FOURIER–JACOBI MODELS FOR REAL UNITARY GROUPS

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ABSTRACT. We prove the local Gan–Gross–Prasad conjecture for Fourier–Jacobi models of real unitary groups.

CONTENTS

1. INTRODUCTION

The goal of this paper is to prove the local Gan–Gross–Prasad (GGP) conjecture for Fourier–Jacobi models on real unitary groups. This completes the proof of [GGP12, Conjecture 17.3] for unitary groups in all cases. In the simplest case, i.e. unitary group are all compact, the Fourier–Jacobi model essentially describes the restriction of the Weil representation of the metaplectic group $\widetilde{\mathrm{Sp}}_{2n}(\mathbb{R})$ to a maximal compact subgroup, which is a very classical subject of representation theory and classical invariant theory.

1.1. **Generic packets.** By a character of $\mathbb{C}^\times$, we mean a unitary character. It is conjugate self-dual if χ is trivial on $\mathbb{R}_{>0}$. Any conjugate self-dual character is of the form

$$\xi_m(z) = z^m (z\overline{z})^{-\frac{m}{2}}.$$

for some integer m. It is of sign $+1$ (resp. -1) if m is even (resp. odd). If χ is a character of $\mathbb{C}^\times$ we put $\chi^c(z) = \chi(\overline{z})$. Put $|z|_{\mathbb{C}} = z\overline{z}$. A quasi-character of $\mathbb{C}^\times$ is a continuous homomorphism $\chi : \mathbb{C}^\times \to \mathbb{C}^\times$. Any quasi-character can be written uniquely as $\chi = \xi_m |\cdot|_{\mathbb{C}}^s$ for some $m \in \mathbb{Z}$ and $s \in \mathbb{C}$. It is a character if s is purely imaginary. Put $\mathrm{Re}\,\chi = \mathrm{Re}\,s$.

Let ψ and $\psi^{\mathbb{C}}$ be additive characters of $\mathbb{C}$ given by

$$\psi(z) = e^{2\pi\sqrt{-1}(z+\overline{z})}, \quad \psi^{\mathbb{C}}(z) = \psi(\sqrt{-1}z) = e^{2\pi(\overline{z}-z)}.$$

Let n be a positive integer. An L-parameter for unitary groups in n variables is an n-dimensional continuous semisimple representation of $\mathbb{C}^\times$. As $\mathbb{C}^\times$ is abelian, we can write

$$\phi = \bigoplus_{i=1}^{k} c_i \xi_i \oplus \bigoplus_{i=1}^{a} \left(\xi_{k+i} \oplus \xi_{k+i}^{c,-1} \right), \tag{1.1}$$

so that

- $\xi_1, \cdots, \xi_k$ are distinct conjugate self-dual characters of $\mathbb{C}^\times$ of sign $(-1)^{n-1}$, and $c_1, \cdots, c_k$ are positive integers;
- $\xi_{k+1}, \cdots, \xi_{k+a}$ are (not necessarily distinct) quasi-characters that are not conjugate self-dual characters of sign $(-1)^{n-1}$ and $\operatorname{Re} \xi_{k+i} \geq 0$;
- $n = c_1 + \cdots + c_k + 2a$.

The Vogan packet attached to ϕ, denoted by Π_ϕ, is the (disjoint) union of all Π_ϕ^V as V ranges over all (isomorphism classes of) hermitian spaces of dimension n,

$$\Pi_\phi = \bigcup_{V:\ \dim V = n} \Pi_\phi^V.$$

Each Π_ϕ^V is a finite set of irreducible representations of $\mathrm{U}(V)$. Throughout this paper, by a representation, we mean a smooth Fréchet representation of moderate growth. Put

$$\phi_0 = \bigoplus_{i=1}^{k} c_i \xi_i,$$

which is an L-parameter of unitary groups in $n - 2a$ variables. There is a Vogan L-packet

$$\Pi_{\phi_0} = \bigcup_{V_0:\ \dim V_0 = n-2a} \Pi_{\phi_0}^{V_0}$$

where $\Pi_{\phi_0}^{V_0}$ is a finite set of limit of discrete series representations of $\mathrm{U}(V_0)$. The packet Π_ϕ can be constructed from the limit of discrete series L-packet Π_{ϕ_0} as follows. If V does not contain an isotropic subspace of dimension a, then $\Pi_\phi^V = \emptyset$. Assume that V contains an isotropic subspace of dimension a, then let $V_0 \subset V$ be a hermitian space so that its orthogonal complement is a split hermitian space of dimension $2a$. We may take a parabolic subgroup P of $\mathrm{U}(V)$ so that its Levi subgroup is isomorphic to $(\mathbb{C}^\times)^a \times \mathrm{U}(V_0)$. Let us temporarily order $\xi_{k+1}, \cdots, \xi_{k+a}$ so that

$$\operatorname{Re} \xi_{k+1} \geq \cdots \geq \operatorname{Re} \xi_{k+a'} > 0 = \operatorname{Re} \xi_{k+a'+1} = \cdots \operatorname{Re} \xi_{k+a}.$$

Let $\pi_0 \in \Pi_{\phi_0}^{V_0}$. Then the parabolically induced representation

$$\operatorname{Ind}_P^{\mathrm{U}(V)} \xi_{k+1} \otimes \cdots \otimes \xi_{k+a} \otimes \pi_0 \tag{1.2}$$

has a unique irreducible Langlands quotient. Then Π_ϕ^V is the collection of all these Langlands quotients where π_0 ranges over $\Pi_{\phi_0}^{V_0}$. In short, taking Langlands quotients of parabolic inductions gives a bijection between Π_ϕ^V and $\Pi_{\phi_0}^{V_0}$.

