BESSEL MODELS FOR UNITARY GROUPS AND SCHWARTZ HOMOLOGY

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Abstract. We prove the local Gan–Gross–Prasad conjecture for generic L-packets of real unitary groups. The proof is to reduce the conjecture to the tempered case which has been treated in our previous paper.

Contents

1. Introduction

The goal of this paper is to prove the local Gan–Gross–Prasad (GGP) conjecture for real unitary groups $\mathrm{U}(n+2t+1) \times \mathrm{U}(n)$, as stated in [GGP12, Theorem 17.3]. Previously in [Xue] we proved this conjecture under the assumption that the representations under consideration are all tempered. In this paper we prove the conjecture in general by reducing it to the tempered case.

Besides the final result, one notable point is the systematic application of Schwartz analysis in the argument. In particular, Schwartz induction [dC91], Borel's lemma, and Schwartz homology introduced in [CS] are extensively used. The method in this paper should have applications to other restriction problems.

In the setting of the local GGP conjecture for p-adic orthogonal groups, Mœglin and Waldspurger [MW12] carried out a reduction process that reduces the conjecture in general to the tempered case. This process was extended to the setting of p-adic unitary groups by Gan and Ichino [GI16]. Various techniques that are only available to p-adic groups, e.g. Jacquet modules, Berstein's second adjointess theorem and Bernstein–Zelevinsky derivatives, were used there. The argument in this paper is quite different. The action of the center of the universal enveloping

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algebra is carefully analyzed and plays a definitive role in the argument. It is not clear whether there is a uniform treatment for both p-adic and real cases.

We now give the precise statement of the theorem and an outline of the argument.

1.1. **Generic packets.** By a character of $\mathbb{C}^\times$, we mean a unitary character. It is conjugate self-dual if χ is trivial on $\mathbb{R}_{>0}$. Any conjugate self-dual character is of the form

$$\omega_m(z) = z^m (z\overline{z})^{-\frac{m}{2}}.$$

for some integer m. It is of sign $+1$ (resp. -1) if m is even (resp. odd). If χ is a character of $\mathbb{C}^\times$ we put $\chi^c(z) = \chi(\overline{z})$. Put $|z|_{\mathbb{C}} = z\overline{z}$. A quasi-character of $\mathbb{C}^\times$ is a continuous homomorphism $\xi : \mathbb{C}^\times \to \mathbb{C}^\times$. Any quasi-character can be written uniquely as

$$\xi = \omega_m |\cdot|_{\mathbb{C}}^s$$

for some $m \in \mathbb{Z}$ and $s \in \mathbb{C}$. It is a character if s is purely imaginary. Put $\operatorname{Re}\xi = \operatorname{Re} s$.

Let ψ and $\psi^{\mathbb{C}}$ be additive characters of $\mathbb{C}$ given by

$$\psi(z) = e^{2\pi\sqrt{-1}(z+\overline{z})}, \quad \psi^{\mathbb{C}}(z) = \psi(\sqrt{-1}z) = e^{2\pi(\overline{z}-z)}.$$

Let n be a positive integer. An L-parameter for unitary groups in n variables is a n-dimensional continuous semisimple representation of $\mathbb{C}^\times$. As $\mathbb{C}^\times$ is abelian, we can write

$$\phi = \bigoplus_{i=1}^{k} c_i \xi_i \oplus \bigoplus_{i=1}^{a} \left(\xi_{k+i} \oplus \xi_{k+i}^{c,-1} \right), \tag{1.1}$$

so that

- $\xi_1, \cdots, \xi_k$ are distinct conjugate self-dual characters of $\mathbb{C}^\times$ of sign $(-1)^{n-1}$, and $c_1, \cdots, c_k$ are positive integers;
- $\xi_{k+1}, \cdots, \xi_{k+a}$ are (not necessarily distinct) quasi-characters that are not conjugate self-dual characters of sign $(-1)^{n-1}$ and $\operatorname{Re}\xi_{k+i} \geq 0$;
- $n = c_1 + \cdots + c_k + 2a$.

The Vogan packet attached to ϕ, denoted by Π_ϕ, is the (disjoint) union of all Π_ϕ^V as V ranges over all (isomorphism classes of) hermitian spaces of dimension n,

$$\Pi_\phi = \bigcup_{V:\ \dim V = n} \Pi_\phi^V.$$

Each Π_ϕ^V is a finite set of irreducible representations of $\mathrm{U}(V)$. Throughout this paper, by a representation, we mean a smooth Fréchet representation of moderate growth. Put

$$\phi_0 = \bigoplus_{i=1}^{k} c_i \xi_i,$$

which is a limit of discrete series L-parameter of unitary groups in $n - 2a$ variables. The packet Π_ϕ can be constructed from the limit of discrete series L-packet Π_{ϕ_0} as follows. First if V does not contain an isotropic subspace of dimension a, then $\Pi_\phi^V = \emptyset$. Assume that V contains an isotopic subspace of dimension a, then let $V_0 \subset V$ be a hermitian space so that its orthogonal complement is a split hermitian space of dimension $2a$. We may take a parabolic subgroup P of $\mathrm{U}(V)$ so that its Levi subgroup is isomorphic to $(\mathbb{C}^\times)^a \times \mathrm{U}(V_0)$. There is a finite set $\Pi_{\phi_0}^{V_0}$ of irreducible limit of discrete series representations of $\mathrm{U}(V_0)$. Let us temporarily order $\xi_{k+1}, \cdots, \xi_{k+a}$ so that

$$\operatorname{Re}\xi_{k+1} \geq \cdots \geq \operatorname{Re}\xi_{k+a'} > 0 = \operatorname{Re}\xi_{k+a'+1} = \cdots \operatorname{Re}\xi_{k+a}.$$

Then the parabolically induced representation

$$\operatorname{Ind}_P^{\mathrm{U}(V)} \xi_{k+1} \otimes \cdots \otimes \xi_{k+a} \otimes \pi_0 \tag{1.2}$$

has a unique irreducible Langlands quotient. Note that we have made used of the fact that the characters $\xi_{k+a'+1}, \cdots, \xi_{k+a}$ are not conjugate self-dual of sign $(-1)^{n-1}$ so that the parabolic induction of $\xi_{k+a'+1} \otimes \cdots \otimes \xi_{k+a} \otimes \pi_0$ is irreducible by [KS80]. Then Π_ϕ^V is the collection of all these Langlands quotients where π_0 ranges over $\Pi_{\phi_0}^{V_0}$. In short, taking Langlands quotients of parabolic inductions gives a bijection between Π_ϕ^V and $\Pi_{\phi_0}^{V_0}$.

The centralizer group A_ϕ is defined to be $A_{\phi_0} = (\mathbb{Z}/2\mathbb{Z})^k$. We label elements in $A_\phi = A_{\phi_0}$ as

$$\bigoplus_{i=1}^{k} (\mathbb{Z}/2\mathbb{Z}) a_i, \tag{1.3}$$

where a_i is a symbol corresponding to ξ_i. Without saying the contrary we will follow this convention of labeling characters in the L-parameter. To each representation $\pi \in \Pi_\phi$ there is a character $\eta : A_\phi \to \langle \pm 1 \rangle$ attached to it and this defines a bijection between Π_ϕ and all characters of A_ϕ. There is a similar bijection between Π_{ϕ_0} and A_{ϕ_0}. The following diagram commutes

$$\begin{array}{ccc} \Pi_{\phi_0} & \longrightarrow & \operatorname{Hom}(A_{\phi_0}, \langle \pm 1 \rangle) \\ \downarrow & & \| \\ \Pi_\phi & \longrightarrow & \operatorname{Hom}(A_\phi, \langle \pm 1 \rangle) \end{array},$$

where the left arrow is the bijection given by the parabolic induction as before. Depending on the context, we refer to either ϕ or (ϕ, η) as the L-parameter of π.

This bijection $\Pi_\phi \to \operatorname{Hom}(A_\phi, \langle \pm 1 \rangle)$ depends on the choice of an equivalence class of Whittaker datum. When n is odd, we choose it to be the unique Whittaker datum (up to equivalence) of $\mathrm{U}(\frac{n+1}{2}, \frac{n-1}{2})$. When n is even as explained in [GGP12, Section 10], this is equivalent to choosing an additive character of $\mathbb{C}$ which is trivial on $\mathbb{R}$. Throughout this paper, we will take this additive character to be $\overline{\psi^{\mathbb{C}}}$.

We say that L-parameter ϕ is generic if Π_ϕ contains a generic representation (with any fixed Whittaker datum). It follows from [Vog78, Theorem 6.2, equivalence of (a) and (e)] that this is equivalent to that

$$L\left(s, \phi, \mathrm{As}^{(-1)^n}\right)$$

is holomorphic at $s = 1$. The result [Vog78, Theorem 6.2] is stated for "large" representations, but it is shown in [Kos78, Theorem 6.8.1] that a representation being large is equivalent to being generic. If $L(s, \phi, \mathrm{As}^{(-1)^n})$ is holomorphic at $s = 1$, then for each $\pi_0 \in \Pi_{\phi_0}^{V_0}$, by [SV80, Theorem 6.19], the parabolic induction (1.2) is irreducible. Thus if ϕ as (1.1) is generic, then Π_ϕ consists of irreducible parabolically induced representations of the form (1.2), where π_0 is a limit of discrete series representation and ranges over Π_{ϕ_0}.

1.2. **Bessel models.** Let t be a nonnegative integer. Let $W \subset V$ be a pair of hermitian spaces of dimensions n and $n + 2t + 1$ respectively. Let $G = \mathrm{U}(V)$ and $H = \mathrm{U}(W)$. We say that the pair (W, V) is relevant if $V = W \oplus^{\perp} Z$ and we can find a basis

$$z_0, z_{\pm 1}, z_{\pm 2}, \cdots, z_{\pm t}$$

of Z with

$$h_V(z_i, z_j) = (-1)^n \delta_{i,-j}, \quad i, j = 0, \pm 1, \cdots, \pm t.$$

Let P be the parabolic subgroup of G stabilizing the flag of isotropic subspaces

$$\langle z_t \rangle \subset \langle z_t, z_{t-1} \rangle \subset \cdots \subset \langle z_t, \cdots, z_1 \rangle,$$

Let N be the unipotent radical of P, and $S = H \ltimes N$ which is a subgroup of G. We define a generic character of N as follows. Let $u \in N$, we define a character

$$\nu(u) = \psi\left(-\operatorname{Tr}_{\mathbb{C}/\mathbb{R}} \sum_{i=0}^{t-1} h_V(z_{-i-1}, u z_i)\right).$$

As ν is invariant under the conjugation action of H, it admits a unique extension to S which is trivial on H. We denote this extension agin by ν.

Let π and σ be representations of G and H respectively. We denote by $\mathrm{Hom}_S(\pi \widehat{\otimes} \sigma, \nu)$ the space of continuous linear forms $\ell : \pi \widehat{\otimes} \sigma \to \mathbb{C}$ with the property that

$$\ell(\pi(h)v \otimes \sigma(h)w) = \nu(h)\ell(v \otimes w), \quad \text{for all } h \in S,\ v \in \pi \text{ and } w \in \sigma.$$

Here σ is viewed as a representation of S via the natural projection $S \to H$. We define the multiplicity

$$m(\pi, \sigma) = \dim \mathrm{Hom}_S(\pi \widehat{\otimes} \sigma, \nu).$$

By [SZ12, JSZ10] if π and σ are irreducible then $m(\pi, \sigma) \leq 1$.

If $t = 0$, then $S = H$ and ν is trivial. Then $\mathrm{Hom}_S(\pi \widehat{\otimes} \sigma, \nu)$ is the space of H-invariant continuous linear forms on $\pi \widehat{\otimes} \sigma$. This Hom-space will be referred to as the spherical model. If $W = 0$ and hence σ is trivial, then G is quasi-split and ν is a generic character of the unipotent radical of a Borel subgroup. Then $\mathrm{Hom}_S(\pi \widehat{\otimes} \sigma, \nu)$ is the space of Whittaker models on π. Thus the Bessel models "interpolate" the spherical model and the Whittaker model.

Let π and σ be irreducible representations of G and H respectively. Assume that they lie in generic packets. Let (ϕ_π, η_π) and $(\phi_\sigma, \eta_\sigma)$ be the parameters of π and σ respectively. Write the L-parameters

$$\phi_\pi = \bigoplus_{i=1}^{k} c_i \chi_i \oplus \bigoplus_{i=1}^{a} \left(\chi_{k+i} \oplus \chi_{k+i}^{c,-1} \right), \quad \phi_\sigma = \bigoplus_{j=1}^{l} d_i \mu_j \oplus \bigoplus_{j=1}^{b} \left(\mu_{l+j} \oplus \mu_{l+j}^{c,-1} \right), \tag{1.4}$$

as (1.1) and

$$A_{\phi_\pi} = \bigoplus_{i=1}^{k} (\mathbb{Z}/2\mathbb{Z}) a_i, \quad A_{\phi_\sigma} = \bigoplus_{j=1}^{l} (\mathbb{Z}/2\mathbb{Z}) b_j.$$

For any L-parameter ϕ, the local root numbers $\epsilon\left(\phi, \psi^{\mathbb{C}}\right)$ are defined, c.f. [GGP12].

Theorem 1.1. *Suppose that π and σ correspond to $\eta_\pi : A_{\phi_\pi} \to \langle \pm 1 \rangle$ and $\eta_\sigma : A_{\phi_\sigma} \to \langle \pm 1 \rangle$ respectively. Then $m(\pi, \sigma) = 1$ if and only if*

$$\eta_\pi(a_i) = \epsilon\left(\chi_i \otimes \phi_\sigma, \psi^{\mathbb{C}}\right), \quad \eta_\sigma(b_j) = \epsilon\left(\phi_\pi \otimes \mu_j, \psi^{\mathbb{C}}\right),$$

for all $i = 1, \cdots, k$ and $j = 1, \cdots, l$.

This confirms [GGP12, Conjecture 17.3] for the unitary groups $\mathrm{U}(n+2t+1) \times \mathrm{U}(n)$. If π and σ are tempered, the theorem has been established in [Xue].

1.3. **Outline of the proof.** The proof consists of two steps. We first prove the case $t = 0$ and then reduce the general case to it. As representations lying in a generic packet are irreducible parabolically induced representations, it is not surprising that a large part of the argument is devoted to the relation between parabolic inductions and multiplicities. Sections 2 and 3 explain the machinery of Schwartz analysis and develop the necessary tools. Section 4 carries out the proof of the case $t = 0$. Sections 5 and 6 are devoted to the reduction to the $t = 0$ case.

Let us first consider the case $t = 0$. Here and below in this paper, following usual practice, we often write the parabolic induction (1.2) as

$$\chi_{k+1} \times \cdots \times \chi_{k+a} \times \pi_0,$$

when there is no confusion with the parabolic subgroup P. Similar notation applies to other parabolic inductions.

Suppose the L-parameters of the representations π and σ are given as in Theorem 1.1. Then we may write

$$\pi = \chi_{k+1} \times \cdots \times \chi_{k+a} \times \pi_0, \quad \sigma = \mu_{l+1} \times \cdots \times \mu_{l+b} \times \sigma_0,$$

where π_0 and σ_0 are limit of discrete series representations. We may write

$$\chi_{k+i} = \omega_{l_i} |\cdot|_{\mathbb{C}}^{s_i}, \quad \mu_{l+j} = \omega_{m_j} |\cdot|_{\mathbb{C}}^{t_j}, \quad i = 1, \cdots, a, \quad j = 1, \cdots, b,$$

where $l_1, \cdots, l_a, m_1, \cdots, m_b$ are integers and $s_1, \cdots, s_a, t_1, \cdots, t_b$ are complex numbers with non-negative real parts. By relabeling we may assume that

$$\mathrm{Re}\,(l_1 + 2s_1) \geq \cdots \geq \mathrm{Re}\,(l_a + 2s_a), \quad \mathrm{Re}\,(-m_1 + 2t_1) \geq \cdots \geq \mathrm{Re}\,(-m_b + 2t_b).$$

These numbers are related to the infinitesimal characters of π and σ. With this ordering, $\mathrm{Re}\,(l_1+2s_1)$ is the largest among $\mathrm{Re}\,(l_i \pm 2s_i)$'s and hence $\mathrm{Re}\,(-l_1 - 2s_1)$ is the smallest among $\mathrm{Re}\,(-l_i \pm 2s_i)$'s, and $\mathrm{Re}\,(-m_1 + 2t_1)$ is the largest among $\mathrm{Re}\,(-m_j \pm 2t_j)$'s and hence $\mathrm{Re}\,(m_1 - 2t_1)$ is the smallest among $\mathrm{Re}\,(m_i \pm 2s_i)$'s.

We distinguish two cases.

- Assume that $\mathrm{Re}\,(l_1 + 2s_1) \geq \mathrm{Re}\,(-m_1 + 2t_1)$ or $b = 0$ first. We put

  $$\pi^- = \chi_{k+2} \times \cdots \times \chi_{k+a} \times \pi_0, \quad \pi_1 = |\cdot|^{s_1'} \times \chi_{k+2} \times \cdots \times \chi_{k+a} \times \pi_0,$$

  where $s_1' \in \sqrt{-1}\mathbb{R}$ is a generic purely imaginary number, i.e. it avoids countably many values. Here π^- is a representation of $\mathrm{U}(V_-)$ where V_- is a subspace of V and $V_-^\perp$ is a hyperplane, and π_1 is a representation of G. We prove that

  $$m(\pi, \sigma) = m(\sigma, \pi^-) = m(\pi_1, \sigma). \tag{1.5}$$

  We will explain the idea of the proof of this below. The net effect of this procedure is to replace the possibly nonunitary χ_{k+1} by a unitary one $|\cdot|^{s_1'}$ in π.
- Assume that $\mathrm{Re}\,(l_1 + 2s_1) < \mathrm{Re}\,(-m_1 + 2t_1)$ or $a = 0$ now. We put

  $$\sigma^+ = |\cdot|^{t_1'} \times \mu_{l+1} \times \cdots \times \mu_{l+b} \times \sigma_0, \quad \sigma_1 = |\cdot|^{t_1'} \times \mu_{l+2} \times \cdots \times \mu_{l+b} \times \sigma_0,$$

  where $t_1' \in \sqrt{-1}\mathbb{R}$ is a generic purely imaginary number. Here σ^+ is a representation of $\mathrm{U}(W^+)$ where W^+ is a hermitian space containing W and $W^\perp$ is a hyperplane, and σ_1 is a representation of H. We prove that

  $$m(\pi, \sigma) = m(\sigma^+, \pi) = m(\pi, \sigma_1). \tag{1.6}$$

  The net effect of this procedure is to replace the possibly nonunitary μ_{l+1} by a unitary one $|\cdot|^{t_1'}$ in σ.

