Wallach, *Real reductive groups. II*, Pure and Applied Mathematics, vol. 132, Academic Press, Inc., Boston, MA, 1992. MR1170566 ↑2

[Wey50] H. Weyl, *The theory of groups and quantum mechanics*, Dover Publications, Inc., New York, 1950. Translated from the second (revised) German edition by H. P. Robertson; Reprint of the 1931 English translation. MR3363447 ↑1

[Xue] H. Xue, *Bessel models for unitary groups and Schwartz homology*. preprint. ↑2

Department of Mathematics, The University of Arizona, Tucson, AZ, 85721

E-mail address: `xuehang@math.arizona.edu`