The centralizer group A_ϕ is defined to be $A_{\phi_0} = (\mathbb{Z}/2\mathbb{Z})^k$. We label elements in $A_\phi = A_{\phi_0}$ as

$$\bigoplus_{i=1}^{k} (\mathbb{Z}/2\mathbb{Z}) a_i, \tag{1.3}$$

where a_i is a symbol corresponding to ξ_i. Without saying the contrary we will follow this convention of labeling characters in an L-parameter. To each representation $\pi \in \Pi_\phi$ there is a character $\eta : A_\phi \to \langle \pm 1 \rangle$ attached to it and this defines a bijection between Π_ϕ and all characters of A_ϕ. There is a similar bijection between Π_{ϕ_0} and A_{ϕ_0}. The following diagram commutes

$$\begin{tikzcd} \Pi_{\phi_0} \arrow[r] \arrow[d] & \mathrm{Hom}(A_{\phi_0}, \langle \pm 1 \rangle) \arrow[d, equal] \\ \Pi_\phi \arrow[r] & \mathrm{Hom}(A_\phi, \langle \pm 1 \rangle) \end{tikzcd} ,$$

where the left arrow is the bijection given by the parabolic induction as before.

This bijection $\Pi_\phi \to \mathrm{Hom}(A_\phi, \langle \pm 1 \rangle)$ depends on the choice of an equivalence class of Whittaker datum. When n is odd, we choose it to be the unique Whittaker datum (up to equivalence) of $\mathrm{U}(\frac{n+1}{2}, \frac{n-1}{2})$. When n is even as explained in [GGP12, Section 10], this is equivalent to choosing an additive character of $\mathbb{C}$ which is trivial on $\mathbb{R}$. Throughout this paper, we will take this additive character to be $\overline{\psi^{\mathbb{C}}}$.

We say that L-parameter ϕ is generic if Π_ϕ contains a generic representation (with any fixed Whittaker datum). We explained in [Xueb, Subsection 1.1] that if ϕ as (1.1) is generic, then Π_ϕ consists of irreducible parabolically induced representations of the form (1.2), where π_0 is a limit of discrete series representation and ranges over Π_{ϕ_0}.

1.2. **Fourier–Jacobi models.** Let us introduce the Fourier–Jacobi models for real unitary groups. Let $W \subset V$ be skew-hermitian spaces so that $V = W \oplus^\perp Z$ where Z is a split skew-hermitian space of dimention $2t$. We fix a basis $z_{\pm 1}, \cdots, z_{\pm t}$ of Z so that

$$q_V(z_i, z_{-j}) = \delta_{ij}, \quad i, j = \pm 1, \cdots, \pm t.$$

Let U be the unipotent radical of the parabolic subgroup of $\mathrm{U}(V)$ stabilizing the flag of completely isotropic subspaces

$$\langle z_t \rangle \subset \langle z_t, z_{t-1} \rangle \subset \cdots \subset \langle z_t, \cdots, z_1 \rangle,$$

We define a character of U by

$$\psi_U(u) = \psi\left(-\mathrm{Tr}_{\mathbb{C}/\mathbb{R}} \sum_{i=1}^{t-1} q_V(z_{-i-1}, u z_i) \right), \quad u \in U.$$

If $t = 0$ or 1 we take ψ_U to be the trivial character. The group $S_V = U \rtimes \mathrm{U}(W)$ is called a Fourier–Jacobi subgroup of $\mathrm{U}(V)$. If $t = 1$ it is simply called a Jacobi subgroup. Then the character ψ_U inflates to a character of S_V.

The same construction also applies to $W^+ = W \oplus^\perp \langle z_1, z_{-1}\rangle$ and we obtain the Jacobi subgroup S_{W^+} of $\mathrm{U}(W^+)$. Let μ be a conjugate self-dual character of $\mathbb{C}^\times$ of sign -1 and $\omega = \omega_{\psi,\mu}$ be the Weil representation of S_{W^+}, cf. Section 2 for a detailed explanation. There is a projection

$$S_V \to S_{W^+}$$

and ω inflates to a representation to S_V which we also denote by ω. Since ψ_U is invariant under the S_{W^+} conjugation action, $\nu = \psi_U \otimes \omega$ is a representation of S_V.

Let π and σ be irreducible representations of $\mathrm{U}(V)$ and $\mathrm{U}(W)$ respectively. We put

$$m(\pi,\sigma) = \dim \mathrm{Hom}_{S_V}(\pi \widehat{\otimes} \sigma \widehat{\otimes} \overline{\nu}, \mathbb{C}).$$

By [SZ12, LS13], we have $m(\pi,\sigma) \leq 1$.

Assume that π and σ lie in generic packets. Let (ϕ_π, η_π) and $(\phi_\sigma, \eta_\sigma)$ be the parameters of π and σ respectively. Let us write

$$\phi_\pi = \bigoplus_{i=1}^{k} c_i \chi_i \oplus \bigoplus_{i=1}^{a} \left(\chi_{k+i} \oplus \chi_{k+i}^{c,-1}\right), \quad \phi_\sigma = \bigoplus_{j=1}^{l} d_i \mu_j \oplus \bigoplus_{j=1}^{b} \left(\mu_{l+j} \oplus \mu_{l+j}^{c,-1}\right),$$

and

$$A_{\phi_\pi} = \bigoplus_{i=1}^{k} (\mathbb{Z}/2\mathbb{Z}) a_i, \quad A_{\phi_\sigma} = \bigoplus_{j=1}^{l} (\mathbb{Z}/2\mathbb{Z}) b_j,$$

following the convention in (1.3).

The main theorem of the paper is the following. It confirms [GGP12, Conjecture 17.3] in this setup. In the theorem ϵ stands for the local root numbers, as defined in [GGP12, Section 5].