These procedures can be repeated. We order the numbers

$$\mathrm{Re}\,(l_i + 2s_i), \quad i = 1, \cdots, a, \quad \mathrm{Re}\,(-m_j + 2t_j), \quad j = 1, \cdots, b$$

from the largest to the smallest and applies the above procedures to $\chi_{k+1}, \cdots, \chi_{k+a}, \mu_{l+1}, \cdots, \mu_{l+b}$ according to this order. The ultimate effect is that we replace these potentially nonunitary quasi-characters by the unitary characters of the form $|\cdot|^s$ with $s \in \sqrt{-1}\mathbb{R}$. More precisely we find generic elements

$$s_1', \cdots, s_a', t_1', \cdots, t_b' \in \sqrt{-1}\mathbb{R}$$

so that we have

$$\pi_a = |\cdot|^{s_1'} \times \cdots \times |\cdot|^{s_a'} \times \pi_0,$$

and

$$\sigma_b = |\cdot|^{t_1'} \times \cdots \times |\cdot|^{t_b'} \times \sigma_0,$$

with

$$m(\pi, \sigma) = m(\pi_a, \sigma_b).$$

The representations π_a and σ_b are irreducible and tempered. Moreover in this process, the L-parameters ϕ_π and ϕ_σ are changed, while ϕ_{π_0} and ϕ_{σ_0}, the centralizers A_{ϕ_π} and A_{ϕ_σ}, the characters η_π and η_σ, and the root numbers appearing in Theorem 1.1 remain intact. Thus Theorem 1.1 for (π, σ) is reduced to that for (π_a, σ_b) which has been established in [Xue]. This argument can be packed into a slightly more souped-up version using a double induction on both a and b. But it seems (to me) that the induction is rather involved and confusing. So we keep the argument as above.

It remains to explain the proof of (1.5) and (1.6). This is the content of Proposition 4.3 and is where the machinery of Schwartz analysis comes into the argument. By definition, Schwartz homologies are derived functors of the H-coinvariance functor $- \to -_H$. Thus the continuous dual of the zeroth homology

$$H_0^{\mathcal{S}}(H, \pi \widehat{\otimes} \sigma) = (\pi \widehat{\otimes} \sigma)_H$$

is $\mathrm{Hom}_H(\pi \widehat{\otimes} \sigma, \mathbb{C})$. We now explain the proof of $m(\pi, \sigma) = m(\sigma, \pi^-)$ in (1.5) under the assumption that $\mathrm{Re}\,(l_1 + 2s_1) \geq \mathrm{Re}\,(-m_1 + 2t_1)$. All other equalities are proved in the same fashion. The point is to study the restriction of π to H. Note that π is a parabolically induced representation, so this restriction is studied via the usual orbit method. We may assume that W is isotropic, the case it being anisotropic is much easier. Let $W = W_0 \oplus^\perp (E \oplus F)$ where E and F are isotropic lines in W and $E \oplus F$ is a hyperplane. Let P_- be a parabolic subgroup of G stabilizing E. Its Levi subgroup is isomorphic to $\mathbb{C}^\times \times \mathrm{U}(V_-)$ and V_- is the orthogonal complement of $E \oplus F$ in V. We write π as

$$\mathrm{Ind}_{P_-}^G \chi_{k+1} \otimes \pi^-$$

where π^- is an irreducible representation of $\mathrm{U}(V_-)$. The intersection $P_- \cap H$ is again a parabolic subgroup of H. There are two H orbits on the flag variety $P_- \backslash G$, the open one and the closed one. There is an H-stable closed subspace π° of π consisting of all Schwartz sections in π over the open orbit. We take the long exact sequence of the Schwartz homologies $H_i^{\mathcal{S}}(H, -)$ associated to

$$0 \to \pi^\circ \to \pi \to \pi/\pi^\circ \to 0$$

to obtain

$$\cdots \to H_1^{\mathcal{S}}(H, \pi/\pi^\circ \widehat{\otimes} \sigma) \to H_0^{\mathcal{S}}(H, \pi^\circ \widehat{\otimes} \sigma) \to H_0^{\mathcal{S}}(H, \pi \widehat{\otimes} \sigma) \to H_0^{\mathcal{S}}(H, \pi/\pi^\circ \widehat{\otimes} \sigma) \to 0.$$

By a version of Borel's lemma, c.f. Proposition 2.5, the quotient π/π° has an H-stable filtration whose graded pieces are of the form

$$\rho^j = \mathrm{Ind}_{P_- \cap H}^H \chi_{k+1} |\cdot|_{\mathbb{C}}^{\frac{1}{2}+j} \otimes (\pi^-|_{H \cap \mathrm{U}(V_-)}), \quad j = 0, 1, 2, \cdots.$$

Here the extra factor $|\cdot|_{\mathbb{C}}^{\frac{1}{2}}$ comes from the comparison of the modulus character for P and P_- and the factors $|\cdot|_{\mathbb{C}}^{j}$ all come from the "transverse derivatives" along the closed orbit. The infinitesimal character of $\sigma^\vee$ is

$$\left(\frac{-m_1+2t_1}{2},\cdots,\frac{-m_b+2t_b}{2};\mu_1,\cdots,\mu_{n-2b};\frac{-m_b-2t_b}{2},\cdots,\frac{-m_1-2t_1}{2}\right),$$

where $\mu_i \in \frac{n+1}{2}+\mathbb{Z}$, $i=1,2,\cdots,b$. Under the assumption that $\mathrm{Re}\,(l_1+2s_1)\geq \mathrm{Re}\,(-m_1+2t_1)$ we have

$$\mathrm{Re}\,\frac{l_1+2s_1+1}{2}+j>\mathrm{Re}\,\frac{-m_i\pm 2t_i}{2},\quad i=1,\cdots,b,\quad j=0,1,2,\cdots.$$

Moreover as π lies in a generic packet, we have

$$\frac{l_1+2s_1+1}{2}+j\notin\frac{n+1}{2}+\mathbb{Z}.$$

Thus the numbers $\frac{l_1+2s_1+1}{2}+j$ do not appear in the infinitesimal character of $\sigma^\vee$. Then we can construct an element z in the center of the universal enveloping algebra of H, annihilating ρ^j, but not $\sigma^\vee$. A construction of this type is given in Section 3. It follows from a general vanishing result of Schwartz homology, i.e. Corolloary 2.8, that

$$H_i^{\mathcal{S}}\left(H,\rho^j\,\widehat{\otimes}\,\sigma\right)=0$$

for all i. Then by taking inverse limit we conclude that

$$H_i^{\mathcal{S}}\left(H,\pi/\pi^\circ\,\widehat{\otimes}\,\sigma\right)=0$$

for all i and hence the long exact sequence implies that

$$H_0^{\mathcal{S}}(H,\pi^\circ\,\widehat{\otimes}\,\sigma)=H_0^{\mathcal{S}}(H,\pi\,\widehat{\otimes}\,\sigma).$$

Finally Frobenius reciprocity yields

$$H_0^{\mathcal{S}}(H,\pi^\circ\,\widehat{\otimes}\,\sigma)=H_0^{\mathcal{S}}(\mathrm{U}(V_-),\sigma\,\widehat{\otimes}\,\pi^-).$$

The desired equality

$$m(\pi,\sigma)=m(\sigma,\pi^-)$$

then follows.

The reason why we choose to work with Schwartz analysis instead of other (co)homology theories is that we are able to describe the filtration on π/π° in this context relatively easily. More importantly, Theorem 1.1 only holds for Fréchet representations of moderate growth. The Theorem would have been false had we worked in the other categories of representations, e.g. the category of Harish-Chandra modules. This is a well-known phenomenon first observed by Kostant [Kos78]. The most suitable (co)homology theory for the category of Fréchet representations of moderate growth is the Schwartz homology theory. To me Schwartz analysis provides us with the best framework in studying restriction problems for parabolically induced representations.

We end this long discussion by commenting briefly on the reduction of the general case to the case $t=0$. The argument is in the same spirit. The notation below is independent from the above explanation for $t=0$. Recall that we have a relevant pair of hermitian spaces $W \subset V$ of dimension n and $n+2t+1$ respectively. Let z_0' be an element with norm $(-1)^{n+1}$ and we put

$$W^+ = V \oplus^\perp \langle z_0' \rangle, \quad Y = X \oplus \langle z_0 + z_0' \rangle.$$

Then Y is a $(t+1)$-dimensional isotropic subspace of W^+. Let P be the parabolic subgroup of $\mathrm{U}(W^+)$ stabilizing Y. The Levi subgroup of P is isomorphic to $\mathrm{GL}_{t+1}(\mathbb{C}) \times \mathrm{U}(W)$. We choose an irreducible principal series representation

$$\tau = |\cdot|_{\mathbb{C}}^{s_1} \times \cdots \times |\cdot|_{\mathbb{C}}^{s_{t+1}}$$

of $\mathrm{GL}_{t+1}(\mathbb{C})$, and consider the parabolically induced representation

$$\sigma^+ = \mathrm{Ind}_P^{\mathrm{U}(W^+)} \tau \,\widehat{\otimes}\, \sigma$$

of $\mathrm{U}(W^+)$. We assume that $s_1, \cdots, s_{t+1} \in \mathbb{C}$ are complex numbers in general position, i.e. $(s_1, \cdots, s_{t+1}) \in \mathbb{C}^{t+1}$ avoids the zeros of countably many nonzero polynomials functions. Again there are two G-orbits on $P \backslash \mathrm{U}(W^+)$, the open one and the closed one. Let $\sigma^{+,\circ}$ be the G-invariant subspace of σ^+ consisting of Schwartz sections over the open orbit. The same argument as in the case $t=0$ gives that

$$m(\sigma^+, \pi) = \dim \mathrm{Hom}_G(\sigma^{+,\circ} \,\widehat{\otimes}\, \pi, \mathbb{C}).$$

The difference is that Frobenius reciprocity does not immediately apply to this Hom space and $\sigma^{+,\circ}$ admits a further filtration. This filtration is closely related to the R-stable filtration on τ where R is the mirabolic subgroup of $\mathrm{GL}_{t+1}(\mathbb{C})$. Section 5 is devoted to the study of this filtration. It is the counterpart in our context to the Bernstein–Zelevinski's theory of derivatives for p-adic general linear groups. In the case of p-adic general linear groups, this filtration is obtained from the exactness and adjointness properties of various induction and Jacquet functors, which are not available in our setting. We obtain the desired filtration by combining the Borel's lemma and a trick of applying the Fourier transform. Once this filtration is understood we can apply the machinery of Schwartz analysis as before and eventually, via Frobenius reciprocity again, obtain that

$$\dim \mathrm{Hom}_G(\sigma^{+,\circ} \,\widehat{\otimes}\, \pi, \mathbb{C}) = m(\pi, \sigma).$$

Therefore Theorem 1.1 for (π, σ) is reduced to that for (σ^+, π) which is in the case $t=0$. This concludes the proof of Theorem 1.1.

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2. Schwartz analysis

2.1. **Schwartz induction.** We work in the setting of almost linear Nash groups, i.e. a Nash group which admits a Nash homomorphism to $\mathrm{GL}_n(\mathbb{R})$ with finite kernel. We denote by gothic letters the Lie algebra of the corresponding group, e.g. if G is a almost linear Nash group, then $\mathfrak{g} = \mathrm{Lie}(G)$ stands for its Lie algebra, and $\mathfrak{g}_{\mathbb{C}}$ its complexification. We denote by $\mathcal{U}(\mathfrak{g}_{\mathbb{C}})$ the universal enveloping algebra and $\mathcal{Z}(\mathfrak{g}_{\mathbb{C}})$ its center. By a representation of G, we mean a Fréchet representation of moderate growth, i.e. a Fréchet representation (π, V) of G so that for every seminorm $\|\cdot\|_1$ on V, there is another seminorm $\|\cdot\|_2$ on V, and some positive Nash function f on G so that

$$\|\pi(g)v\|_1 \leq f(g)\|v\|_2, \quad g \in G, v \in V.$$

Let $\mathcal{S}\mathrm{mod}_G$ be the category of Fréchet representations of moderate growth of G.

In what follows, if V and W are topological vector spaces, we write $V = W$ to indicate that they are isomorphic as topological vector spaces.

Let G be an almost linear Nash group and H a Nash subgroup. Let (σ, V) be a representation of H. View V as the trivial representation of G and let $\mathcal{S}(G, V)$ (resp. $C^\infty(G, V)$) be the space of Schwartz functions (resp. smooth functions) on G valued in V. Let I be the continuous map

$$I : \mathcal{S}(G, V) \to C^\infty(G, V), \quad f \mapsto \left(g \mapsto \int_H \sigma(h) f(h^{-1}g) \mathrm{d}_L h\right), \tag{2.1}$$

where $\mathrm{d}_L h$ is the left invariant measure. Let $\mathrm{ind}_H^G V$ or $\mathrm{ind}_H^G \sigma$ be the image of I. Under the quotient topology of $\mathcal{S}(G, V)$ and the right translation of G, it is a Fréchet representation of moderate growth of G and is called the Schwartz induction of (σ, V). If H is a parabolic subgroup of G, then this is the usual unnormalized parabolic induction.

We have the following geometric interpretation of Schwartz inductions. We define a right action of H on $G \times V$ by

$$(g, v) \cdot h = (h^{-1}g, \sigma(h)^{-1}v), \tag{2.2}$$

and let $\mathcal{V} = H\backslash(G \times V)$ be the quotient of $G \times V$ this H-action. It is a tempered (right) G-bundle over $H\backslash G$, c.f. [CS, Section 6.1]. By [CS, Proposition 6.7],

$$\mathrm{ind}_H^G V = \Gamma^{\mathcal{S}}(H\backslash G, \mathcal{V})$$

where $\Gamma^{\mathcal{S}}(H\backslash G, \mathcal{V})$ stands for the space of all Schwartz sections of $\mathcal{V}$ in the sense of [CS, Definition 6.1].

We remark that left actions are used throughout in [CS]. To be consistent with the usual convention of parabolic inductions, we choose to use right actions of G on $H\backslash G$. The fiber at 1 of the bundle $\mathcal{V}$ is then the vector space V with a right H action. This right action and the representation σ is connected by

$$\sigma(h)v = v \cdot h^{-1}. \tag{2.3}$$

The representation σ will be referred to as the representation on V that the bundle $\mathcal{V}$ gives rise to.

We recall several results from [CS].

Proposition 2.1 ([CS, Proposition 7.1]). *Schwartz induction is an exact functor from $\mathcal{S}\mathrm{mod}_H$ to $\mathcal{S}\mathrm{mod}_G$.*

This proposition will mostly be used implicitly and will be referred to as "exactness of Schwartz induction".

Proposition 2.2 ([CS, Proposition 7.2]). *Let H' be a closed Nash subgroup of H and V be a representation of H'. Then*

$$\mathrm{ind}_H^G \left(\mathrm{ind}_{H'}^H V\right) = \mathrm{ind}_{H'}^G V.$$

This proposition will be referred to as "induction by stages".

Proposition 2.3 ([CS, Proposition 7.4]). *Let W and V be representations of H and G respectively. Assume that one of W and V is nuclear. Then*

$$\mathrm{ind}_H^G(W \widehat{\otimes} V|_H) = (\mathrm{ind}_H^G W) \widehat{\otimes} V.$$

Here $\widehat{\otimes}$ stands for the completed tensor product of complete locally convex topological vector spaces. In any completed tensor product appearing in this paper, at least one (most of the time both) of the topological vector spaces will be nuclear and thus there is no ambiguity which tensor product we use, c.f. [Trè06, Theorem 50.1(f)]. Moreover the completed tensor product of two nuclear Fréchet spaces is again a nuclear Fréchet space.

Proposition 2.4 ([CS, Proposition 6.6, Theorem 6.8]). *Let V be a representation of H. Then*

$$\left((\mathrm{ind}_H^G(V \otimes \delta_H)) \otimes \delta_G^{-1}\right)_G = V_H,$$

where δ_H and δ_G stand for the modulus characters of H and G respectively.

This proposition will be referred to as the "Frobenius reciprocity".

2.2. **Borel lemma.** The classical Borel's lemma asserts that the quotient

$$\mathcal{S}(\mathbb{R}^n)/\mathcal{S}(\mathbb{R}^n \backslash \{0\})$$

is isomorphic to the power series ring

$$\mathbb{C}[[x_1, \cdots, x_n]],$$

with the canonical quotient map $\mathcal{S}(\mathbb{R}^n) \to \mathbb{C}[[x_1, \cdots, x_n]]$ being given by taking the Taylor expansion at origin.