Theorem 1.1. *Let the notation be as above. Then $m(\pi,\sigma) \neq 0$ if and only if*

$$\eta_\pi(a_i) = \epsilon(\chi_i \otimes \phi_\sigma \otimes \mu^{-1}, \overline{\psi^{\mathbb{C}}}), \quad \eta_\sigma(b_j) = \epsilon(\phi_\pi \otimes \mu_j \otimes \mu^{-1}, \overline{\psi^{\mathbb{C}}}).$$

The proof of this theorem is standard. We first treat the basic case: $t = 0$ and π and σ both being tempered. We prove it by reducing it to the local GGP conjecture for Bessel models, which we have established in [Xuea]. In fact the theorem in this case has been almost proved in [Xuea], with the condition $m(\pi,\sigma) \neq 0$ replaced by the nonvanishing of a certain explicit intertwining map. We prove in Section 3 that these two conditions are equivalent. Once we have establish this basic case, we reduce all the other cases to the basic case, by making use the Schwartz homology theory [CS21] and the techniques developed in [Xueb].

The same method can be used to treat the local GGP conjecture for symplectic–metaplectic groups. The lack of analogous results from [Xuea, Xueb] nevertheless makes the argument longer. It seems hard to write both unitary and symplectic cases uniformly in one article while maintaining good readability. Thus the symplectic case will be treated in subsequent work.

1.3. **Conventions and notation.** Let G be a group, we denote by $\mathfrak{g}_{\mathbb{R}}$ the Lie algebra, $\mathfrak{g}$ the complexification, $\mathcal{U}(\mathfrak{g})$ the universal enveloping algebra of $\mathfrak{g}$, and $\mathcal{Z}(\mathfrak{g})$ the center of $\mathcal{U}(\mathfrak{g})$.

Let $X(G)$ be the group of (algebraic) characters of G, and $\mathcal{A}_G = \mathrm{Hom}_{\mathbb{Z}}(X(G), \mathbb{R})$, $\mathcal{A}_G^* = X(G) \otimes_{\mathbb{Z}} \mathbb{R}$. There is a map

$$H_G : G \to \mathcal{A}_G \quad g \mapsto (\chi \mapsto \log|\chi(g)|)\,.$$

For $\alpha \in \sqrt{-1}\mathcal{A}_G^*$ and a representation π of G, we put

$$\pi_\alpha(g) = \pi(g) e^{\langle H_G(g), \alpha \rangle}.$$

By a representation of G, we mean a smooth Frechet representation of moderate growth. If $(\pi, \mathcal{V})$ the such a representation and is of finite length, then it is admissible, and we denote by $(\pi^\vee, \mathcal{V}^\vee)$ the admissible dual.

The representation π is unitary if there is a G-invariant inner product on $\mathcal{V}$. Thus $\pi^\vee \simeq \overline{\pi}$. We denote by $\langle -, - \rangle$ and $\|-\|$ the inner product and the induced norm on $\mathcal{V}$ respectively. Then $\mathcal{V}$ completes to a Hilbert space $\mathcal{V}^{(0)}$ with respect to this inner product, and the action π continuously extends to this space, which we denote by $\pi^{(0)}$. For any $k \geq 0$ we denote by $(\pi^{(k)}, \mathcal{V}^{(k)})$ the space of order k smooth vectors in $\mathcal{V}^{(0)}$ and the continuous action $\pi^{(k)}$ on it. Then $\mathcal{V} = \bigcap_k \mathcal{V}^{(k)}$. Each space $\mathcal{V}^{(k)}$ is a Hilbert space itself, and $\mathcal{V}$ is a Frechet–Hilbert space.

The tensor product $\widehat{\otimes}$ means completed projective tensor product. For two Frechet–Hilbert spaces we denote by $\widehat{\otimes}_{\mathrm{h}}$ the completed Hilbert space tensor product.

By a tempered representation of a reductive group G, we mean a (unitary) representation weakly contained in $L^2(G)$.

All groups we encounter in this paper are almost linear Nash groups, cf. [CS21, Section 1.3], and thus we can speak of Schwartz functions on it. If G is a group and H is a subgroup, we let ind_H^G be the unnormalized Schwartz induction functor, cf. [CS21, Section 6.2]. We denote by Ind_H^G the normalized induction functor.

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2. Representation of the Jacobi group

2.1. **The oscillating representation.** Let $\mathbb{V}$ be a symplectic space of dimension $2n$ over $\mathbb{R}$. A Heisenberg group $H(\mathbb{V})$ is the unipotent group of the form $\mathbb{V} \oplus \mathbb{R}$, whose elements are denoted by (v, z) and the addition law is given by

$$(v,z)(v',z') = \left(v + v', z + z' + \frac{1}{2} q_{\mathbb{V}}(v, v') \right).$$

We denote by Z the center of $H(\mathbb{V})$. Fix maximal isotropic subspaces $\mathbb{X}$ and $\mathbb{X}'$ of $\mathbb{V}$ with

$$\mathbb{V} = \mathbb{X} \oplus \mathbb{X}'.$$

We denote by $\rho = \rho_\psi$ the oscillator representation of $H(\mathbb{V})$ on $\mathcal{S}(\mathbb{X})$, namely

$$\rho((v+v',z))\phi(x) = \psi\left(z + q_V(x,v') + \frac{1}{2}q_V(v,v')\right)\phi(x+v), \quad x, v \in \mathbb{X}, \quad v' \in \mathbb{X}^\vee. \tag{2.1}$$

This is the unique irreducible infinite dimensional unitary representation of $H(\mathbb{V})$ with the central character ψ.