Now let $\mathcal{X}$ be a Nash manifold. Assume that $\mathcal{Z}$ is a closed Nash submanifold. Let $\mathcal{U}$ be the open complement of Z. Extension by zero gives a natural injective map $\Gamma^{\mathcal{S}}(\mathcal{U}, \mathcal{E}) \to \Gamma^{\mathcal{S}}(\mathcal{X}, \mathcal{E})$. Put

$$\Gamma^{\mathcal{S}}_{\mathcal{Z}}(\mathcal{X}, \mathcal{E}) = \Gamma^{\mathcal{S}}(\mathcal{X}, \mathcal{E})/\Gamma^{\mathcal{S}}(\mathcal{U}, \mathcal{E}).$$

Let $\mathcal{N}^\vee_{\mathcal{Z}/\mathcal{X}}$ be the complexification of the conormal bundle. This is a tempered bundle over $\mathcal{Z}$ in the sense of [CS, Section 6.1]. Let $\mathcal{E}$ be a tempered bundle over $\mathcal{X}$. The following is the Borel's lemma in this setting.

Proposition 2.5 ([CS, Proposition 8.2, 8.3])**.** *There is a decreasing filtration on $\Gamma^{\mathcal{S}}_{\mathcal{Z}}(\mathcal{X},\mathcal{E})$, denoted by $\Gamma^{\mathcal{S}}_{Z}(\mathcal{X},\mathcal{E})_k$, $k = 0, 1, 2, \cdots$, so that*

$$\Gamma^{\mathcal{S}}_{\mathcal{Z}}(\mathcal{X},\mathcal{E}) = \varprojlim \Gamma^{\mathcal{S}}_{\mathcal{Z}}(\mathcal{X},\mathcal{E})/\Gamma^{\mathcal{S}}_{\mathcal{Z}}(\mathcal{X},\mathcal{E})_k.$$

Moreover the graded pieces are isomorphic to

$$\Gamma^{\mathcal{S}}(\mathcal{Z}, \mathrm{Sym}^k \mathcal{N}^\vee_{\mathcal{Z}/\mathcal{X}} \otimes \mathcal{E}|_{\mathcal{Z}}), \quad k = 0, 1, 2, \cdots,$$

If $\mathcal{X}$ is a G-Nash manifold, $\mathcal{Z}$ is stable under the action of G and $\mathcal{E}$ is a tempered G-bundle, then this filtration is stable under the action of G.

2.3. **Schwartz homology.** If V is a representation of G, we denote by V_G the G-coinvariance. There are a number of equivalent ways to define it. According to [CS, Theorem 1.7], one definition is

$$V_G = V \Big/ \sum_{g\in G}(g-1)V.$$

The space V_G is given the quotient topology and is a locally convex topological space. Note that the subspace $\sum_{g\in G}(g-1)V$ might not be closed and thus V_G might not be Hausdorff.

If $l : V \to \mathbb{C}$ is a continuous G-invariant linear form, then it descends to a continuous linear form on V_G and to the maximal Hausdorff quotient V^0_G. Conversely any continuous linear form of V_G automatically factors through V^0_G and we obtain a continuous linear form on V by composing it with the natural map $V \to V_G$. It follows that $\mathrm{Hom}_G(V, \mathbb{C})$ is the dual space of V_G or V^0_G. Therefore if $\mathrm{Hom}_G(V, \mathbb{C})$ is finite dimensional, then so is V^0_G and

$$\dim \mathrm{Hom}_G(V, \mathbb{C}) = \dim V^0_G.$$

A Schwartz homology theory $H^{\mathcal{S}}_i(G, -)$ has been introduced in [CS]. The functor $V \mapsto V_G$ is a right exact functor from $\mathcal{S}\mathrm{mod}_G$ to the category of locally convex topological vector spaces. The Schwartz homology $H^{\mathcal{S}}_i(G, -)$ is the derived functor of $V \mapsto V_G$. It can be calculated using strong projective resolutions as follows. Let

$$\cdots \to P_1 \to P_0 \to V \to 0$$

be a strong projective resolution of V in the sense of [CS, Definition 5.11], i.e. it is an exact sequence in $\mathcal{S}\mathrm{mod}_G$, homomorphisms are strong and P_i's are relatively projective. Then $H^{\mathcal{S}}_i(G, V)$ is the i-th homology of the complex

$$\cdots \to (P_1)_G \to (P_0)_G \to 0.$$

The homologies $H^{\mathcal{S}}_i(G, V)$ may or may not be Hausdorff. By [CS, Proposition 1.9], it is Hausdorff when it is finite dimensional.

Let

$$0 \to V_1 \to V_2 \to V_3 \to 0$$

be a short exact sequence of representations of G, then by [CS, Corollary 7.8], we have a long exact sequence

$$(2.4) \qquad \cdots \to H^{\mathcal{S}}_{i+1}(G, V_3) \to H^{\mathcal{S}}_i(G, V_1) \to H^{\mathcal{S}}_i(G, V_2) \to H^{\mathcal{S}}_i(G, V_3) \to H^{\mathcal{S}}_{i-1}(G, V_1) \to \cdots,$$

of locally convex (not necessarily Hausdorff) topological vector spaces.

Proposition 2.6 ([CS, Theorem 7.5])**.** *Let H be a closed Nash subgroup of G and V a representation of H. Then*

$$H^{\mathcal{S}}_i(G, (\mathrm{ind}_H^G(V \otimes \delta_H)) \otimes \delta_G^{-1}) = H^{\mathcal{S}}_i(H, V),$$

for all $i \geq 0$.

This proposition will be referred to as "Shapiro's lemma". It recovers Forbenius reciprocity when $i = 0$.

Proposition 2.7. *Let (π, V) and (σ, W) be representations of G. Assume that both of them are nuclear. Let $X \mapsto \check{X}$ be the canonical anti-automorphism of $\mathcal{U}(\mathfrak{g}_{\mathbb{C}})$ such that $\check{X} = -X$ if $X \in \mathfrak{g}$. Assume that there is an element $X \in \mathcal{Z}(\mathfrak{g}_{\mathbb{C}})$ such that X acts on π as a constant λ_1 and $\check{X}$ acts on σ as a constant λ_2. If $\lambda_1 \neq \lambda_2$, then*

$$H^{\mathcal{S}}_i(G, \pi \,\widehat{\otimes}\, \sigma) = 0$$

for all $i \geq 0$.

Proof. Let $P_{-2} = 0$, $P_{-1} = V$ and we inductively define P_i to be $\mathcal{S}(G, K_{i-1})$ where K_{i-1} stands for the kernel of $P_{i-1} \to P_{i-2}$, $i = 0, 1, 2, \cdots$. Let $P_i \to P_{i-1}$ be the composition

$$P_i = \mathcal{S}(G, K_{i-1}) \to K_{i-1} \to P_{i-1}$$

where the first map is given by integration over G. By [CS, Proposition 5.4],

$$\cdots \to P_1 \to P_0 \to V \to 0$$

is a strong projective resolution where the maps are given by integration along G. We let $P_\bullet$ be the complex

$$\cdots \to P_1 \to P_0 \to 0.$$

Since V is nuclear, all P_i's are nuclear. As W is nuclear

$$\cdots \to P_1 \,\widehat{\otimes}\, W \to P_0 \,\widehat{\otimes}\, W \to V \,\widehat{\otimes}\, W \to 0$$

is a strong projective resolution of $V \,\widehat{\otimes}\, W$, c.f. Lemma 2.9 below. The element $\check{X} \in \mathcal{Z}(\mathfrak{g})$ gives a morphism $W \to W$ and hence a morphism of the complexes

$$1 \otimes \check{X} : P_\bullet \,\widehat{\otimes}\, W \to P_\bullet \,\widehat{\otimes}\, W.$$

It descends to a morphism of the complexes

$$1 \otimes \check{X} : (P_\bullet \widehat{\otimes} W)_G \to (P_\bullet \widehat{\otimes} W)_G. \tag{2.5}$$

By assumption $\check{X}$ acts as a scalar λ_2 on W, therefore it acts as as the scalar λ_2 on the complex and hence on the homologies $H_i^{\mathcal{S}}(G, V \widehat{\otimes} W)$.

By [CS, Lemma 4.6], the element $1 \otimes \check{X} - X \otimes 1$ acts trivially on the complex $(P_\bullet \widehat{\otimes} W)_G$. Thus on the one hand $X \otimes 1$ acts on $H_i^{\mathcal{S}}(G, V \widehat{\otimes} W)$ as a scalar λ_2. On the other hand, X acts as a scalar λ_1 on V. Since W is nuclear, the tensor functor $- \widehat{\otimes} W$ is exact. It follows that $H_i^{\mathcal{S}}(G, - \widehat{\otimes} W)$ are the derived functors of $(- \widehat{\otimes} W)_G$. Therefore $X \otimes 1$ acts as a scalar λ_1 on $H_i^{\mathcal{S}}(G, V \widehat{\otimes} W)$. Since $\lambda_1 \neq \lambda_2$, we conclude that

$$H_i^{\mathcal{S}}(G, V \widehat{\otimes} W) = 0.$$

This finishes the proof. □

We observe that if σ is irreducible and $\check{X}$ acts on σ by λ_2, then X acts on $\sigma^\vee$ by λ_2. Thus we have the following corollary.

Corollary 2.8. *Assume in addition that σ is irreducible in Proposition 2.7. If there is an element $X \in \mathcal{Z}(\mathfrak{h}_\mathbb{C})$ so that X acts on π by λ_1 and acts on $\sigma^\vee$ by λ_2 with $\lambda_1 \neq \lambda_2$. Then*

$$H_i^{\mathcal{S}}(G, \pi \widehat{\otimes} \sigma) = 0$$

for all $i \geq 0$.

2.4. **Remarks on linear algebra.** We add some remarks regarding linear algebra. We begin with the following well-known lemma on nuclear Fréchet spaces. It is so fundamental that we mostly use it without reference. The cited reference also contains useful information on nuclear Fréchet spaces and their duals.

Lemma 2.9 ([CHM00, Lemma A.3])**.** *Let W be a nuclear Fréchet space and*

$$0 \to V_1 \to V_2 \to V_3 \to 0$$

be a short exact sequence of nuclear Fréchet spaces. Then

$$0 \to V_1 \widehat{\otimes} W \to V_2 \widehat{\otimes} W \to V_3 \widehat{\otimes} W \to 0$$

is a short exact sequence of nuclear Fréchet spaces.

We will frequently use the following result. Again it is mostly used without reference.

Lemma 2.10. *Let G be a Nash group and*

$$0 \to V_1 \to V_2 \to V_3 \to 0$$

be an exact sequence of representation of G. If

$$H_i^{\mathcal{S}}(G, V_3) = 0$$

for all i, then $H_i^{\mathcal{S}}(G, V_1) = H_i^{\mathcal{S}}(G, V_2)$ for all i.

Proof. It follows from the long exact sequence that

$$H_i^{\mathcal{S}}(G, V_1) \to H_i^{\mathcal{S}}(G, V_2)$$

is a continuous isomorphism of vector spaces. It follows from [CS, Lemma 8.6] that it is an open map. Therefore it is an isomorphism of topological vector spaces. □

Let I be some index set and V be a Fréchet space with a descending filtration of closed subspaces indexed by I. We say this filtration is complete if I is countable well-ordered and if the canonical map

$$V \to \varprojlim V/V_\alpha$$

is an isomorphism of topological vector spaces. The successive quotient $V_\alpha/V_{\alpha+}$, $\alpha \in I$, are called the graded pieces. Here if $\alpha \in I$ we denote by $\alpha+$ its successor. The filtration in Proposition 2.5 is complete.

Lemma 2.11. *Let W be a closed subspace of V and Z the quotient. Assume that W and Z admit complete filtrations indexed by the countable well-ordered set I and J respectively,*

$$W_\alpha, \quad \alpha \in I; \quad Z_\beta, \quad \beta \in J.$$

Then V admits a complete filtration indexed by $I \cup J$ with graded pieces

$$W_\alpha/W_{\alpha+}, \quad Z_\beta/Z_{\beta+}, \quad \alpha \in I, \quad \beta \in J.$$

If V is a representation of some Nash group G and W is G-stable, then the filtration is also G-stable.

Proof. The index set $I \cup J$ is ordered so that it does not change the ordering in I and J and $\alpha > \beta$ if $\alpha \in I$ and $\beta \in J$. Then $I \cup J$ is well-ordered. Let $\varphi : V \to Z$ be the natural map. The filtration V_γ, indexed by $\gamma \in I \cup J$, given by

$$V_\gamma = \begin{cases} W_\gamma, & \text{if } \gamma \in J; \\ \varphi^{-1}(Z_\gamma), & \text{if } \gamma \in I. \end{cases}$$

is a filtration with required graded pieces. It gives rise to an inverse system

$$V_\bullet : \cdots \to V/V_{\gamma+} \to V/V_\gamma \to \cdots .$$

Moreover we extend the inverse system

$$\cdots \to W/W_{\alpha+} \to W/W_\alpha \to \cdots \to W/W_{\alpha_0} = 0$$

to

$$W_\bullet : \cdots \to W/W_{\alpha+} \to W/W_\alpha \to \cdots \to W/W_{\alpha_0} = 0 \to \cdots \to 0 \to 0.$$

where the 0's are indexed by J, and the inverse system

$$\cdots \to Z/Z_{\beta+} \to Z/Z_\beta \to \cdots \to Z/Z_{\beta_0} = 0$$

to

$$Z_\bullet : \cdots \to Z \to Z \to \cdots \to Z/Z_{\beta+} \to Z/Z_\beta \to \cdots \to Z/Z_{\beta_0} = 0,$$

where the Z's are indexed by I. By these definitions we have an exact sequence of inverse systems

$$0 \to W_\bullet \to V_\bullet \to Z_\bullet \to 0$$

where the index set is countable and $W_\bullet$ satisfies the Mittag-Leffler condition. It follows that

$$0 \to \varprojlim W_\bullet \to \varprojlim V_\bullet \to \varprojlim Z_\bullet \to 0.$$

Since the filtration on W and Z are both complete, we have $W \to \varprojlim W_\bullet$ and $Z \to \varprojlim Z_\bullet$ are isomorphism of topological vector spaces. Therefore

$$V \to \varprojlim V_\bullet$$

is an isomorphism of topological vector spaces by the snake lemma and the filtration on V is complete. □

The situation we encounter in this paper is usually that we have a finite filtration on a Fréchet space and each graded piece admits a complete filtration. Then applying the above lemma several times we see that the whole vector space admits a complete filtration. In other words, to construct a complete filtration on the vector space, it is enough to construct it on each graded piece.

Lemma 2.12. *Let V be a nuclear Fréchet space with a complete filtration V_α, $\alpha \in I$. Let W be a nuclear Fréchet space. Then*

$$V_\alpha \widehat{\otimes} W, \quad \alpha \in I$$

is a complete filtration on $V \widehat{\otimes} W$ whose graded pieces are

$$V_\alpha / V_{\alpha+} \widehat{\otimes} W.$$

Proof. The space V is nuclear Fréchet, so are all its graded pieces. We conclude from Lemma 2.9 that

$$V_\alpha \widehat{\otimes} W, \quad \alpha \in I$$

is indeed a filtration on $V \widehat{\otimes} W$ with the graded pieces described in the lemma. It remains to prove that the filtration is complete. Indeed by the Schwartz kernel theorem [Trè06, (50.17)], we have

$$(V/V_\alpha) \widehat{\otimes} W = L(W^\vee, V/V_\alpha), \quad V \widehat{\otimes} W = L(W^\vee, V),$$

where $W^\vee$ stands for the strong dual of W and L stands for the continuous linear maps. Then

$$V \widehat{\otimes} W = \varprojlim (V/V_\alpha) \widehat{\otimes} W, \tag{2.6}$$

by the definition of the inverse limit. This proves the lemma. □

Proposition 2.13. *Let G be a Nash group and V be a representation of G. Assume that V admits a complete filtration V_α, $\alpha \in I$. Assume that for some i, the Schwartz homologies $H^{\mathcal{S}}_{i+1}(G, V/V_\alpha)$ are finite dimensional for all α. Then the canonical map*

$$H^{\mathcal{S}}_i(G, V) \to \varprojlim H^{\mathcal{S}}_i(G, V/V_\alpha),$$

is a continuous isomorphism of vector spaces.

Proof. This is [Gro61, Proposition 13.2.3], with cochain complexes $K^\bullet_\alpha$ in [Gro61, Proposition 13.2.3] replaced by chain complexes $P_{\alpha,\bullet}$ where

$$P_{\alpha,\bullet} \to V/V_\alpha \to 0$$

is a strong projective resolution. Recall that a inverse system

$$\cdots \to U_{\alpha+} \to U_\alpha \to \cdots$$

satisfies the Mittag-Leffler condition if for any $\alpha \in I$ there is some $\beta \geq \alpha$ so that the image of

$$U_\gamma \to U_\alpha$$

are the same for all $\gamma \geq \beta$. We note that the system

$$\cdots \to V/V_{\alpha+} \to V/V_\alpha \to \cdots$$

satisfies the Mittag-Leffler condition since the maps are all surjective. The system

$$\cdots \to H^{\mathcal{S}}_{i+1}(G, V/V_{\alpha+}) \to H^{\mathcal{S}}_{i+1}(G, V/V_\alpha) \to \cdots$$

also satisfies the Mittag-Leffler condition as they are all finite dimensional by assumption. The proof of [Gro61, Proposition 13.2.3] can now be copied word by word. □

Corollary 2.14. *Let G be a Nash group and V, W be representations of G. Assume that V and W are both nuclear. Assume that V admits a complete filtration V_α, $\alpha \in I$, and*

$$H^{\mathcal{S}}_i(G, (V_\alpha/V_{\alpha+}) \widehat{\otimes} W) = 0$$

for $\alpha \in I$. Then

$$H^{\mathcal{S}}_i(G, V \widehat{\otimes} W) = 0.$$

Proof. As W and V are nuclear Fréchet space, by Lemma 2.12,

$$V_\alpha \widehat{\otimes} W, \quad \alpha \in I$$

is a complete filtration on $V \widehat{\otimes} W$ whose graded pieces are

$$V_\alpha/V_{\alpha+} \widehat{\otimes} W, \quad \alpha \in I.$$

Moreover

$$H^{\mathcal{S}}_i(G, (V/V_\alpha) \widehat{\otimes} W) = 0,$$

for all i and all $\alpha \in I$ by the usual dimension shifting argument via the long exact sequence (2.4). The we obtain the corollary from Proposition 2.13. □

Lemma 2.15. *Let G be a Nash group and H be a closed Nash subgroup. Let V be a representation of H that admits an H-stable complete filtration V_α, $\alpha \in I$. Assume that V is nuclear. Then*

$$\operatorname{ind}_H^G V_\alpha, \quad \alpha \in I$$

is a G-stable complete filtration on $\operatorname{ind}_H^G V$ whose graded pieces are

$$\operatorname{ind}_H^G V_\alpha / V_{\alpha+}, \quad \alpha \in I.$$

Proof. We only need to show that the filtration $\operatorname{ind}_H^G V_\alpha$ is complete, all other assertions follow from Proposition 2.1, i.e. Schwartz induction is an exact functor.