We define a partial Fourier transform

$$\mathcal{S}(\mathbb{X}) \widehat{\otimes} \mathcal{S}(\mathbb{X}) \to \mathcal{S}(\mathbb{V}), \quad \phi \mapsto \phi^\dagger,$$

where $\phi \in \mathcal{S}(\mathbb{X}) \widehat{\otimes} \mathcal{S}(\mathbb{X}) \simeq \mathcal{S}(\mathbb{X} \times \mathbb{X})$ and

$$\phi^\dagger(v) = \int_{\mathbb{X}} \phi(x + \frac{l}{2}, x - \frac{l}{2})\psi(q_{\mathbb{V}}(x,l'))\mathrm{d}x, \quad v = l + l', \ l \in \mathbb{X}, \ l' \in \mathbb{X}^\vee.$$

Then

$$\langle \rho(h(v,0))\phi_1, \phi_2 \rangle = (\phi_1 \otimes \phi_2)^\dagger(v). \tag{2.2}$$

In particular

$$v \mapsto \langle \rho(h(v,0))\phi_1, \phi_2 \rangle$$

is a Schwartz function on $\mathbb{V}$. Since partial Fourier transform preserves the inner product, we also conclude that if $\phi_1, \phi_2, \phi_3, \phi_4 \in \mathcal{S}(\mathbb{X})$, then

$$\int_{H(\mathbb{V})/Z} \langle \rho(h)\phi_1, \phi_2 \rangle \overline{\langle \rho(h)\phi_3, \phi_4 \rangle} \mathrm{d}h = \langle \phi_1, \phi_3 \rangle \langle \phi_4, \phi_2 \rangle. \tag{2.3}$$

Let us define $\mathcal{S}(H(\mathbb{V}), \psi)$ be the space of all smooth functions f on $H(\mathbb{V})$ such that $f(zh) = \psi(z)f(h)$ for all $h \in H(\mathbb{V})$ and $z \in Z$, and

$$\sup_{v \in \mathbb{V}} |Df(h)| < \infty$$

for all algebraic diffirential operators D on $\mathbb{V}$. This is an algebra under the usual convolution, which acts on $\mathcal{S}$ via $\overline{\rho_\psi}$ as follows. If $\phi_1, \phi_2, \phi_3 \in \mathcal{S}(\mathbb{X})$ then we define

$$\int_{H(\mathbb{V})/Z} \langle \rho_\psi(h)\phi_1, \phi_2 \rangle \overline{\rho_\psi(h)\phi_3} \mathrm{d}h$$

to be the unique element in $\mathcal{S}(\mathbb{X})$ such that

$$\left\langle \int_{H(\mathbb{V})/Z} \langle \rho_\psi(h)\phi_1, \phi_2 \rangle \overline{\rho_\psi(h)\phi_3} \mathrm{d}h, \overline{\phi_4} \right\rangle = \int_{H(\mathbb{V})/Z} \langle \rho(h)\phi_1, \phi_2 \rangle \overline{\langle \rho(h)\phi_3, \phi_4 \rangle} \mathrm{d}h.$$

We introduce more notation. We fix a basis $X_1, \cdots, X_n$ of $\mathbb{X}$ and a dual basis $X'_1, \cdots, X'_n$ in $\mathbb{X}'$. Using this basis we identify $\mathbb{V}$ with $\mathbb{R}^{2n}$. If $v = (x_1, \cdots, x_{2n}) \in \mathbb{V}$, then we put

$$\|v\| = (x_1^2 + \cdots + x_{2n}^2)^{\frac{1}{2}}.$$

The elements X_i, X_i' are naturally viewed as elements in $\mathrm{Lie}(H(V))$. Define the Sobolev norm of order k on $\mathcal{S}$ by

$$\|\phi\|_k = \sum_{a_1+\cdots+a_n+b_1+\cdots+b_n\le k} \|\rho_\psi(X_1^{a_1}\cdots X_n^{a_n}X_1^{',b_1}\cdots X_n^{',b_n})\phi\|$$

Note that if we write $x=(x_1,\cdots,x_n)\in\mathbb{X}$, and we view ϕ as a Schwartz function on $\mathbb{R}^n$, then

$$\rho_\psi(X_i)\phi = \partial_{x_i}\phi, \quad \rho_\psi(X_i')\phi = (-2\pi x_i)\phi.$$

Thus $\|\phi\|_k$ really is the usual L^2 Sobolev norm for function on $\mathbb{R}^n$. The norm $\|\cdot\|_k$ depends on the choice of the basis, but different choices lead to equivalent norms.

For later use we need a slightly more precise estimate.

Lemma 2.1. *For any integer $d>0$ there is a constant C, such that for all $\phi_1,\phi_2\in\mathcal{S}(\mathbb{X})$ we have*

$$|\langle\rho_\psi(h(v,0))\phi_1,\phi_2\rangle| \le C(1+\|v\|^2)^{-d}\|\phi_1\|_{4d}\|\phi_2\|_{4d}.$$

Since $\|\cdot\|_{4d}$ continuously extends to $\mathcal{S}(\mathbb{X})^{(k)}$ for $k\ge 4d+1$, we conclude a posteriori by continuity that the inequality holds for $\phi_1,\phi_2\in\mathcal{S}(\mathbb{X})^{(k)}$ when $k\ge 4d+1$.

Proof. Write $v=l+l'$, $l\in\mathbb{X}$, $l'\in\mathbb{X}$. Since

$$1+\|v\|^2 \le 2(1+\|l\|^2)(1+\|l'\|^2)$$

we need to prove that

$$(1+\|l\|^2)^d(1+\|l'\|^2)^d\left|\int_{\mathbb{X}}\phi_1(x+l)\phi_2(x)\psi(q_{\mathbb{V}}(x,l'))\mathrm{d}l\mathrm{d}l'\right| \le C\|\phi_1\|_{4d}\|\phi_2\|_{4d}. \tag{2.4}$$

Since $(1+\|l'\|^2)^d$ is a polynomial of degree $2d$ in l', using integration by parts, we conclude that there is a degree $2d$ polynomial $D_{\mathbb{X}}$ in $X_1,\cdots,X_n$, depending only on d such that the left hand side of (2.4) equals

$$(1+\|l\|^2)^d\left|\int_{\mathbb{X}}D_{\mathbb{X}}(\phi_1(x+l)\phi_2(x))\psi(q_{\mathbb{V}}(x,l'))\mathrm{d}x\right|.$$

By the chain rule we conclude that this is a linear combination of the terms of the form

$$(1+\|l\|^2)^d\left|\int_{\mathbb{X}}\int_{\mathbb{X}^\vee}D_1\phi_1(x+l)D_2\phi_2(x)\psi(q_{\mathbb{V}}(x,l'))\mathrm{d}x\right|$$

where D_1 and D_2 are polynomials of in $X_1,\cdots,X_n$, and the degree of D_1D_2 is at most $2d$. The coefficients in this linear combination depends only on d. Each term is bounded by

$$(1+\|l\|^2)^d\left|\int_{\mathbb{X}}D_1\phi_1(x+l)D_2\phi_2(x)\mathrm{d}x\right|.$$

Since

$$1+\|l\|^2 \le 2(1+\|x+l\|^2)(1+\|x\|^2),$$

and by Cauchy-Schwartz inequality, we conclude that the last integral is bounded by

$$\left(\int_{\mathbb{X}}|D_1\phi_1(x+l)|^2(1+\|x+l\|^2)^{2d}\mathrm{d}x\right)^{\frac12}\left(\int_{\mathbb{X}}|D_2\phi_2(x)|^2(1+\|x\|^2)^{2d}\mathrm{d}x\right)^{\frac12},$$

which is bounded by a constant multiple (depending only on d) of

$$\|\phi_1\|_{4d}\|\phi_2\|_{4d}.$$

This proves the desired estimate. $\square$

2.2. **Representation of the Jacobi group.** Let (V, q_V) be skew-hermitian space of dimension n, and $\mathrm{Res}\, V$ be the symplectic space over $\mathbb{R}$ whose underline space is V viewed as a real vector space and the symplectic form is $\mathrm{Re}\, q_V$. Then we have the Heisenberg group $H(\mathrm{Res}\, V)$. Let $\mathrm{Sp}(\mathrm{Res}\, V)$ be the symplectic group attached to $\mathrm{Res}\, V$ and $\widetilde{\mathrm{Sp}}(\mathrm{Res}\, V)$ the double metaplectic cover. It is well-known that the oscillating representation ρ_ψ extends to a representation, which we still denote by ρ_ψ, of $H(\mathrm{Res}\, V) \rtimes \widetilde{\mathrm{Sp}}(\mathrm{Res}\, V)$. This is called the Weil representation.

There is a natural homomorphism $\mathrm{U}(V) \to \mathrm{Sp}(\mathrm{Res}\, V)$. Put $J(V) = H(\mathrm{Res}\, V) \rtimes \mathrm{U}(V)$. Let μ be a character of $\mathbb{C}^\times$ whose restriction to $\mathbb{R}^\times$ is the sign character. Then there is a splitting map $\mathrm{U}(V) \to \widetilde{\mathrm{Sp}}(\mathrm{Res}\, V)$ of the metaplectic cover (depending on μ). Thus we get a homomorphism $J(V) \to H(\mathrm{Res}\, V) \rtimes \widetilde{\mathrm{Sp}}(\mathrm{Res}\, V)$. The Weil representation ρ_ψ pulls back to the Weil representation of $J(V)$, which we denote by $\omega_{\psi,\mu}$. For most part of this paper, the characters of ψ and μ will be fixed, so we write only ω.

By a representation of $J(V)$, we mean a smooth representation of moderate growth with central character ψ. By [Sun12], representation of $J(V)$ with central character ψ are all of the form

$$\pi \mathbin{\widehat{\otimes}} \omega$$

where π is a representation of $\mathrm{U}(V)$.

Assume now that π is a unitary representation. By [Sun12, Theorem 1.1] we have

$$(\pi^{(0)} \mathbin{\widehat{\otimes}}_{\mathrm{h}} \omega^{(0)})^\infty = \pi \mathbin{\widehat{\otimes}}_{\mathrm{h}} \omega = \pi \mathbin{\widehat{\otimes}} \omega.$$

The last equality is because the space on which ω is realized is a space of Schwartz functions and hence is nuclear. It follows that for any $k > 0$ if l is sufficiently large then

$$(\pi^{(0)} \mathbin{\widehat{\otimes}}_{\mathrm{h}} \omega^{(0)})^{(l)} \subset \pi^{(k)} \mathbin{\widehat{\otimes}}_{\mathrm{h}} \omega^{(k)}. \tag{2.5}$$

We now describe the realization of ω on a mixed model following [GI16, Section 7.4]. It will be used to deduce various estimates in the next subsection. Though [GI16] considers only the nonarchimedean local fields, the formulae for the Weil representation presented there are valid for all local fields of characteristic zero. Let V_0 be subspace of V of codimension $2r$ such that $V_0^\perp$ has a basis $\{v_i, v_i' \mid i = 1, \cdots, r\}$ with,

$$q_V(v_i, v_j) = q_V(v_i', v_j') = 0, \quad q_V(v_i, v_j') = \delta_{ij}, \quad i, j = 1, \cdots, r.$$

Fix any basis of V_0. Then we identify V with $\mathbb{C}^n$ (row vectors), and write elements in $\mathrm{U}(V)$ as matrices. Let X and X' be the span of $v_1, \cdots, v_r$ and $v_1', \cdots, v_r'$ respectively, and identify them with $\mathbb{C}^r$ using these bases. Let $P = MN$ be the parabolic subgroup of $\mathrm{U}(V)$ stabilizing X. Let $a \in \mathrm{GL}_r(\mathbb{C})$, $b \in M_{(n-2r)\times r}(\mathbb{C})$, $c \in M_{r\times r}(\mathbb{C})$ with ${}^t\overline{c} = c$, and put

$$m(a) = \begin{pmatrix} a & & \\ & 1 & \\ & & {}^t\overline{a}^{-1} \end{pmatrix}, \quad n(b) = \begin{pmatrix} 1 & b & \frac{1}{2}bb^* \\ & 1 & b^* \\ & & 1 \end{pmatrix}, \quad z(c) = \begin{pmatrix} 1 & & c \\ & 1 & \\ & & 1 \end{pmatrix},$$

where $b^* = \sqrt{-1}\,{}^t\overline{b}$. We view b as a column vector whose rows are elements in V_0.