By [CS, Proposition 6.5], Schwartz induction has the following equivalent interpretation. Let H act on $\mathcal{S}(G,V)$ by

$$h \cdot f(g) = h \cdot f(h^{-1}g), \quad f \in \mathcal{S}(G,V), \quad h \in H, g \in G,$$

then

$$\operatorname{ind}_H^G V = (\mathcal{S}(G,V) \otimes \delta_H^{-1})_H,$$

where δ_H is the modulus character of H. Since V is a nuclear Fréchet space, so is $\mathcal{S}(G,V)$. We have

$$\mathcal{S}(G,V) = \mathcal{S}(G) \widehat{\otimes} V = \mathcal{S}(G) \widehat{\otimes} \left(\varprojlim V/V_\alpha\right) = \varprojlim \left(\mathcal{S}(G) \widehat{\otimes} (V/V_\alpha)\right) = \varprojlim \mathcal{S}(G, V/V_\alpha)$$

by Lemma 2.12. Note that $\mathcal{S}(G, V/V_\alpha) \otimes \delta_H^{-1}$, as a representation of H, is relatively projective by [CS, Proposition 5.2]. Therefore all Schwartz homologies of $\mathcal{S}(G, V/V_\alpha) \otimes \delta_H^{-1}$ vanish except for the zeroth homology. By Proposition 2.13, we conclude that

$$(\mathcal{S}(G,V) \otimes \delta_H^{-1})_H = \left(\varprojlim \mathcal{S}(G, V/V_\alpha) \otimes \delta_H^{-1}\right)_H \to \varprojlim \left((\mathcal{S}(G, V/V_\alpha) \otimes \delta_H^{-1})_H\right)$$

is a continuous isomorphism of vector spaces. All spaces involved are Fréchet spaces by [CS, Theorem 5.9], so by the open mapping theorem, this is an isomorphism of topological spaces. Thus

$$\operatorname{ind}_H^G V = \varprojlim \left(\operatorname{ind}_H^G V/V_\alpha\right).$$

This proves the lemma. □

3. Action of the universal enveloping algebra

We recall some results on the action of the universal enveloping algebra on parabolically induced representations. It is well-known that parabolic induction preserves infinitesimal characters. In our later applications the inducing data usually do not have infinitesimal characters. We will however construct elements in the center of universal enveloping algebra which annihilate these parabolic induced representations. This construction will be used repeatedly later. Together with Corollary 2.8, they provide the key tools of deducing vanishing results of Schwartz homologies.

3.1. **Complex general linear groups.** Let us first consider the character $\omega_l|\cdot|_{\mathbb{C}}^u$ of $\mathbb{C}^\times$. The Lie algebra of $\mathbb{C}^\times$ is $\mathbb{C}$ and its complexification is identified with $\mathbb{C}\oplus\mathbb{C}$ where the Lie algebra of $\mathbb{C}^\times$ is a Lie subalgebra consisting of $(x, -\overline{x})$. View the character $\omega_l|\cdot|_{\mathbb{C}}^u$ as a one dimensional representation. Direct computation shows that the complexified Lie algebra $\mathbb{C}\oplus\mathbb{C}$ acts as follows. The element $(1,-1)$ acts as multiplication by $2u$ and $(\sqrt{-1},\sqrt{-1})$ acts as multiplication by $\sqrt{-1}l$. Thus the infinitesimal character of this representation is

$$\left(\frac{l+2u}{2}, \frac{l-2u}{2}\right) \in \mathbb{C}\oplus\mathbb{C},$$

if we naturally identify the dual space of $\mathbb{C}\oplus\mathbb{C}$ with itself.

Let $L = \operatorname{Res}_{\mathbb{C}/\mathbb{R}} \mathrm{GL}_r(\mathbb{C})$ and $\mathfrak{l} = \operatorname{Lie}(L)$. Let T be the diagonal torus and $\mathfrak{t} = \operatorname{Lie}(T)$ the Lie algebra. We identify $\mathfrak{l}_{\mathbb{C}}$ with $\mathfrak{gl}_n(\mathbb{C})\oplus\mathfrak{gl}_n(\mathbb{C})$ so that $\mathfrak{l}$ is a Lie subalgebra consisting of matrices

$$(X, -w_r\overline{{}^tX}w_r), \quad X \in M_{r\times r}(\mathbb{C})$$

where w_r is the anti-diagonal matrix whose anti-diagonal entries are all 1's. The complexified Cartan subalgebra $\mathfrak{t}_{\mathbb{C}}$ consists of

$$\left(\begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_r\end{pmatrix}, \begin{pmatrix} b_1 & & \\ & \ddots & \\ & & b_r\end{pmatrix}\right), \quad a_1, \cdots, a_r, b_1, \cdots, b_r \in \mathbb{C}.$$

The Weyl group is isomorphic to $\mathfrak{S}_r \times \mathfrak{S}_r$ and the two factors acts by permuting the a_i's and b_i's respectively. By the Harish-Chandra isomorphism, $\mathcal{Z}(\mathfrak{l}_{\mathbb{C}})$ is isomorphic to

$$\mathbb{C}[x_1,\cdots,x_r]^{\mathfrak{S}_r} \otimes \mathbb{C}[y_1,\cdots,y_r]^{\mathfrak{S}_r}.$$

Let $l_1,\cdots,l_r$ be integers and $s_1,\cdots,s_r$ be complex numbers. We consider the principal series representation

$$\tau = \omega_{l_1}|\cdot|_{\mathbb{C}}^{s_1} \times \cdots \times \omega_{l_r}|\cdot|_{\mathbb{C}}^{s_r},$$

induced from the usual upper triangular Borel subgroup.

Let $A = \mathbb{C}^\times \times\cdots\times\mathbb{C}^\times$ be the diagonal torus. In the above choice of coordinates, the infinitesimal character of the character $\omega_{l_1}|\cdot|_{\mathbb{C}}^{s_1} \otimes\cdots\otimes \omega_{l_r}|\cdot|_{\mathbb{C}}^{s_r}$ of A is

$$\left(\frac{l_1+2s_1}{2},\cdots,\frac{l_r+2s_r}{2},\frac{l_r-2s_r}{2},\cdots,\frac{l_1-2s_1}{2}\right). \tag{3.1}$$

It is well-known that parabolic induction preserves infinitesimal characters. Thus the infinitesimal character of τ is an $\mathfrak{S}_r\times\mathfrak{S}_r$-orbit of (3.1) in $\mathbb{C}^{2r}$, where two $\mathfrak{S}_r$'s act as permuting the first r coordinates and the last r coordinates respectively.

Let ξ be an irreducible finite dimensional representation of L. Let $\Delta(\xi)$ be the set of all weights appearing in ξ. Then $\Delta(\xi)$ is a finite subset of $\mathbb{C}^{2r}$, stable under the action of $\mathfrak{S}_r\times\mathfrak{S}_r$. The representation $\tau\otimes\xi$ itself might not admit an infinitesimal character if $\dim\xi \geq 2$. But by [Kos75, Corollary 5.6], it has a finite length filtration whose graded pieces have infinitesimal characters and all possible infinitesimal characters are of the form

(3.2)
$$(c_1, \cdots, c_r, d_1, \cdots, d_r) = \left(\frac{l_1 + 2s_1}{2} + a_1, \cdots, \frac{l_r + 2s_r}{2} + a_r, \frac{l_r - 2s_r}{2} + b_1, \cdots, \frac{l_1 - 2s_1}{2} + b_r\right),$$

where $(a_1, \cdots, a_r, b_1, \cdots, b_r) \in \Delta(\xi)$.

Let

$$p(x_1, \cdots, x_r, y_1, \cdots, y_r) \in \mathbb{C}[x_1, \cdots, x_r, y_1, \cdots, y_r]^{\mathfrak{S}_r \times \mathfrak{S}_r}$$

be a polynomial, symmetric in $x_1, \cdots, x_r$ and $y_1, \cdots, y_r$ respectively. Then by [Kos75, Theorem 5.1], the element in $\mathcal{Z}(\mathfrak{l}_\mathbb{C})$ given by

$$\alpha = \prod_{(a_1, \cdots, a_r, b_1, \cdots, b_r) \in \Delta(\xi)} (p(x_1, \cdots, x_r, y_1, \cdots, y_r) - p(c_1, \cdots, c_r, d_1, \cdots, d_r)), \tag{3.3}$$

where $(c_1, \cdots, c_r, d_1, \cdots, d_r)$ is given by (3.2), annihilates $\tau \otimes \xi$.

3.2. **Parabolic inductions.** Let us consider the following setup.

- Let V be a hermitian space of dimension n and V_0 a subspace whose orthogonal complement has a basis
$$v_{\pm 1}, \cdots, v_{\pm r},$$
where
$$h_V(v_i, v_{-j}) = \delta_{ij}.$$
- Put $X = \langle v_1, \cdots, v_r \rangle$ and $X^\vee = \langle v_{-1}, \cdots, v_{-r} \rangle$.
- Let $G = \mathrm{U}(V)$. Let $P = MU$ be the parabolic subgroup of G stabilizing X. Here and below the notation $P = MU$ always mean that M is a Levi subgroup and U is the unipotent radical. Then
$$M \simeq \mathrm{GL}_r(\mathbb{C}) \times \mathrm{U}(V_0)$$
where we have identified $\mathrm{GL}(X)$ with $\mathrm{GL}_r(\mathbb{C})$ using the basis $v_1, \cdots, v_r$ of X. We write $L = \mathrm{Res}_{\mathbb{C}/\mathbb{R}} \mathrm{GL}_r(\mathbb{C})$.
- Let $l_1, \cdots, l_r \in \mathbb{Z}$ and $s_1, \cdots, s_r \in \mathbb{C}$. Let τ be the principal series representation
$$\omega_{l_1} |\cdot|^{s_1} \times \cdots \times \omega_{l_r} |\cdot|^{s_r},$$
induced from the usual upper triangular Borel subgroup.
- Let ξ be a finite dimensional representation of L.
- Let σ be a representation of $\mathrm{U}(V_0)$, not necessarily irreducible.
- Let
$$\pi = \mathrm{Ind}_P^G \left((\tau \otimes \xi) \widehat{\otimes} \sigma \right)$$
be a representation of G.

Let us choose a basis $e_1, \cdots, e_{n-2t}$ of V_0 so that the matrix representing the hermitian form of V_0 is $\begin{pmatrix} 1_p & \\ & -1_q \end{pmatrix}$ where (p, q) is the signature of $\mathrm{U}(V_0)$. We use the basis

$$v_1, \cdots, v_r, e_1, \cdots, e_{n-2t}, v_{-r}, \cdots, v_{-1}$$

of V to realize G and its subgroups as matrix groups. Under this basis P corresponds to the blocked upper triangular matrices.

Let T be the diagonal torus of G and $\mathfrak{t} = \mathrm{Lie}(T)$ which consists of diagonal matrices

$$\begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{pmatrix}$$

with the properties that $a_i \in \mathbb{C}$, $\overline{a_i} = -a_{2n+1-i}$, $i = 1, \cdots, r$ and $a_{r+j} \in \sqrt{-1}\mathbb{R}$, $j = 1, \cdots, n-2r$. The complexification $\mathfrak{t}_{\mathbb{C}}$ is an n-dimensional vector space consisting of diagonal matrices and the Weyl group acts by permuting these entries. Then $\mathcal{Z}(\mathfrak{g}_{\mathbb{C}})$ is identified as usual with

$$\mathbb{C}[x_1, \cdots, x_n]^{\mathfrak{S}_n}$$

where $\mathfrak{S}_n$ acts by permuting the x_i's.

From the proof of [Kna01, Proposition 8.22], we have

$$\mathcal{Z}(\mathfrak{g}_{\mathbb{C}}) \subset \mathfrak{u}_{\mathbb{C}}\mathcal{U}(\mathfrak{g}_{\mathbb{C}}) \oplus \mathcal{U}(\mathfrak{m}_{\mathbb{C}}), \tag{3.4}$$

where we recall that $P = MU$, $\mathfrak{m} = \mathrm{Lie}(M)$ and $\mathfrak{u} = \mathrm{Lie}(U)$. Define $\mu : \mathcal{Z}(\mathfrak{g}_{\mathbb{C}}) \to \mathcal{U}(\mathfrak{m}_{\mathbb{C}})$ to be the projection to the $\mathcal{U}(\mathfrak{m}_{\mathbb{C}})$ component. Then the image of μ lies in $\mathcal{Z}(\mathfrak{m}_{\mathbb{C}})$ and thus μ gives a well-defined algebra homomorphism

$$\mu : \mathcal{Z}(\mathfrak{g}_{\mathbb{C}}) \to \mathcal{Z}(\mathfrak{m}_{\mathbb{C}}).$$

Moreover by [Kna01, (8.34)] this algebra homomorphism is compatible with the Harish-Chandra isomorphism. More precisely Harish-Chandra isomorphism gives

$$\mathcal{Z}(\mathfrak{g}_{\mathbb{C}}) \simeq \mathbb{C}[x_1, \cdots, x_n]^{\mathfrak{S}_n}, \quad \mathcal{Z}(\mathfrak{m}_{\mathbb{C}}) \simeq \mathbb{C}[x_1, \cdots, x_n]^{\mathfrak{S}_r \times \mathfrak{S}_{n-2r} \times \mathfrak{S}_r},$$

where in the second isomorphism the three groups $\mathfrak{S}_r$, $\mathfrak{S}_{n-2r}$ and $\mathfrak{S}_r$ act as permuting $x_1, \cdots, x_r$, and $x_{r+1}, \cdots, x_{n-r}$, and $x_{n-r+1}, \cdots, x_{2n}$ respectively. The homomorphism μ is then given by the natural inclusion

$$\mathbb{C}[x_1, \cdots, x_n]^{\mathfrak{S}_n} \to \mathbb{C}[x_1, \cdots, x_n]^{\mathfrak{S}_r \times \mathfrak{S}_{n-2r} \times \mathfrak{S}_r}.$$

We have constructed in (3.3) for every

$$p(x_1, \cdots, x_r, x_{n-r+1}, \cdots, x_n) \in \mathbb{C}[x_1, \cdots, x_r, x_{n-r+1}, \cdots, x_n]^{\mathfrak{S}_r \times \mathfrak{S}_r},$$

an element α in $\mathcal{Z}(\mathfrak{m}_{\mathbb{C}})$ that annihilates $\tau \otimes \xi$. Let us put

$$z = \prod_{\gamma \in \mathfrak{S}_n / \mathfrak{S}_r \times \mathfrak{S}_{n-2r} \times \mathfrak{S}_r} \gamma \cdot \alpha. \tag{3.5}$$

Then z gives rise to an element in $\mathcal{Z}(\mathfrak{g}_{\mathbb{C}})$.

Lemma 3.1. *The element z constructed in (3.5) annihilates π.*

Proof. Let us first observe that $\mu(z)$ annihilates $\tau \otimes \xi$. Indeed we can write

$$z = z'\alpha, \quad z' = \prod_{\gamma \neq 1} \gamma \cdot \alpha,$$

where $\gamma \neq 1$ means that γ is not the coset represented by 1. Note that both z' and α are invariant under the action of $\mathfrak{S}_r \times \mathfrak{S}_{n-2r} \times \mathfrak{S}_r$, and thus are elements in $\mathcal{Z}(\mathfrak{m}_{\mathbb{C}})$. So $\mu(z)$ is the product of α and another element in $\mathcal{Z}(\mathfrak{m}_{\mathbb{C}})$, and hence annihilates $\tau \otimes \xi$ because α does so.

Let f be an element in π. Since $X \cdot f(1) = 0$ for all $X \in \mathfrak{u}_{\mathbb{C}}$, we conclude that $X \cdot f(1) = 0$ for all $X \in \mathfrak{u}_{\mathbb{C}}\mathcal{U}(\mathfrak{g}_{\mathbb{C}})$. It follows from (3.4) that if $z \in \mathcal{Z}(\mathfrak{g}_{\mathbb{C}})$, then

$$(z \cdot f)(1) = \mu(z) \cdot (f(1)),$$

for all $f \in \pi$. As $z \in \mathcal{Z}(\mathfrak{g}_{\mathbb{C}})$, for all $g \in G$ we have

$$z \cdot f(g) = \pi(g)(z \cdot f)(1) = z \cdot (\pi(g)f)(1) = \mu(z) \cdot (\pi(g)f)(1) = 0$$

for all $f \in \pi$ and $g \in G$. Thus z annihilates π. □

Lemma 3.2. *Choose the polynomial p as*

$$x_1 + x_2 + \cdots + x_r.$$

Let z be the element constructed as in (3.5) from p. Let π' be an irreducible representation of G and z acts on π' by a constant λ. Then λ is a nonconstant polynomial function in $s_1, \cdots, s_r$.