The Weil representation is realized on the mixed model. Let us denote the Weil representation of $J(V_0)$ by ω_0 and fix a realization $\mathcal{S}_0$ of the Weil representation of $J(V_0)$. Elements of the Heisenberg group $H(V_0)$ are denoted by $h_0(v, z)$, $v \in V_0, z \in \mathbb{R}$. The Weil representation of $J(V)$ is then realized on $\mathcal{S} = \mathcal{S}(\mathbb{C}^r)\,\widehat{\otimes}\,\mathcal{S}_0$. We view elements in $\mathcal{S}$ as Schwartz functions on $\mathbb{C}^r$ (row vector) valued in $\mathcal{S}_0$. We do not need the fully detailed description of the action as in [GI16, Section 7.4], but only the following. Let $a \in \mathrm{GL}_r(\mathbb{C})$, $b \in M_{(n-2r)\times r}(\mathbb{C})$ and $c \in M_{r\times r}(\mathbb{C})$ with ${}^t\overline{c} = c$. We view b as an element in V_0^r, or more precisely a column vector whose entries are elements in V_0. Then

$$\begin{aligned} \omega(m(a))\phi(x) &= \mu(\det a)|\det a|_{\mathbb{C}}^{\frac{1}{2}}\phi(xa), \\ \omega(n(b))\phi(x) &= \omega_0(h_0(xb, 0))(\phi(x)), \\ \omega(z(c))\phi(x) &= \psi(xc\,{}^t\overline{x})\phi(x). \end{aligned} \tag{2.6}$$

Moreover it follows from (2.1) that if $v = l + v_0 + l' \in V$, $l \in X, l' \in X', v_0 \in V_0$, then

$$\omega(h(v, 0))\phi(x) = \omega_0(h_0(v_0, 0))(\phi(x + l))\psi(q_V(x, l') + \frac{1}{2}q_V(l, l')). \tag{2.7}$$

Consider the closed subspace of $\mathcal{S}$

$$\mathcal{S}^0 = \mathcal{S}(\mathbb{C}^r\backslash\{0\})\,\widehat{\otimes}\,\mathcal{S}_0.$$

It is P-invariant by the formulae (2.6). Let $R \subset \mathrm{GL}_r(\mathbb{C})$ be the mirabolic subgroup, i.e. the subgroup whose last row equals $(0, \cdots, 0, 1)$. Put $Q = N \rtimes (R \times \mathrm{U}(V_0))$, which is a subgroup of P and has a quotient isomorphic to $J(V_0)$.

Lemma 2.2. *As a representation of P we have*

$$\mathcal{S}^0 = \mathrm{ind}_Q^P\,\mu|\cdot|_{\mathbb{C}}^{\frac{1}{2}} \otimes \mathcal{S}_0.$$

Here

- $\mu|\cdot|_{\mathbb{C}}^{\frac{1}{2}}$ *is a character of R;*
- *the Weil representation $\mathcal{S}_0$ of $J(V_0)$ is viewed as a representation of Q via the quotient map $Q \to J(V_0)$.*

Proof. Define the map

$$\mathrm{ind}_Q^P\left(\mu|\cdot|_{\mathbb{C}}^{\frac{1}{2}} \otimes \mathcal{S}_0\right) \to \mathcal{S}(\mathbb{C}^r\backslash\{0\})\,\widehat{\otimes}\,\mathcal{S}_0, \quad f \mapsto \phi,$$

where

$$\phi(x) = \mu(\det a)^{-1}|\det a|_{\mathbb{C}}^{-\frac{1}{2}} f\left(\begin{pmatrix} a & & \\ & 1 & \\ & & {}^t\overline{a}^{-1}\end{pmatrix}\right),$$

and a is an element in $\mathrm{GL}_r(\mathbb{C})$ such that $(0, \cdots, 0, 1)a = x$. It is straightforward to check, using the description of the mixed model (2.6), that this map is independent of the choice of a and is indeed an isomorphism. $\square$

We now study the quotient $\mathcal{S}/\mathcal{S}^0$. Since $\mathcal{S}_0$ is nuclear, by Borel's lemma, the quotient $\mathcal{S}/\mathcal{S}^0$ is isomorphic to

$$\mathbb{C}[[z_1, \cdots, z_r, \overline{z_1}, \cdots, \overline{z_r}]] \,\widehat{\otimes}\, \mathcal{S}_0.$$

Any $m(a), a \in \mathrm{GL}_r(\mathbb{C})$ acts on it by

$$\omega(m(a))\phi(z_1, \cdots, z_r, \overline{z_1}, \cdots, \overline{z_r}) = \mu(\det a)|\det a|_{\mathbb{C}}^{\frac{1}{2}}\phi((z_1, \cdots, z_r)a, (\overline{z_1}, \cdots, \overline{z_r})\overline{a}). \tag{2.8}$$

The space of power series is filtered by the (lowest) degree. We denote

$$\mathcal{W}_k = \{\phi \in \mathbb{C}[[z_1, \cdots, z_r, \overline{z_1}, \cdots, \overline{z_r}]] \mid \deg \phi \geq k\}.$$

Lemma 2.3. *Each $\mathcal{W}_k \,\widehat{\otimes}\, \mathcal{S}_0$ is stable under the action of N, and N acts trivially on the graded pieces $(\mathcal{W}_k/\mathcal{W}_{k+1}) \,\widehat{\otimes}\, \mathcal{S}_0$.*

Proof. Let b and c be as in the description of the mixed model. We will prove that if $\phi \in \mathcal{W}_k$, then $\omega(n(b))\phi - \phi$ and $\omega(z(c))\phi - \phi$ are both in $\mathcal{W}_{k+1} \,\widehat{\otimes}\, \mathcal{S}_0$. For this we need to show that if $\phi \in \mathcal{S}(\mathbb{C}^r) \,\widehat{\otimes}\, \mathcal{S}_0$ and the partial derivatives of ϕ vanish at $x = 0$ up to order k, then the derivatives of $\omega(n(b))\phi - \phi$ and $\omega(z(c))\phi - \phi$ vanish up to order $k + 1$.