Proof. The element z constructed in (3.5) with this choice of the p is of the form

$$\prod_{\sigma \in \mathfrak{S}_n / \mathfrak{S}_r \times \mathfrak{S}_{n-2r} \times \mathfrak{S}_r} \prod_{(a_1, \cdots, a_r, b_1, \cdots, b_r) \in \Delta(\xi)} \left(x_{\sigma(1)} + \cdots + x_{\sigma(r)} - \frac{l_1 + 2s_1}{2} - \cdots - \frac{l_r + 2s_r}{2} - a_1 - \cdots - a_r \right).$$

For any fixed value of $x_1, \cdots, x_n$, it is clearly a nonconstant polynomial in $s_1, \cdots, s_r$. □

4. Spherical models

The goal of this section is to prove Theorem 1.1 in the case $t = 0$.

4.1. **A filtration on parabolic induction.** Let us consider the following setup.

- Let $W \subset V$ be a relevant pair of hermitian space of dimensions n and $n+1$ respectively. Recall that this means that V/W is a line of sign $(-1)^n$.
- Assume that W is isotropic. Choose a decomposition $W = W_0 \oplus^{\perp} (E \oplus F)$ where E and F are isotropic lines and $E \oplus F$ is a hyperplane perpendicular to W_0.
- Let L be a line of sign $(-1)^n$ in $E \oplus F$. Put $V_0 = W_0 \oplus^{\perp} L$.

- We have

$$W_0 \subset V_0 \subset W \subset V,$$

and each successive inclusion gives a relevant pair of hermitian spaces of codimension one.
- Put $G = \mathrm{U}(V)$, $H = \mathrm{U}(W)$, $G_0 = \mathrm{U}(V_0)$, and $H_0 = \mathrm{U}(W_0)$.
- Let $P \subset G$ and $Q = M_Q U_Q \subset H$ be the respective parabolic subgroups stabilizing E. The Levi subgroup M_Q of Q is isomorphic to $\mathbb{C}^\times \times H_0$ where we identify $\mathrm{GL}(E)$ with $\mathbb{C}^\times$ naturally.

We have an embedding of flag varieties

$$\mathcal{Z} = Q\backslash H \to \mathcal{X} = P\backslash G.$$

There are two H-orbits in $\mathcal{X}$, the closed orbit $\mathcal{Z}$ consisting of all isotropic lines contained in W. Its complement $\mathcal{U}$ is the open orbit which consists of all isotropic lines not contained in W.

The complexified conormal bundle $\mathcal{N}^\vee_{\mathcal{Z}/\mathcal{X}}$ is a tempered H-bundle over $\mathcal{Z}$. To describe it, we only need to describe the fiber of it over the point $[E]$ of $\mathcal{Z}$ represented by $E \subset W$, as a representation of Q. We remind the readers that G acts on $\mathcal{X}$ from the right and thus the fiber over $[E]$ is a vector space with a right Q action. The representation of Q that this bundle gives rise to and this right action are connected by (2.3).

Lemma 4.1. *The fiber of over $[E]$ of the $\mathcal{N}^\vee_{\mathcal{Z}/\mathcal{X}}$ is the two dimensional representation of Q where U_Q and $\mathrm{U}(W_0)$ acts trivially and $\mathbb{C}^\times$ acts as*

$$z \mapsto \begin{pmatrix} z & \\ & \overline{z} \end{pmatrix}.$$

Proof. Let us first show that that U_Q acts trivially. We take a basis of W_0

$$e_1, \cdots, e_{n-2},$$

a nonzero vector e_0 of L with $\langle e_0, e_0\rangle = 1$, a nonzero element f_1 of E and a nonzero element f_{-1} of F so that $\langle f_1, f_{-1}\rangle = 1$. In terms of the basis

$$f_1, e_1, \cdots, e_{n-2}, e_0, f_{-1}$$

of V we write elements in $\mathrm{U}(V)$ as matrices. Then $\mathrm{U}(W)$ consists of elements of the form

$$\begin{pmatrix} a & & b \\ & 1 & \\ c & & d \end{pmatrix}, \quad a \in M_{(n-1)\times(n-1)}(\mathbb{C}), \quad b, {}^t c \in \mathbb{C}^{n-1}, \quad d \in \mathbb{C}.$$

Let ${}^tU_P$ be the unipotent radical of the opposite parabolic subgroup of P. Then $P\backslash P\,{}^tU_P$ is the open cell in the flag variety $\mathcal{X}$. Take

$$g = \begin{pmatrix} 1 & v^* & 0 & t \\ & 1_{n-2} & & v \\ & & 1 & 0 \\ & & & 1 \end{pmatrix} \in U_Q, \quad x = \begin{pmatrix} 1 & & & \\ 0 & 1_{n-2} & & \\ x & & 1 & \\ \frac{1}{2}xx^* & 0 & x^* & 1 \end{pmatrix} \in {}^tU_P.$$

Here $v \in \mathbb{C}^{n-2}$, $t, x \in \mathbb{C}$, and v^*, x^* are the corresponding elements which make g and u lie in $\mathrm{U}(V)$. Elementary computation gives that the xg and

$$\begin{pmatrix} 1 & & & \\ * & 1_{n-2} & & \\ \frac{x}{1+\frac{1}{2}xx^*t} & & 1 & \\ * & * & * & 1 \end{pmatrix}$$

represent the same point on $\mathcal{X}$. Taking derivative at $x = 0$ of $\frac{x}{1+\frac{1}{2}xx^*t}$ with respect to x yields that the right action of g on the conormal space at the point $[E] \in Z$ is trivial.

The action of $\mathrm{U}(W_0)$ being trivial is straightforward. From the coordinates above we see that this right action of $\mathbb{C}^\times$ is given by that

$$\begin{pmatrix} z & & \\ & 1_{n-2} & \\ & & \overline{z}^{-1} \end{pmatrix}$$

acts as $\begin{pmatrix} z & \\ & \overline{z} \end{pmatrix}^{-1}$. Then by (2.3) the representation on this vector space is given by $\begin{pmatrix} z & \\ & \overline{z} \end{pmatrix}$. □

Let π_0 be an irreducible representation of G_0 and χ a character of $\mathbb{C}^\times$. Consider the parabolic induction

$$\pi = \mathrm{Ind}_P^G(\chi|\cdot|_{\mathbb{C}}^u \otimes \pi_0).$$

Let π° be H-invariant subspace consisting of Schwartz sections over $\mathcal{U}$. Then as representations of H we have

$$\pi^\circ \simeq \mathrm{ind}_{G_0}^H \pi_0. \tag{4.1}$$

By Proposition 2.5, the quotient $\pi_{\mathcal{Z}} = \pi/\pi^\circ$ has a decreasing complete filtration $\pi_{\mathcal{Z},k}$, $k = 0, 1, 2, \cdots$, whose graded pieces, as representations of H, are direct sums of

$$\mathrm{Ind}_Q^H\left(\chi\omega_{2j-k}|\cdot|_{\mathbb{C}}^{u+\frac{k+1}{2}} \otimes \pi_0|_{H_0}\right), \quad j = 0, \cdots, k. \tag{4.2}$$

The factors $\omega_{2j-k}|\cdot|_{\mathbb{C}}^{\frac{k}{2}}$ all come from symmetric powers of the complexified conormal bundle. The extra factor $|\cdot|_{\mathbb{C}}^{\frac{1}{2}}$ comes from the comparison of the modulus characters of P and Q.

4.2. **Induction and multiplicity.** Let σ be a representation of H of the form

$$\operatorname{Ind}_{Q_b}^{H}\left(\omega_{m_1}|\cdot|_{\mathbb{C}}^{t_1}\otimes\cdots\otimes\omega_{m_b}|\cdot|_{\mathbb{C}}^{t_b}\otimes\sigma_b\right), \tag{4.3}$$

where

- $W_b\subset W$ be a hermitian space so that its orthogonal complement is a split hermitian space of dimension $2b$,
- Q_b is a parabolic subgroup of H so that its Levi component is isomorphic to $(\mathbb{C}^\times)^b\times \mathrm{U}(W_b)$,
- $m_1,\cdots,m_b\in\mathbb{Z}$ and $t_1,\cdots,t_b$ are complex numbers with nonnegative real parts,
- σ_b is an irreducible limit of discrete series representation of $\mathrm{U}(W_b)$.

Let

$$\rho=\operatorname{Ind}_{Q}^{H}\left(\omega_l|\cdot|_{\mathbb{C}}^{u}\otimes\rho_0\right),$$

where $l\in\mathbb{Z}$, $u\in\mathbb{C}$ with $\operatorname{Re}u\geq 0$ and ρ_0 is a representation of H_0.

Lemma 4.2. *If either $l+2u$ or $l-2u$ does not equal to any $-m_i\pm 2t_i$, $i=1,\cdots,b$, and if $l+2u$ (and hence $l-2u$) is not an integer with the same parity as $n+1$, then*

$$H_i^{\mathcal{S}}(H,\rho\,\widehat{\otimes}\,\sigma)=0$$

for all i.

Proof. We will construct an element $z\in\mathcal{Z}(\mathfrak{h}_{\mathbb{C}})$ so that it annihilates ρ but not $\sigma^\vee$. We then conclude by Corollary 2.8.

Let choose the Cartan subgroup $\mathfrak{t}$ of $\mathfrak{h}$ and identify $\mathcal{Z}(\mathfrak{h}_{\mathbb{C}})$ with

$$\mathbb{C}[x_1,\cdots,x_n]^{\mathfrak{S}_n},$$

as in Subsection 3.2. The infinitesimal character of $\sigma^\vee$ is

$$\left(\frac{-m_1+2t_1}{2},\cdots,\frac{-m_b+2t_b}{2};\mu_1,\cdots,\mu_{n-2b};\frac{-m_b-2t_b}{2},\cdots,\frac{-m_1-2t_1}{2}\right),$$

where $(\mu_1,\cdots,\mu_{n-2b})$ is the infinitesimal character of $\sigma_b^\vee$. In particular $\mu_i\in\frac{n+1}{2}+\mathbb{Z}$.

Let $\mathfrak{m}$ be the Lie algebra of M_Q. We apply the construction in Subsection 3.2 and in particular (3.5) to the current situation, i.e. $r=1$, $\tau=\omega_l|\cdot|^u$, ξ being trivial. Take the polynomial p there to be either $p(x_1,\cdots,x_n)=x_1$ or $p(x_1,\cdots,x_n)=x_n$. Then we conclude that

$$z_1=\prod_{i=1}^{n}\left(x_i-\frac{l+2u}{2}\right),\quad z_2=\prod_{i=1}^{n}\left(x_i-\frac{l-2u}{2}\right)$$

both being in $\mathcal{Z}(\mathfrak{h}_{\mathbb{C}})$, annihilate ρ. Let us prove that at least one of them does not annihilate $\sigma^\vee$. First we note that

$$\frac{l\pm 2u}{2}\neq\mu_j$$

for any j, as $\mu_j\in\frac{n+1}{2}+\mathbb{Z}$. If both z_1 and z_2 annihilate $\sigma^\vee$, we have

$$\frac{l+2u}{2}=\frac{-m_i\pm 2t_i}{2}$$

for some i, and

$$\frac{l-2u}{2} = \frac{-m_j \pm 2t_j}{2}$$

for some j. But at least one of them does not happen by the assumptions of the lemma. □

Put

$$\pi = \mathrm{Ind}_P^G \left(\omega_l |\cdot|_{\mathbb{C}}^u \otimes \pi_0\right).$$

and assume that $l+2u$ (and hence $l-2u$) is not an integer having the same parity with $n+1$.

Proposition 4.3. *Assume that $l+2u$ is not an integer with the same parity as n. Assume that for any nonnegative integer j, either*

$$l + 2u + 2j + 1 \neq -m_i \pm 2t_i, \quad i = 1, \cdots, b$$

or

$$l - 2u - 2j - 1 \neq -m_i \pm 2t_i, \quad i = 1, \cdots, b.$$

Then

$$m(\pi, \sigma) = m(\sigma, \pi_0).$$

Proof. Let π° be the subspace of Schwartz sections in π over the open H-orbit. By (4.2), the quotient $\pi_{\mathcal{Z}} = \pi/\pi^\circ$ has an H-stable complete filtration $\pi_{\mathcal{Z},k}$, $k = 0, 1, 2, \cdots$, whose graded pieces are

$$\rho_k^j = \mathrm{Ind}_Q^H \left(\omega_{2j-k+l} |\cdot|^{u+\frac{1+k}{2}} \otimes \pi_0|_{H_0}\right), \quad j = 0, 1, \cdots, k, \quad k = 0, 1, 2, \cdots.$$

By assumption we have either

$$l + 2j - k + (2u + 1 + k) = l + 2u + 2j + 1 \neq -m_i \pm 2t_i, \quad i = 1, \cdots, b.$$

or

$$l + 2j - k - (2u + 1 + k) = l - 2u + 2(j-k) - 1 \neq -m_i \pm 2t_i, \quad i = 1, \cdots, b.$$

Then by Lemma 4.2, for all $i = 0, 1, 2, \cdots$, $k = 0, 1, \cdots$, and $j = 0, 1, \cdots, k$, we have

$$H_i^{\mathcal{S}}(H, \rho_k^j \widehat{\otimes} \sigma) = 0. \tag{4.4}$$

It follows that

$$H_i^{\mathcal{S}}\left(H, \pi_Z/\pi_{Z,k} \widehat{\otimes} \sigma\right) = 0$$

for all k. As $\pi_Z/\pi_{Z,k}$ and σ are all nuclear Fréchet spaces, by Corollary 2.14, we have

$$H_i^{\mathcal{S}}(H, \pi_Z \widehat{\otimes} \sigma) = 0, \quad i = 0, 1, 2, \cdots.$$

Therefore we conclude that

$$H_i^{\mathcal{S}}(H, \pi \widehat{\otimes} \sigma) = H_i^{\mathcal{S}}(H, \pi^\circ \widehat{\otimes} \sigma), \quad i = 0, 1, \cdots$$

As in (4.1) we have

$$\pi^\circ \simeq \mathrm{ind}_{G_0}^H \pi_0.$$

Therefore by Propositions 2.3 and Shapiro's lemma we conclude

$$H_i^{\mathcal{S}}(H, \pi^\circ \widehat{\otimes} \sigma) = H_i^{\mathcal{S}}(H, \mathrm{ind}_{G_0}^{H}(\pi_0 \widehat{\otimes}(\sigma|_{G_0}))) = H_i^{\mathcal{S}}(G_0, \pi_0 \widehat{\otimes} \sigma),$$

for all $i \geq 0$. In particular when $i = 0$, we conclude that

$$(\pi \widehat{\otimes} \sigma)_H = (\sigma \widehat{\otimes} \pi_0)_{G_0}$$

and so are their maximal Hausdorff quotient. It follows that

$$m(\pi, \sigma) = m(\sigma, \pi_0).$$

This proves the proposition. □

4.3. **Spherical models.** Let π be an irreducible representation of G and assume that π lies in a generic packet. Then π can be written as an irreducible parabolic induction

$$\omega_{l_1}|\cdot|_{\mathbb{C}}^{s_1} \times \cdots \times \omega_{l_a}|\cdot|_{\mathbb{C}}^{s_a} \times \pi_0, \tag{4.5}$$

where

- $l_1, \cdots, l_a \in \mathbb{Z}$,
- $s_1, \cdots, s_a \in \mathbb{C}$ with nonnegative real part,
- $\omega_{l_i}|\cdot|_{\mathbb{C}}^{s_i}$ is not conjugate self-dual of sign $(-1)^n$,
- $V_a \subset V$ is a hermitian subspace such that $V_a^\perp$ is a split hermitian space of dimension $2a$,
- π_0 is a limit of discrete series representation of $\mathrm{U}(V_a)$.

Lemma 4.4. *Assume that π is an irreducible representation of G that lies in a generic packet as above. Then $l_i \pm 2s_i$ is not an integer having same parity with n for any $i = 1, \cdots, a$.*

Proof. The L-parameter ϕ_π of π is

$$\phi_{\pi_0} \oplus \bigoplus_{i=1}^{a} \left(\omega_{l_i}|\cdot|_{\mathbb{C}}^{s_i} \oplus \omega_{l_i}|\cdot|_{\mathbb{C}}^{-s_i}\right),$$

where ϕ_{π_0} is the L-parameter of π_0. We recall that Π_{ϕ_π} is a generic packet if and only if

$$L\left(s, \phi_\pi, \mathrm{As}^{(-1)^{n+1}}\right)$$

is holomorphic at $s = 1$ (note $\dim V = n + 1$). This Asai L-function is a product of

$$\prod_{i=1}^{a} L_{\mathbb{R}}(s + 2s_i, \mathrm{sgn}^{l_i+n+1}) L_{\mathbb{R}}(s - 2s_i, \mathrm{sgn}^{l_i+n+1})$$

and some other local L-functions. Recall that

$$L_{\mathbb{R}}(s) = \pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right), \quad L_{\mathbb{R}}(s, \eta) = \pi^{-\frac{s+1}{2}} \Gamma\left(\frac{s+1}{2}\right),$$

and the poles of $\Gamma(s)$ are at $s = 0, -1, -2, \cdots$. Thus if $l_i + n + 1$ is even, then

$$L_{\mathbb{R}}(1 + 2s_i, \mathrm{sgn}^{l_i+n+1}) L_{\mathbb{R}}(1 - 2s_i, \mathrm{sgn}^{l_i+n+1}) = \pi^{-1} \Gamma\left(\frac{1}{2} + s_i\right) \Gamma\left(\frac{1}{2} - s_i\right).$$

It is holomorphic precisely when $s_i \notin \frac{1}{2} + \mathbb{Z}$. Similarly if $l_i + n + 1$ is odd, then

$$L_{\mathbb{R}}(1 + 2s_i, \mathrm{sgn}^{l_i+n+1}) L_{\mathbb{R}}(1 - 2s_i, \mathrm{sgn}^{l_i+n+1}) = \pi^{-2} \Gamma\left(1 + s_i\right) \Gamma\left(1 - s_i\right).$$

It is holomorphic precisely when s_i is a nonzero integer. Moreover s_i cannot be zero in this case since otherwise $\omega_{l_i} |\cdot|^{s_i}$ would be a conjugate self-dual character of sign $(-1)^n$. This proves the lemma. □

Let σ be an irreducible representation of H lying in a generic packet. We can then write σ as a parabolic induction

$$\sigma = \omega_{m_1} |\cdot|_{\mathbb{C}}^{t_1} \times \cdots \times \omega_{m_b} |\cdot|_{\mathbb{C}}^{t_b} \times \sigma_0,$$

where

- $m_1, \cdots, m_b \in \mathbb{Z}$,
- $t_1, \cdots, t_b \in \mathbb{C}$ with nonnegative real part,
- $\omega_{m_i} |\cdot|_{\mathbb{C}}^{t_i}$ is not conjugate self-dual of sign $(-1)^{n-1}$,
- $W_b \subset W$ is a hermitian subspace such that $W_b^{\perp}$ is a split hermitian space of dimension $2b$,
- σ_0 is a limit of discrete series representation of $\mathrm{U}(W_b)$.

In this case we have

$$m(\pi, \sigma) = \dim \mathrm{Hom}_H(\pi \mathbin{\widehat{\otimes}} \sigma, \mathbb{C}).$$

Proof of Theorem 1.1 assuming $t = 0$. Recall that Theorem 1.1 says that $m(\pi, \sigma) = 1$ if and only if the L-parameters of π and σ are given as in the theorem. The strategy is to make use of Proposition 4.3 repeatedly to replace the characters $\omega_{l_i} |\cdot|^{s_i}$'s and $\omega_{m_i} |\cdot|^{t_i}$'s by unitary characters. Then we are reduced to the tempered case proved in [Xue].

By relabeling, we may assume that

$$\mathrm{Re}\, \frac{l_i + 2s_i}{2} \geq \mathrm{Re}\, \frac{l_{i+1} + 2s_{i+1}}{2}, \quad i = 1, \cdots, a-1$$

and

$$\mathrm{Re}\, \frac{-m_i + 2t_i}{2} \geq \mathrm{Re}\, \frac{-m_{i+1} + 2t_{i+1}}{2}, \quad i = 1, \cdots, b-1.$$

If $a = b = 0$, then we are in the tempered case so the theorem is proved.

Assume that

$$\mathrm{Re}\, \frac{l_1 + 2s_1}{2} \geq \mathrm{Re}\, \frac{-m_1 + 2t_1}{2},$$

or $b = 0$. Then $l_1 + 2s_1 + 2j + 1 > -m_i \pm 2t_i$ for all $i = 1, \cdots, b$ and all nonnegative integer j (this is a vacuum statement if $b = 0$). This is because by our ordering, $-m_1 + 2t_1$ has the maximal real part among $-m_i \pm 2t_i$'s, $i = 1, \cdots, b$. Moreover $l_1 + 2s_1$ is not an integer having the same parity with n by Lemma 4.4. Thus the conditions in Proposition 4.3 are verified. Put

$$\pi_1^- = \omega_{l_2} |\cdot|^{s_2} \times \cdots \times \omega_{l_a} |\cdot|^{s_a} \times \pi_0.$$

Then by Proposition 4.3, we have

$$m(\pi, \sigma) = m(\sigma, \pi_1^-).$$

Choose $s_1' \in \sqrt{-1}\mathbb{R}$ such that $\mathrm{Im}\, s_1' \neq 0, \pm \mathrm{Im}\, t_i$, $i = 1, \cdots, b$, and

$$\pi_1 = |\cdot|^{s_1'} \times \omega_{l_2} |\cdot|^{s_2} \times \cdots \times \omega_{l_a} |\cdot|^{s_a} \times \pi_0.$$

still lies in the generic packet. The conditions in Proposition 4.3 are verified again and we conclude

$$m(\pi_1, \sigma) = m(\sigma, \pi_1^-).$$

The net effect is that we replace a possibly nonunitary character $\omega_{l_1}|\cdot|^{s_1}$ by a unitary character $|\cdot|^{s_1'}$. We can of course repeat this process for $\omega_{l_i}|\cdot|_{\mathbb{C}}^{s_i}$, $i = 1, 2, \cdots, c$, as long as

$$\mathrm{Re}\, \frac{l_i + 2s_i}{2} \geq \mathrm{Re}\, \frac{-m_1 + 2t_1}{2}, \quad i = 1, 2, \cdots, c,$$

or $b = 0$. The net effect is that we replace by $\omega_{l_i}|\cdot|_{\mathbb{C}}^{s_i}$ by $|\cdot|_{\mathbb{C}}^{s_i'}$, $i = 1, 2, \cdots, c$, where $s_i' \in \sqrt{-1}\mathbb{R}$ is a generic purely imaginary number. Put

$$\pi_i = |\cdot|_{\mathbb{C}}^{s_1'} \times \cdots |\cdot|_{\mathbb{C}}^{s_i'} \times \omega_{l_{i+1}} |\cdot|_{\mathbb{C}}^{s_{i+1}} \times \cdots \times \omega_{l_a} |\cdot|_{\mathbb{C}}^{s_a} \times \pi_0, \quad i = 1, 2, \cdots, c.$$

We have

$$m(\pi, \sigma) = m(\pi_1, \sigma) = \cdots = m(\pi_c, \sigma),$$

Suppose now that we have

$$\mathrm{Re}\, \frac{l_{c+1} + 2s_{c+1}}{2} < \mathrm{Re}\, \frac{-m_1 + 2t_1}{2},$$

or we are in the case either $c = 0$ or $a = 0$. The case $c = 0$ or $a = 0$ simply means that we have

$$\mathrm{Re}\, \frac{l_1 + 2s_1}{2} < \mathrm{Re}\, \frac{-m_1 + 2t_1}{2}$$

or $a = 0$ to begin with, and thus did not implement the procedures as described above. We let $\pi_c = \pi$ if this is the case.

Let us choose $t_1' \in \sqrt{-1}\mathbb{R}$ so that

$$t_1' \neq \pm s_i', \quad i = 1, \cdots, c, \quad t_1' \neq \pm \mathrm{Im}\, s_i, \quad i = c+1, \cdots, a,$$

and

$$\sigma_1^+ = |\cdot|^{t_1'} \times \omega_{m_1} |\cdot|^{t_1} \times \cdots \times \omega_{m_b} |\cdot|^{t_b} \times \sigma_0.$$

Applying Proposition 4.3 to σ_1^+ and π, by our choice of t_1' we have

$$m(\sigma_1^+, \pi) = m(\pi, \sigma).$$

Let us put

$$\sigma_1 = |\cdot|^{t_1'} \times \omega_{m_2} |\cdot|^{t_2} \times \cdots \times \omega_{m_b} |\cdot|^{t_b} \times \sigma_0.$$

By assumption we have

$$\mathrm{Re}\, \frac{m_1 - 2t_1}{2} < \mathrm{Re}\, \frac{-l_{c+1} - 2s_{c+1}}{2},$$

or $a = 0$. It follows that

$$m_1 - 2t_1 - 2j - 1 < -l_i \pm 2s_i, \quad i = c+1, \cdots, a,$$

for all nonnegative integer j. This is because by our ordering, the real part of $-l_{c+1}-2s_{c+1}$ is the smallest among all $-l_i \pm 2s_i$'s, $i=c+1,\cdots,a$. Moreover by our choice of $s_1',\cdots,s_c'$, we have

$$m_1-2t_1-2j-1\neq \pm 2s_i', \quad i=1,\cdots,c,$$

as their imaginary parts are different. Finally m_1+2t_1 is not an integer of the same parity as $n+1$ since σ lies in a generic packet. Thus applying Proposition 4.3 as before, we obtain

$$m(\sigma_1^+,\pi_c)=m(\pi_c,\sigma_1).$$

The net effect of this process is to replace a possibly nonunitary character $\omega_{m_1}|\cdot|^{t_1}$ by a unitary one $|\cdot|^{t_1'}$. We may repeat this process for $\omega_{m_1}|\cdot|_{\mathbb{C}}^{t_1},\cdots,\omega_{m_d}|\cdot|_{\mathbb{C}}^{t_d}$ as long as

$$\mathrm{Re}\,\frac{l_{c+1}+2s_{c+1}}{2}<\mathrm{Re}\,\frac{-m_i+2t_i}{2}, \quad i=1,\cdots,d.$$

The net effect is that we replace $\omega_{m_i}|\cdot|_{\mathbb{C}}^{t_i}$ by $|\cdot|_{\mathbb{C}}^{t_i'}$, $i=1,\cdots,d$, where $t_i'\in\sqrt{-1}\mathbb{R}$ is a generic purely imaginary number. Put

$$\sigma_i=|\cdot|_{\mathbb{C}}^{t_1'}\times\cdots\times|\cdot|_{\mathbb{C}}^{t_i'}\times\omega_{m_{i+1}}|\cdot|_{\mathbb{C}}^{t_{i+1}}\times\cdots\times\omega_{m_b}|\cdot|_{\mathbb{C}}^{t_b}\times\sigma_0.$$

We have

$$m(\pi_c,\sigma)=m(\pi_c,\sigma_1)=\cdots=m(\pi_c,\sigma_d).$$

Suppose that we have

$$\mathrm{Re}\,\frac{l_{c+1}+2s_{c+1}}{2}\geq \mathrm{Re}\,\frac{-m_{d+1}+2t_{d+1}}{2}.$$

Then we can switch back to π_c and make modifications of it in the same way as to π. We do the modification to π_c until we are not able to, and then switch to σ_d and make modifications to it in the same way as σ. We keep repeating this process and switching back and forth between π and σ. The process terminates after $a+b$ steps and the ultimate effect is that we find generic purely imaginary numbers

$$s_1',\cdots,s_a',t_1',\cdots,t_b'\in\sqrt{-1}\mathbb{R}$$

so that we have

$$\pi_a=|\cdot|^{s_1'}\times\cdots\times|\cdot|^{s_a'}\times\pi_0, \quad \sigma_b=|\cdot|^{t_1'}\times\cdots\times|\cdot|^{t_b'}\times\sigma_0,$$

with

$$m(\pi,\sigma)=m(\pi_a,\sigma_b).$$

Since π_a and σ_b are both tempered, Theorem 1.1 holds for (π_a,σ_b). Theorem 1.1 then holds for (π,σ) as

$$A_{\phi_\pi}=A_{\phi_{\pi_a}}, \quad A_{\phi_\sigma}=A_{\phi_{\sigma_b}},$$

and

$$\eta_\pi=\eta_{\pi_a}, \quad \eta_\sigma=\eta_{\sigma_b}.$$

This proves Theorem 1.1 when $t=0$. □

5. Restrictions to the mirabolic subgroups

In goal of this section is to study the restriction of the principal series representations of $\mathrm{GL}_{t+1}(\mathbb{C})$ to the mirabolic subgroup. This will be used in reducing Theorem 1.1 to the case $t = 0$. Even though we work with $\mathrm{GL}_{t+1}(\mathbb{C})$ the same method and conclusions also hold for $\mathrm{GL}_{t+1}(\mathbb{R})$ with only minor modifications.

5.1. **The mirabolic subgroup stable filtration.** Let us first fix some notation.

- Put $G_t = \mathrm{GL}_t(\mathbb{C})$, B_t the upper triangular Borel subgroup of G_t. We take the convention that G_0 is the trivial group.
- Let $P_{k,t-k} = M_{k,t-k}U_{k,t-k}$ be the (upper triangular) parabolic subgroup of G_t corresponding to the partition $(k, 1, 1, \cdots, 1)$.
- Let $R_{k,t-k} = G_kU_{k,t-k}$ be the subgroup of $P_{k,t-k}$. Then $R_{t-1,1}$ is the usual mirabolic subgroup of G_t.
- $Q_{a,b,c}$ be the intersection of the parabolic subgroup of G_{a+b+c} corresponding to the partition (a, b, c) and the mirabolic subgroup. Let $L_{a,b,c}$ be the subgroup of $Q_{a,b,c}$ consisting of "diagonal blocks", i.e. the subgroup consisting of elements of the form
$$\begin{pmatrix} g_1 & & \\ & g_2 & \\ & & g_3 \end{pmatrix}, \quad g_1 \in G_a,\ g_2 \in G_b,\ g_3 \in R_{c-1,1}$$
- Recall that $\psi : \mathbb{C} \to \mathbb{C}^\times$ is the additive character $z \mapsto e^{2\pi\sqrt{-1}\,\mathrm{Re}\, z}$ and let $\psi_{t-k} : U_{k,t-k} \to \mathbb{C}^\times$ be a generic character of the form
$$\psi_{t-k}(u) = \psi\left(\sum_{i=k}^{t-1} u_{i,i+1}\right).$$
We observe that ψ_{t-k} extends to a character of $R_{k,t-k}$ as it is invariant under the conjugation action of G_k. This character is again denoted by ψ_{t-k}.

Let $s_1, \cdots, s_{t+1} \in \mathbb{C}^\times$. We consider the principal series representation
$$\tau = |\cdot|_{\mathbb{C}}^{s_1} \times \cdots \times |\cdot|_{\mathbb{C}}^{s_{t+1}} = \mathrm{ind}_{B_{t+1}}^{G_{t+1}} |\cdot|_{\mathbb{C}}^{s_1+\frac{t}{2}} \otimes |\cdot|_{\mathbb{C}}^{s_2+\frac{t-2}{2}} \otimes \cdots \otimes |\cdot|_{\mathbb{C}}^{s_{t+1}-\frac{t}{2}}$$
of G_{t+1}. For our purposes it is more convenient to use unnormalized inductions.

Proposition 5.1. *As a representation of $R_{t,1}$, the representation τ has a subrepresentation isomorphic to*
$$\mathrm{ind}_{N_{t+1}}^{R_{t,1}} \psi_t$$
The quotient admits an $R_{t,1}$-stable complete filtration whose graded pieces are of the shape
$$\mathrm{ind}_{Q_{a,b,c}}^{R_{t,1}} \tau_a \mathbin{\widehat{\otimes}} \tau_b \mathbin{\widehat{\otimes}} \tau_c$$

where $a+b+c=t+1$, $a+b\neq 0$, τ_a is a representation of G_a, τ_b is a representation of G_b and τ_c is a representation of $R_{c,1}$, which are described below. The representation $\tau_a\widehat{\otimes}\tau_b\widehat{\otimes}\tau_c$ of $L_{a,b,c}$ is naturally viewed as a representation of $Q_{a,b,c}$.

- *The representation τ_a is of the form*
$$\mathrm{ind}_{B_a}^{G_a}\omega_{l_1}|\cdot|_{\mathbb{C}}^{s_{i_1}+k_1}\otimes\cdots\otimes\omega_{l_a}|\cdot|_{\mathbb{C}}^{s_{i_a}+k_a},$$
where $1\leq i_1,\cdots,i_a\leq t+1$ are integers, $l_1,\cdots,l_a\in\mathbb{Z}$ and $k_1,\cdots,k_a\in\frac{1}{2}\mathbb{Z}$.
- *The representation τ_b is $\tau_b'\otimes\rho$ where τ_b' is a representation of the same form as τ_a and ρ is a finite dimensional representation of G_b.*
- *The representation τ_c is isomorphic to*
$$\mathrm{ind}_{N_c}^{R_{c-1,1}}\psi_{c-1}$$

The proof occupies the rest of this section.

5.2. **Filtration by the support.** Let $\mathcal{X}=B_{t+1}\backslash G_{t+1}$ be the usual flag variety. The (right) action of $R_{t,1}$ on X has $t+1$ orbit, $\mathcal{Z}_0=\mathcal{U},\mathcal{Z}_1,\cdots,\mathcal{Z}_t$. We have $\mathcal{Z}_0=\mathcal{U}$ is open, and (as Nash manifolds)
$$\operatorname{codim}\mathcal{Z}_i=2i,\quad \mathcal{Z}_{i+1}\subset\overline{\mathcal{Z}_i}.$$
They are represented by $\eta_0,\eta_1,\cdots,\eta_t$ respectively with
$$\eta_i=\begin{pmatrix}1_i & \\ & w_{t+1-i}\end{pmatrix}\begin{pmatrix}1_i & & \\ & w_{t-i} & \\ & & 1\end{pmatrix},$$
where w_i is an $i\times i$ matrix whose antidiagonal entries are 1's and zero elsewhere. Elements of the group $\eta_i^{-1}B_{t+1}\eta_i$ take the shape
$$\begin{pmatrix}A & {}^tu\\ v & d\end{pmatrix}$$
where $A\in B_t$, $u=(*,\cdots,*,0,\cdots,0)$ where last $t-i$ entries being zero, $v=(0,\cdots,0,*,\cdots,*)$ where first i entries being zero, and $d\in\mathbb{C}$. Therefore elements in $S_i=\eta_i^{-1}B_{t+1}\eta_i\cap R_{t,1}$ are of the form
$$\begin{pmatrix}A & {}^tu\\ & 1\end{pmatrix},$$
where $A\in B_t$, $u=(*,\cdots,*,0,\cdots,0)$ where last $t-i$ entries being zero.

We denote by ν_i the right action of S_i on $\mathbb{C}^i$ (row vectors) given by first projecting to the upper left $i\times i$ block and multiply on $\mathbb{C}^i$ from the right.