The action of $z(c)$ is straightforward. Since

$$\omega(z(c))\phi(x) - \phi(x) = (e^{-2\pi\sqrt{-1}xc\,{}^t\overline{x}} - 1)\phi(x).$$

The power series expansion of $e^{-2\pi\sqrt{-1}xc\,{}^t\overline{x}} - 1$ is

$$-2\pi\sqrt{-1}xc\,{}^t\overline{x} + \text{ higher order terms.}$$

We now consider $\omega(n(b))\phi$. Let $\mathrm{Res}\, V_0 = Y \oplus Y^\vee$ be a decomposition where Y and $Y^\vee$ are maximal isotropic spaces of $\mathrm{res}\, V_0$, and we take $\mathcal{S}_0$ to be $\mathcal{S}(Y)$. We may assume that ϕ is of the form $\phi_1 \otimes \phi_2$ where $\phi_1 \in \mathcal{S}(X)$ and $\phi_2 \in \mathcal{S}(Y)$ such that the partial derivatives of ϕ_1 vanish to the order k at $x = 0$. By (2.6) we have

$$\omega(n(b))\phi(x) = \phi_1(x)\omega_0(h_0(xb, 0))\phi_2(y).$$

If $b \in Y^r$ we have

$$\omega_0(h_0(xb, 0))\phi_2(y) = \phi_2(y + xb) = \phi_2(y) + xb \cdot J\phi_2(y) + \text{ higher order terms in } x.$$

Therefore

$$\omega(n(b))\phi(x) - \phi(x) = \phi_1(x)\left(xb \cdot J\phi_2(y) + \text{ higher order terms in } x\right),$$

which implies that the partial derivatives of $\omega(n(b))\phi - \phi$ vanish to the order at least $k+1$. If $b \in Y^{\vee,r}$ we have

$$\begin{aligned}\omega_0(h_0(xb,0))\phi_2(y) &= \phi_2(y)\psi(q_{V_0}(y,xb)) \\ &= \phi_2(y)\left(1 + (-2\pi\sqrt{-1}q_{V_0}(y,xb)) + \text{ higher order terms in } x\right).\end{aligned}$$

It follows that

$$\omega(n(b))\phi(x) - \phi(x) = \phi_1(x)\phi_2(y)\left((-2\pi\sqrt{-1}q_{V_0}(y,xb)) + \text{ higher order terms in } x\right),$$

which implies that the partial derivatives of $\omega(n(b))\phi - \phi$ vanish to the order at least $k+1$. This proves the lemma. □

Lemma 2.4. *As a representation of* $M = \mathrm{GL}_r(\mathbb{C}) \times \mathrm{U}(V_0)$, *the graded pieces* $\mathcal{W}_k/\mathcal{W}_{k+1} \widehat{\otimes} \mathcal{S}_0$ *are isomorphic to*

$$\left(\mu|\cdot|_{\mathbb{C}}^{\frac{1}{2}} \otimes \mathrm{Sym}^k\left(\mathbb{C}^r \oplus \overline{\mathbb{C}^r}\right)\right) \widehat{\otimes} \mathcal{S}_0,$$

where $\mathrm{GL}_r(\mathbb{C})$ *acts on* $\mathbb{C}^r$ *via the standard right multiplication, and* $\mathrm{U}(V_0)$ *acts on* $\mathcal{S}_0$.

Proof. This follows directly from (2.8) and Lemma 2.3. □

2.3. **Some estimates.** We derive some estimates which will be needed in the next section. We follow the notation from the mixed model of ω. Let r be the Witt index of V and V_0 is the anisotropic kernel of V. Let $P_0 = M_0N_0$ be the minimal parabolic subgroup stabilizing the flag

$$\mathbb{C}\{v_1\} \subset \mathbb{C}\{v_1, v_2\} \subset \cdots \subset \mathbb{C}\{v_1, \cdots, v_r\}.$$

We let $A \simeq (\mathbb{R}_{>0})^r$ be the identity component of the maximal split torus in M. Let Δ_P be the roots of A in N and

$$A^+ = \{a \in A \mid |\alpha(a)| \leq 1, \forall \alpha \in \Delta_P\} = \{(a_1, \cdots, a_r) \mid 0 < a_1 \leq \cdots \leq a_r \leq 1\}.$$

We fix a maximal compact subgroup K of $\mathrm{U}(V)$, and we have the Cartan decomposition

$$\mathrm{U}(V) = KA^+K.$$

The measure $\mathrm{d}g$ on $\mathrm{U}(V)$ decomposes as

$$\mathrm{d}g = \varphi(a)\mathrm{d}a\mathrm{d}k_1\mathrm{d}k_2.$$

We will use the estimate that when $a \in A^+$ we have

$$\varphi(a) \leq C\delta_{P_0}^{-1}(a) \tag{2.9}$$

where C is a constant and δ_{P_0} is the modulus character of P_0.