Lemma 5.2. *The representation of S_i on the fiber of the complexified conormal bundle over the point $B_{t+1}\eta_i\in X$ is isomorphic to ν_i.*

The proof is similar to Lemma 4.1 and we omit the details. Again recall that the representation of S_i and the right action of S_i on the fiber over $B_{t+1}\eta_i \in X$ are connected by (2.3). Eventually we will only need the fact that this is a finite dimensional representation of S_i.

It follows that the complexified conormal bundle $\mathcal{N}_i^\vee = \mathcal{N}^\vee_{\mathcal{Z}_i/\mathcal{X}}$ has a filtration

$$\mathcal{N}_i^\vee = \mathcal{N}_i^{\vee,0} \supset \mathcal{N}_i^{\vee,1} \supset \cdots \supset \mathcal{N}_i^{\vee,i-1} \supset \mathcal{N}_i^{\vee,i} = 0$$

so that the fiber of $\mathcal{N}_i^{\vee,j}/\mathcal{N}_i^{\vee,j+1}$ at $B_{t+1}\eta_i$ gives rise two a two dimension representation of S_i of the form

$$g \mapsto \begin{pmatrix} z_j & \\ & \overline{z_j} \end{pmatrix},$$

where z_j is the j-th diagonal element of $g \in S_i$. It is important that the unipotent part of S_i acts trivially on these graded pieces though not on the whole fiber.

Let τ_i be the subspace consisting of Schwartz sections over $\mathcal{X}\backslash\overline{\mathcal{Z}_i}$. This gives rise to an $R_{t,1}$-stable filtration

$$\tau = \tau_{t+1} \supset \tau_t \supset \cdots \supset \tau_0 = 0$$

Lemma 5.3. *The successive quotient τ_i/τ_{i-1} has an $R_{t,1}$-stable complete filtration whose graded pieces are all of the form*

$$\operatorname{ind}_{S_i}^{R_{t,1}} \omega_{l_1}|\cdot|_{\mathbb{C}}^{s_1+\frac{t}{2}+k_1} \otimes \cdots \otimes \omega_{l_i}|\cdot|_{\mathbb{C}}^{s_i+\frac{t-2i+2}{2}+k_i} \otimes |\cdot|_{\mathbb{C}}^{s_{i+2}+\frac{t-2i-2}{2}} \otimes \cdots \otimes |\cdot|_{\mathbb{C}}^{s_{t+1}-\frac{t}{2}}$$

where l_i's are integers and k_i's are nonnegative integers or half integers.

Proof. This is just the application of Proposition 2.5 to the current situation. □

5.3. **Fourier transform.** The bottom piece τ_1 of this filtration admits a further $R_{t,1}$-stable filtration. Note that S_0 is isomorphic to B_t. We have

$$\tau_1 = \operatorname{ind}_{B_t}^{R_{t,1}} |\cdot|_{\mathbb{C}}^{s_2+\frac{t-2}{2}} \otimes \cdots \otimes |\cdot|_{\mathbb{C}}^{s_{t+1}-\frac{t}{2}}.$$

Observe that the $R_{t,1}$-variety $B_t\backslash R_{t,1}$ is naturally isomorphic to $B_t\backslash G_t \times \mathbb{C}^t$ as (right) $R_{t,1}$-varieties where

- $U_{t,1}$ is identified with $\mathbb{C}^t$ (column vectors) and acts on $B_t\backslash G_t$ trivially and on $\mathbb{C}^t$ by translation,
- G_t acts on $B_t\backslash G_t$ by right multiplication and on $\mathbb{C}^t$ by inverse left multiplication.

It follows that the representation τ_1 is realized on

$$\left(\operatorname{ind}_{B_t}^{G_t} |\cdot|_{\mathbb{C}}^{s_2+\frac{t-2}{2}} \otimes \cdots \otimes |\cdot|_{\mathbb{C}}^{s_{t+1}-\frac{t}{2}}\right) \widehat{\otimes}\, \mathcal{S}(\mathbb{C}^t),$$

where

- $U_{t,1}$ acts on $\operatorname{ind}_{B_t}^{G_t} |\cdot|^{s_2+\frac{t-2}{2}} \otimes \cdots \otimes |\cdot|^{s_{t+1}-\frac{t}{2}}$ trivially and on $\mathcal{S}(\mathbb{C}^t)$ by translation,
- $g \in G_t$ acts on the component $\operatorname{ind}_{B_t}^{G_t} |\cdot|^{s_2+\frac{t-2}{2}} \otimes \cdots \otimes |\cdot|^{s_{t+1}-\frac{t}{2}}$ as usual and on $\mathcal{S}(\mathbb{C}^t)$ as

$$g \cdot \phi(x) = \phi(g^{-1}x).$$

Concretely a Schwartz section f over $B_t\backslash R_{t,1}$ is viewed as an element in this realization via

$$(g,u) \mapsto f\left(\begin{pmatrix} g & \\ & 1 \end{pmatrix}\begin{pmatrix} 1_t & u \\ & 1 \end{pmatrix}\right).$$

We may take a Fourier transform $\phi \mapsto \widehat{\phi}$ of $\mathcal{S}(\mathbb{C}^t)$ with respect to the character $\overline{\psi}$. The Fourier transform is given by

$$\widehat{\phi}(y) = \int_{\mathbb{C}^n} \phi(x)\overline{\psi({}^t xy)}\mathrm{d}x$$

where $\mathrm{d}x$ is the seld-dual measure. We also note that for $g \in G_t$ and $\phi \in \mathcal{S}(\mathbb{C}^t)$ we have

$$\widehat{g \cdot \phi}(y) = |\det g|\widehat{\phi}({}^t gy). \tag{5.1}$$

Thus we conclude that τ_1 can also be realized as follows. The space is

$$\left(\mathrm{ind}_{B_t}^{G_t} |\cdot|^{s_2+\frac{t}{2}} \otimes \cdots \otimes |\cdot|^{s_{t+1}-\frac{t-2}{2}}\right) \widehat{\otimes}\, \mathcal{S}(\mathbb{C}^t),$$

but the group $U_{t,1}$ acts on $\mathcal{S}(\mathbb{C}^t)$ as

$$\begin{pmatrix} 1_t & b \\ & 1 \end{pmatrix} \cdot \phi(u) = \psi({}^t bu)\phi(u), \tag{5.2}$$

and $g \in G_t$ acts on $\mathcal{S}(\mathbb{C}^t)$ as

$$g \cdot \phi(u) = \phi({}^t gu). \tag{5.3}$$

Note that we have absorbed the factor $|\det g|$ in (5.1) into the induced representation.

The subspace $\mathcal{S}(\mathbb{C}^t\backslash\{0\})$ is stable under these actions. Recall from Subsection 2.2 that the classical Borel's lemma asserts that the quotient

$$\mathcal{S}(\mathbb{C}^t)/\mathcal{S}(\mathbb{C}^t\backslash\{0\})$$

is isomorphic to $\mathbb{C}[[x_1, \cdots, x_t, y_1, \cdots, y_t]]$ where $z_i = x_i + \sqrt{-1}y_i$, $i = 1, \cdots, t$ are the usual coordinates of $\mathbb{C}^t$. The map

$$\mathcal{S}(\mathbb{C}^t) \to \mathbb{C}[[x_1, \cdots, x_t, y_1, \cdots, y_t]]$$

is just mapping a Schwartz function on $\mathbb{C}^t$ to its Taylor expansion at 0. The space of formal power series $\mathbb{C}[[x_1, \cdots, x_t, y_1, \cdots, y_t]]$ admits complete filtration by the degree, and the graded pieces are given by homogeneous polynomials of a fixed degree. This filtration is $R_{t,1}$-stable. The unipotent radical $U_{t,1}$ acts trivially on these graded pieces, though its action on the whole space of formal power series is not. The action of G_t on $\mathbb{C}[[x_1, \cdots, x_t, y_1, \cdots, y_t]]$ is given again by the formula (5.3).

The topological vector spaces under consideration are all nuclear Fréchet spaces. Therefore by Lemma 2.12 we conclude that

$$\left(\mathrm{ind}_{B_t}^{G_t} |\cdot|^{s_2+\frac{t}{2}} \otimes \cdots \otimes |\cdot|^{s_{t+1}-\frac{t-2}{2}}\right) \widehat{\otimes}\, \mathcal{S}(\mathbb{C}^t)$$

has an $R_{t,1}$-stable subspace

$$\left(\mathrm{ind}_{B_t}^{G_t} |\cdot|^{s_2+\frac{t}{2}} \otimes \cdots \otimes |\cdot|^{s_{t+1}-\frac{t-2}{2}}\right) \widehat{\otimes}\, \mathcal{S}(\mathbb{C}^t\backslash\{0\})$$

whose quotient has an $R_{t,1}$-stable complete filtration with the graded pieces isomorphic to

$$\left(\operatorname{ind}_{B_t}^{G_t}|\cdot|^{s_2+\frac{t}{2}}\otimes\cdots\otimes|\cdot|^{s_{t+1}-\frac{t-2}{2}}\right)\widehat{\otimes}\,\rho_i,\quad i=0,1,2,\cdots,$$

where ρ_i is the representation of G_t on the degree i homogeneous polynomials via the formula (5.3). The unipotent part $U_{t,1}$ acts trivially on these graded pieces, though not trivially on the whole quotient. We will only need the fact that ρ_i's are finite dimensional representations of G_t.

Let us observe that $\mathcal{S}(\mathbb{C}^n\backslash\{0\})$ with the actions given by (5.2) and (5.3) is isomorphic to

$$\operatorname{ind}_{R_{t-1,1}U_{t,1}}^{R_{t,1}}\psi_1,$$

as representations of $R_{t,1}$. Indeed given $\phi\in\mathcal{S}(\mathbb{C}^n\backslash\{0\})$, we define

$$f\left(\begin{pmatrix}1_t & u\\ & 1\end{pmatrix}\begin{pmatrix}g & \\ & 1\end{pmatrix}\right)=\psi({}^tue_t)\phi({}^tge_t)$$

where $e_t={}^t(0,\cdots,0,1)\in\mathbb{C}^t$. Then $f\in\operatorname{ind}_{R_{t-1,1}U_{t,1}}^{R_{t,1}}\psi_1$ and the map $\phi\mapsto f$ defines an $R_{t,1}$-equivariant map

$$\mathcal{S}(\mathbb{C}^t\backslash\{0\})\to\operatorname{ind}_{R_{t-1,1}U_{t,1}}^{R_{t,1}}\psi_1.$$

Conversely given $f\in\operatorname{ind}_{R_{t-1,1}U_{t,1}}^{R_{t,1}}\psi_1$, we put

$$\phi(u)=f\left(\begin{pmatrix}g & \\ & 1\end{pmatrix}\right),$$

where $g\in G_t$ is any element with ${}^tge_t=u$. The invariance property of f ensures that this definition is independent of the choice of g. The map $f\mapsto\phi$ then defines an $R_{t,1}$-equivariant map

$$\operatorname{ind}_{R_{t-1,1}U_{t,1}}^{R_{t,1}}\psi_1\to\mathcal{S}(\mathbb{C}^t\backslash\{0\}).$$

These two maps are inverse to each other by definition.

With this observation, we have that

$$\left(\operatorname{ind}_{B_t}^{G_t}|\cdot|^{s_2+\frac{t}{2}}\otimes\cdots\otimes|\cdot|^{s_{t+1}+\frac{t-2}{2}}\right)\widehat{\otimes}\,\mathcal{S}(\mathbb{C}^n\backslash\{0\})$$

is isomorphic

$$\left(\operatorname{ind}_{B_t}^{G_t}|\cdot|^{s_2+\frac{t}{2}}\otimes\cdots\otimes|\cdot|^{s_{t+1}-\frac{t-2}{2}}\right)\widehat{\otimes}\left(\operatorname{ind}_{R_{t-1,1}U_{t,1}}^{R_{t,1}}\psi_1\right),$$

where $U_{t,1}$ acts trivially on the first factor. Applying Proposition 2.3, we conclude that this is further isomorphic to

$$\operatorname{ind}_{R_{t-1,1}U_{t,1}}^{R_{t,1}}\left(\left(\operatorname{ind}_{B_t}^{G_t}|\cdot|^{s_2+\frac{t}{2}}\otimes\cdots\otimes|\cdot|^{s_{t+1}-\frac{t-2}{2}}\right)\Big|_{R_{t-1,1}}\otimes\psi_1\right),$$

as representations of $R_{t,1}$.

We summarize the discussion in this subsection in the following lemma.

Lemma 5.4. *The representation τ_1 of $R_{t,1}$ has a subrepresentation which is isomorphic to*

$$\operatorname{ind}_{R_{t-1,1}U_{t,1}}^{R_{t,1}}\left(\left(\operatorname{ind}_{B_t}^{G_t}|\cdot|^{s_2+\frac{t}{2}}\otimes\cdots\otimes|\cdot|^{s_{t+1}-\frac{t-2}{2}}\right)\Big|_{R_{t-1,1}}\otimes\psi_1\right)$$

and the quotient admits an $R_{t,1}$-stable complete filtration whose graded pieces are

$$\left(\operatorname{ind}_{B_t}^{G_t}|\cdot|^{s_2+\frac{t}{2}}\otimes\cdots\otimes|\cdot|^{s_{t+1}-\frac{t-2}{2}}\right)\widehat{\otimes}\,\rho_j,\quad j=0,1,2,\cdots,$$

where ρ_j is a finite dimensional representation of G_t. The unipotent part $U_{t,1}$ acts trivially on these graded pieces.

5.4. **Proof of Proposition 5.1.** We prove Proposition 5.1 by induction on t.

In view of Lemma 5.3, it is enough to prove that

$$\operatorname{ind}_{S_i}^{R_{t,1}}\omega_{l_1}|\cdot|^{s_1+\frac{t}{2}+k_1}\otimes\cdots\otimes\omega_{l_i}|\cdot|^{s_i+\frac{t-2i+2}{2}+k_i}\otimes|\cdot|^{s_{i+2}+\frac{t-2i-2}{2}}\otimes\cdots\otimes|\cdot|^{s_{t+1}-\frac{t}{2}}\tag{5.4}$$

has a filtration whose graded pieces are of the form described in Proposition 5.1. Note that the representation with $i=0$ is the subrepresentation. By induction by stages, the representation (5.4) can be written as

$$\operatorname{ind}_{Q_{i,0,t+1-i}}^{R_{t,1}}\tau_i\,\widehat{\otimes}\,\tau_{t+1-i}$$

where

$$\tau_i=\operatorname{ind}_{B_i}^{G_i}\omega_{l_1}|\cdot|^{s_1+\frac{t}{2}+k_1}\otimes\cdots\otimes\omega_{l_i}|\cdot|^{s_i+\frac{t-2i+2}{2}+k_i},$$

and

$$\tau_{t+1-i}=\operatorname{ind}_{B_{t-i}}^{R_{t-i,1}}|\cdot|^{s_{i+2}+\frac{t-2i-2}{2}}\otimes\cdots\otimes|\cdot|^{s_{t+1}-\frac{t}{2}}.$$

By induction by stages, it is enough to prove that τ_{t+1-i} has an $R_{t-i,1}$-stable filtration whose graded pieces are as described in Proposition 5.1.

Apply Lemma 5.4 to the representation τ_{t+1-i}, we conclude that τ_{t+1-i} has a subrepresentation isomorphic to

$$\operatorname{ind}_{R_{t-i-1,1}U_{t-i,1}}^{R_{t-i,1}}\left(\left(\operatorname{ind}_{B_{t-i}}^{G_{t-i}}|\cdot|^{s_{i+2}+\frac{t-2i}{2}}\otimes\cdots\otimes|\cdot|^{s_{t+1}-\frac{t-2}{2}}\right)\Big|_{R_{t-i-1,1}}\otimes\psi_1\right),\tag{5.5}$$

and the quotient has an $R_{t-i,1}$-stable complete filtration whose graded pieces are isomorphic to

$$\left(\operatorname{ind}_{B_{t-i}}^{G_{t-i}}|\cdot|^{s_{i+2}+\frac{t-2i}{2}}\otimes\cdots\otimes|\cdot|^{s_{t+1}-\frac{t-2}{2}}\right)\otimes\rho_{t-i}\tag{5.6}$$

where ρ_{t-i} is a finite dimensional representation of G_{t-i} and the unipotent part $U_{t-i,1}$ acts trivially. The representations (5.6) is of the form described in Proposition 5.1.

It remains to treat the representation (5.5). Apply the induction hypothesis to

$$\left(\operatorname{ind}_{B_{t-i}}^{G_{t-i}}|\cdot|^{s_{i+2}+\frac{t-2i}{2}}\otimes\cdots\otimes|\cdot|^{s_{t+1}-\frac{t-2}{2}}\right)\Big|_{R_{t-i-1,1}}$$

we conclude that this representation has an $R_{t-i-1,1}$-stable filtration whose graded pieces of the form

$$\operatorname{ind}_{Q_{a,b,c}}^{R_{t-i-1,1}}\tau_a\,\widehat{\otimes}\,\tau_b\,\widehat{\otimes}\,\tau_c$$

where $a+b+c=t-i$ and τ_a, τ_b, τ_c are as described in Proposition 5.1. In particular τ_c is a representation of $R_{c-1,1}$ of the form

$$\mathrm{ind}_{N_c}^{R_{t-1,1}} \psi_{c-1},$$

and the piece with $a=b=0$ and $c=t-i$ is the subrepresentation. Plug this back into the representation (5.5) we obtain

$$\mathrm{ind}_{R_{t-i-1,1}U_{t-i,1}}^{R_{t-i,1}} \left(\left(\mathrm{ind}_{Q_{a,b,c}}^{R_{t-i-1,1}} \tau_a \widehat{\otimes} \tau_b \widehat{\otimes} \left(\mathrm{ind}_{N_c}^{R_{c-1,1}} \psi_{c-1} \right) \right) \otimes \psi_1 \right).$$

By induction by stages again we conclude that this is isomorphic to

$$\mathrm{ind}_{Q_{a,b,c+1}}^{R_{t-i,1}} \left(\tau_a \widehat{\otimes} \tau_b \widehat{\otimes} \left(\mathrm{ind}_{N_{c+1}}^{R_{c,1}} \psi_c \right) \right).$$

This is of the form described in Proposition 5.1.

We finally note that the graded piece with $i=0$, $a=b=0$ and $c=t+1$ appears as the subrepresentation in every step. Therefore it is the subrepresentation of τ. This finishes the proof of Proposition 5.1.

6. Bessel models

6.1. **Induction and multiplicity.** Let t be a nonnegative integer. We consider the following setup.

- Let $W \subset V$ be a relevant pair of hermitian spaces of dimensions n and $n+2t+1$ respectively. Recall that this means $W^\perp$ has a basis

  $$z_0, z_{\pm 1}, \cdots, z_{\pm t},$$

  with

  $$h_V(z_i, z_j) = (-1)^n \delta_{i,-j}, \quad i,j = 0, \pm 1, \cdots, \pm t.$$

- Let

  $$X = \langle z_1, \cdots, z_t \rangle, \quad X^\vee = \langle z_{-1}, \cdots, z_{-t} \rangle.$$

- Let z_0' be an element of norm $(-1)^{n+1}$ and $W^+ = V \oplus^\perp \langle z_0' \rangle$. Then $V \subset W^+$ is a relevant pair of codimension one.
- Put $z_{t+1} = z_0 + z_0'$ and $z_{-t-1} = z_0 - z_0'$. Put

  $$Y = X \oplus \langle z_{t+1} \rangle, \quad Y^\vee = X^\vee \oplus \langle z_{-t-1} \rangle.$$

  Then $W^+ = W \oplus Y \oplus Y^\vee$.
- Put $G = \mathrm{U}(V)$, $H = \mathrm{U}(W)$ and $H^+ = \mathrm{U}(W^+)$.
- Let $P = MU$ be the parabolic subgroup of H^+ stabilizing the isotropic subspace Y of W^+. The Levi subgroup M is isomorphic to $\mathrm{GL}_{t+1}(\mathbb{C}) \times \mathrm{U}(W)$.

Let $s_1, \cdots, s_{t+1}$ be complex numbers. We say that they are in general position, if $(s_1, \cdots, s_{t+1}) \in \mathbb{C}^{t+1}$ does not lie in the set of zeros of countably many polynomial functions. The goal of this section is to prove the following proposition.

Proposition 6.1. *Let $s_1, \cdots, s_{t+1} \in \mathbb{C}$ be complex numbers in general position. Let*

$$\tau = |\cdot|_{\mathbb{C}}^{s_1} \times \cdots \times |\cdot|_{\mathbb{C}}^{s_{t+1}}$$

be a principal series representation of $\mathrm{GL}_{t+1}(\mathbb{C})$*. Put*

$$\sigma^+ = \mathrm{Ind}_P^{H^+} \tau \widehat{\otimes} \sigma.$$

Then

$$m(\sigma^+, \pi) = m(\pi, \sigma).$$

The hyperplanes in $\mathbb{C}^{t+1}$ that $(s_1, \cdots, s_{t+1})$ has to avoid depends on π and σ. They can be made explicit from the argument below. However it seems that the explicit description is rather messy and is not absolutely necessary so we will not try to describe it.

This proposition implies Theorem 1.1.

Proof of Theorem 1.1 assuming Proposition 6.1. We have already proved Theorem 1.1 in the case $t = 0$. Thus the theorem holds for (σ^+, π). Theorem 1.1 then holds for (π, σ) as

$$A_{\phi_{\sigma^+}} = A_{\phi_\sigma}, \quad \eta_{\sigma^+} = \eta_\sigma.$$

This finishes the proof of Theorem 1.1. □

Proposition 6.1 strengthens the main theorem of [JSZ10] which asserts that

$$m(\pi, \sigma) \leq m(\sigma^+, \pi).$$

This inequality is sufficient to reduce the multiplicity one theorem for Bessel models in general to the case $t = 0$. The proof in [JSZ10] is analytic. A nonzero element in $\mathrm{Hom}_G(\sigma^+ \widehat{\otimes} \pi, \mathbb{C})$ is constructed from a nonzero element in $\mathrm{Hom}_S(\pi \widehat{\otimes} \sigma, \nu)$, using an explicit integral. This integral is closely related to the integral representation of the tensor product L-function for unitary groups. Our argument is very different and is representation-theoretic in nature.

The rest of this section is devoted to the proof of Proposition 6.1.

6.2. **Closed orbit.** If W is anistropic then G acts transitively on $P\backslash H^+$. We assume that W is isotropic in this subsection. In this case, there are two G-orbits in the flag variety $\mathcal{X} = P\backslash H^+$. The closed orbit $\mathcal{Z}$ consists of all $(t+1)$-dimensional isotropic subspaces of W^+ contained in V. Its complement, the open orbit $\mathcal{U}$, consists of all $(t+1)$-dimensional isotropic subspaces of W^+ not contained in V. In the later case, the intersection with V is t-dimensional.

Let $\sigma^{+,\circ}$ be the subspace of σ^+ consisting of Schwartz sections over $\mathcal{U}$. The goal of this subsection is to study the filtration on $\sigma^+/\sigma^{+,\circ}$. The argument is very similar to that in Subsection 4.1 where we treated the same problem with $t = 0$. So we will be brief.

Let w_1, w_{-1} be an isotropic vector in W, $h_W(w_1, w_{-1}) = 1$, and $W_0 = \langle w_1, w_{-1}\rangle^\perp$, $V_0 = W_0 \oplus \langle z_0 \rangle$. Let $Y' = X \oplus \langle w_1 \rangle$. Then Y' is a $(t+1)$-dimensional isotropic subspace of W^+ contain in V and thus gives rise to a point $[Y']$ in the closed orbit in $\mathcal{X}$. Let $\eta \in H^+$ be an element with $Y \cdot \eta = Y'$

(recall that H^+ acts on $\mathcal{X}$ from the right). Then $P\backslash P\eta G$ is the closed orbit in $\mathcal{X}$. We also have $W \cdot \eta = W_0 \oplus \langle z_0, z_0' \rangle$ and thus $\eta^{-1}\,\mathrm{U}(W)\eta = \mathrm{U}(W_0 \oplus \langle z_0, z_0' \rangle)$, $\eta^{-1}\,\mathrm{U}(W)\eta \cap G = \mathrm{U}(V_0)$. Let

$$Q = \eta^{-1} P \eta \cap G = M_Q U_Q$$

be the parabolic subgroup of G stabilizing $Y' \subset V$. It Levi subgroup M_Q is isomorphic to $\mathrm{GL}_{t+1}(\mathbb{C}) \times \mathrm{U}(V_0)$ where we have naturally identified $\mathrm{GL}(Y')$ with $\mathrm{GL}_{t+1}(\mathbb{C})$. Let ${}^{\eta}\sigma$ be the representation of $\eta^{-1}\,\mathrm{U}(W)\eta$ by ${}^{\eta}\sigma(g) = \sigma(\eta g \eta^{-1})$.

Lemma 6.2. *The codimension of $\mathcal{Z}$ in $\mathcal{X}$ is $2(t+1)$ (as Nash manifolds). The complexified conormal bundle $\mathcal{N}^{\vee}_{\mathcal{Z}/\mathcal{X}}$ gives rise to a representation of Q on its fiber over $[Y']$, where U_Q and $\mathrm{U}(V_0)$ acts trivially and $\mathrm{GL}_{t+1}(\mathbb{C})$ acts as*

$$g \mapsto \begin{pmatrix} g & \\ & \overline{g} \end{pmatrix}.$$

The proof of this lemma is essentially the same as Lemma 4.1 and we omit the details. We denote the representation of $\mathrm{GL}_{t+1}(\mathbb{C})$ in the lemma by ρ. This is a finite dimensional representation of $\mathrm{Res}_{\mathbb{C}/\mathbb{R}}\,\mathrm{GL}_{t+1}(\mathbb{C})$.

Applying Proposition 2.5, we conclude that $\sigma^+_{\mathcal{Z}} = \sigma^+/\sigma^{+,\circ}$ has a complete decreasing filtration $\sigma^+_{\mathcal{Z},k}$ and the graded pieces are given by

$$\sigma^k_{\mathcal{Z}} = \mathrm{ind}_Q^G \left(\left(|\det|^{\frac{1}{2}} \tau \otimes \mathrm{Sym}^k \rho \right) \widehat{\otimes} \left({}^{\eta}\sigma|_{\mathrm{U}(V_0)} \right) \right).$$

Lemma 6.3. *Suppose that $s_1, \cdots, s_{t+1}$ are in general position, then*

$$m(\sigma^+, \pi) = \dim \mathrm{Hom}_G(\sigma^{+,\circ} \widehat{\otimes} \pi, \mathbb{C}).$$

Proof. By Corollary 2.14, as in the proof of Proposition 4.3, it is enough to prove that if $s_1, \cdots, s_{t+1}$ are in general position, then

$$H_i^{\mathcal{S}}\left(G, \sigma^k_{\mathcal{Z}} \widehat{\otimes} \pi\right) = 0 \tag{6.1}$$

for all i and k.

We have constructed elements in $\mathcal{Z}(\mathfrak{g}_{\mathbb{C}})$ in Subsection 3.2, c.f. (3.5), that annihilates $\sigma^k_{\mathcal{Z}}$. Let z_G be the one of them given in Lemma 3.2. Since π is irreducible, the element z_G acts on $\pi^{\vee}$ by a constant λ_π which is a nonzero polynomial function in $s_1, \cdots, s_{t+1}$. Thus if $(s_1, \cdots, s_{t+1})$ avoids the zeros of this polynomial, we have $\lambda_\pi \neq 0$ and thus obtain the desired vanishing (6.1) for this k from Corollary 2.8. Since there are only countably many k's, we conclude that if $s_1, \cdots, s_{t+1}$ are in general position, then we have (6.1) for all k. ☐

6.3. **Open orbit.** If W is isotropic, by Lemma 6.3, it remains to study

$$\mathrm{Hom}_G(\sigma^{+,\circ} \widehat{\otimes} \pi, \mathbb{C}).$$

If W is anisotropic, we have $H^+ = PG$ and to unify notation we simply write $\sigma^{+,\circ} = \sigma^+$. We have

$$\sigma^{+,\circ} = \mathrm{ind}_{P\cap G}^G \left(\left(\tau|\cdot|^{\frac{n+t+1}{2}} \widehat{\otimes} \sigma \right) \Big|_{P\cap G} \right).$$

By [GGP12, Lemma 15.2] (the archimedean case is explicitly excluded in this section, but this lemma carries over without trouble), the intersection $P \cap G$ can be describe as follows.

- Let $w_1, \cdots, w_n$ be a basis of W. We use the basis

$$z_t, \cdots, z_1, w_1, \cdots, w_n, z_0, z_{-1}, \cdots, z_{-t}$$

  of V. We realize $\mathrm{U}(V)$ as a matrix group using this basis.
- Let $V^- = W \oplus X \oplus X^\vee$ be the subspace of V and $\mathrm{U}(V^-)$ is a subgroup of $\mathrm{U}(V)$. Let U_X be the unipotent radical of parabolic subgroup of $\mathrm{U}(V^-)$ stabilizing X. View it as a subgroup of G.
- Let R be the mirabolic subgroup of $\mathrm{GL}(X \oplus \langle z_0 \rangle) = \mathrm{GL}_{t+1}(\mathbb{C})$.
- Then $P \cap G$ is isomorphic to a semi-direct product

$$P \cap G = U_X \rtimes (R \times H).$$

Therefore we have

$$\pi^{+,\circ} = \mathrm{ind}_{U_X \rtimes (R\times H)}^G (\tau|\cdot|^{\frac{n+t+1}{2}})|_R \widehat{\otimes} \sigma.$$

We now apply Proposition 5.1 and follow the notation there. The mirabolic subgroup R is denoted by $R_{t,1}$ there. Recall that we identify $\mathrm{GL}(X \oplus \langle z_0 \rangle)$ with $\mathrm{GL}_{t+1}(\mathbb{C})$ using the basis

$$z_t, \cdots, z_1, z_0,$$

and the mirabolic subgroup consists of elements fixing z_0. The representation $(\tau|\cdot|^{\frac{n+t+1}{2}})|_R$ has a subrepresentation isomorphic to

$$\mathrm{ind}_{N_{t+1}}^R \psi_t$$

and the quotient has an R-stable complete filtration whose graded pieces are of the form

$$\mathrm{ind}_{Q_{a,b,c}}^R \tau_a \widehat{\otimes} \tau_b \otimes \tau_c,$$

where τ_a, τ_b, τ_c are as described in Proposition 5.1 and $a + b \geq 1$. Since σ is irreducible and hence nuclear, we conclude by Lemma 2.12 and 2.15 that $\sigma^{+,\circ}$ has a subrepresentation isomorphic to

$$\mathrm{ind}_{U_X \rtimes (R\times H)}^G \left(\mathrm{ind}_{N_{t+1}}^R \psi_t \widehat{\otimes} \sigma \right),$$

and the quotient has a complete filtration whose graded pieces are of the form

$$\mathrm{ind}_{U_X \rtimes (R\times H)}^G \left(\left(\mathrm{ind}_{Q_{a,b,c}}^R \tau_a \widehat{\otimes} \tau_b \widehat{\otimes} \tau_c \right) \widehat{\otimes} \sigma \right).$$

If $a \geq 1$, we let $X_a \subset X$ be the isotropic subspace of V spanned

$$z_t, \cdots, z_{t-a+1},$$

and $P_a = M_a U_a$ be the parabolic subgroup of G stabilizing X_a. We have

$$M_a = \mathrm{GL}_a(\mathbb{C}) \times \mathrm{U}(V_a)$$

where V_a is the subspace of V spanned by

$$z_{t-a}, \cdots, z_1, w_1, \cdots, w_n, z_0, z_{-1}, \cdots, z_{-t+a}.$$

Let

$$\sigma_a = \mathrm{ind}_{Q_{a,b,c} \cap \mathrm{U}(V_a)}^{\mathrm{U}(V_a)} (\tau_b \widehat{\otimes} \tau_c) \widehat{\otimes} \sigma.$$

Then by induction by stages,

$$\mathrm{ind}_{U_X \rtimes (R \times H)}^{G} \left(\left(\mathrm{ind}_{Q_{a,b,c}}^{R} \tau_a \widehat{\otimes} \tau_b \widehat{\otimes} \tau_c \right) \widehat{\otimes} \sigma \right).$$

is a (unnormalized) parabolic induction

$$\mathrm{ind}_{M_a U_a}^{G} \tau_a \widehat{\otimes} \sigma_a.$$

The rest of the argument is the same as Lemma 6.3. We have constructed in Subsection 3.2 an element in $\mathcal{Z}(\mathfrak{g}_{\mathbb{C}})$ which annihilates $\mathrm{ind}_{M_a U_a}^{G} \tau_a \widehat{\otimes} \sigma_a$. It acts on $\pi^\vee$ by a constant which depends polynomially on $s_1, \cdots, s_{t+1}$. There are only countably many graded pieces. It follows that when $s_1, \cdots, s_{t+1}$ is in general position, this constant is not zero. Therefore

$$H_i^{\mathcal{S}} \left(G, \mathrm{ind}_{U_X \rtimes (R \times H)}^{G} \left(\left(\mathrm{ind}_{Q_{a,b,c}}^{R} \tau_a \widehat{\otimes} \tau_b \widehat{\otimes} \tau_c \right) \widehat{\otimes} \sigma \right) \widehat{\otimes} \pi \right) = 0 \tag{6.2}$$

for all i if $a \geq 1$.

If $a = 0$ then $b \geq 1$. In this case one can prove similarly, using the fact that there are only countably many irreducible finite dimensional representation of $\mathrm{GL}_b(\mathbb{C})$, that (6.2) holds for all i if $b \geq 1$.

We now finish the proof of Proposition 6.1. It follows from (6.2) and Lemma 2.14 that

$$\mathrm{Hom}_G(\sigma^{+,\circ} \widehat{\otimes} \pi, \mathbb{C}) = \mathrm{Hom}_G \left(\mathrm{ind}_{U_X \rtimes (R \times H)}^{G} \left(\mathrm{ind}_{N_{t+1}}^{R} \psi_t \widehat{\otimes} \sigma \right) \widehat{\otimes} \pi, \mathbb{C} \right).$$

Note that $U_X \rtimes (N_{t+1} \times H)$ is the Bessel subgroup S of G and $\overline{\psi_t}$ is precisely the generic character appearing in the Bessel model. Recall that in the definition of the Bessel model, the character $\overline{\psi_t}$ is has a unique extension to S which is trivial on H. This extension was denoted by ν there. By induction by stages we have

$$\mathrm{ind}_{U_X \rtimes (R \times H)}^{G} \left(\mathrm{ind}_{N_{t+1}}^{R} \psi_t \widehat{\otimes} \sigma \right) = \mathrm{ind}_{S}^{G} \overline{\nu} \otimes \sigma.$$

Finally by Frobenius reciprocity we obtain

$$\mathrm{Hom}_G(\sigma^{+,\circ} \widehat{\otimes} \pi, \mathbb{C}) = \mathrm{Hom}_S(\pi \widehat{\otimes} \sigma \otimes \overline{\nu}, \mathbb{C}) = \mathrm{Hom}_S(\pi \widehat{\otimes} \sigma, \nu).$$

This finishes the proof of Proposition 6.1.

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