We choose any basis of V_0 and then identify V with $\mathbb{C}^n$. If $v = (x_1, \cdots, x_n) \in V = \mathbb{C}^n$ then we put

$$\|v\| = (x_1\overline{x_1} + \cdots + x_n\overline{x_n})^{\frac{1}{2}}.$$

Let Ξ be the Harish-Chandra Xi function on $\mathrm{U}(V)$. We also fix a logarithmic height function ς on $\mathrm{U}(V)$, cf. [BP20, Section 1.2]. We will use the standard estimate that when $a \in A^+$ we have

$$\Xi(a) \leq C\delta_{P_0}^{\frac{1}{2}}(a)\varsigma(a)^d \tag{2.10}$$

for some constant C and d.

Matrix coefficients of tempered representations satisfy the weak inequality. Moreover precisely, let $Y_1, \cdots Y_{\dim K}$ be a basis of $\mathfrak{k}_{\mathbb{R}}$ and $\Delta_K = 1 - Y_1^2 - \cdots - Y_{\dim K}^2 \in \mathcal{U}(\mathfrak{k})$. If π is a finite length tempered representation of $\mathrm{U}(V)$ and $e, f \in \pi$, then there are constants C and d, and an element Δ_K in $\mathcal{U}(\mathfrak{k})$ such that

$$|\langle \pi(g)e, e' \rangle| \leq C\Xi(g)\varsigma(g)^d \|e\|_{\Delta_K^{\dim K}} \|e'\|_{\Delta_K^{\dim K}}.$$

By continuity the weak inequality holds for $e, e' \in \pi^{(k)}$ for $k \geq 2\dim K + 1$. We denote by $\mathcal{C}(\mathrm{U}(V))$ the Harish-Chandra Schwartz space of $\mathrm{U}(V)$, i.e. the space of smooth functions f on $\mathrm{U}(V)$ with the property that for any $d > 0$ there is a constant C such that

$$|f(g)| \leq C\Xi(g)\varsigma(g)^{-d}.$$

The Weil representation is realized on the mixed model $\mathcal{S}$ as described in the previous subsection. We fix an inner product on $\mathcal{S}_0$. Then an inner product on $\mathcal{S}$ is given by

$$\langle \phi, \phi' \rangle = \int_{\mathbb{C}^r} \langle \phi(w), \phi'(w) \rangle \mathrm{d}w.$$

Lemma 2.5. *Let $\phi, \phi' \in \mathcal{S}$. Let $g \in \mathrm{U}(V)$ and $v \in V$. Then for any $d > 0$ there is a continuous seminorm ν on $\mathcal{S}$ such that*

$$\left|\langle \omega(h(v,0))\phi, \omega(a)\phi' \rangle\right| \leq (a_1 \cdots a_r)(1 + \|la + l' + v_0\|^2)^{-d}\nu(\phi)\nu(\phi')$$

holds for all $a \in A^+$ and $v = l + v_0 + l' \in V$, $l \in X$, $l' \in X'$, $v_0 \in V_0$.

Proof. Since a does not act on the coordinates in V_0, by (2.2) we are reduced to the case $V_0 = 0$. Thus ϕ, ϕ' are merely Schwartz functions on $\mathbb{C}^r$.

We need to prove that for any $d > 0$ we can find a continuous seminorm ν such that

$$\sup_{l,l'}(1 + \|la\|^2)^d(1 + \|l'\|^2)^d \left|\int_{\mathbb{C}^r} \phi_1(x+l)\phi_2(xa)\psi(q_V(x,l'))\mathrm{d}x\right| \leq \nu(\phi)\nu(\phi').$$

First using integration by parts, we conclude that there is a differential operator ∇_x on $\mathbb{C}^r$ (with variable x) such that

$$(1 + \|l'\|^2)^d \int_{\mathbb{C}^r} \phi_1(x+l)\phi_2(xa)\psi(q_V(x,l'))\mathrm{d}x = \int_{\mathbb{C}^r} \nabla_x(\phi_1(x+l)\phi_2(xa))\psi(q_V(x,l'))\mathrm{d}x.$$

By the chain rule, we conclude that $\nabla_x(\phi_1(x+l)\phi_2(xa))$ is linear combination of functions of the form

$$\phi_3(x+l)\phi_4(xa)$$

where ϕ_3 and ϕ_4 are again Schwartz functions and the coefficient involves polynomials of $a_1, \cdots, a_r$. Since $0 < a_i \leq 1$ for all i, we conclude that in order to prove the lemma it is enough to prove that there is a continuous seminorm ν such that

$$(1 + \|la\|^2)^d \int_{\mathbb{C}^r} |\phi_3(x+l)\phi_4(xa)| \mathrm{d}x \leq \nu(\phi_3)\nu(\phi_4)$$

for all Schwartz functions ϕ_3, ϕ_4 on $\mathbb{C}^r$. Indeed we have

$$1 + \|la\|^2 \leq 2(1 + \|(x+l)a\|^2)(1 + \|xa\|^2) \leq 2(1 + \|x+l\|^2)(1 + \|xa\|^2)$$

and thus

$$(1 + \|la\|^2)^d \int_{\mathbb{C}^r} |\phi_3(x+l)\phi_4(xa)| \mathrm{d}x \leq 2^d \int_{\mathbb{C}^r} (1 + \|x\|^2)^d \phi_3(x) \mathrm{d}x \cdot \sup_x (1 + \|xa\|^2)^d \phi_4(xa).$$

We can find a seminorm ν on $\mathcal{S}$ such that

$$\int_{\mathbb{C}^r} (1 + \|x\|^2)^d \phi_3(x) \mathrm{d}x \cdot \sup_x (1 + \|xa\|^2)^d \phi_4(xa) \leq \nu(\phi_3)\nu(\phi_4').$$

This is what we need. $\square$

Let π and σ be finite length tempered representations of $\mathrm{U}(V)$, and let

$$\ell \in \mathrm{Hom}_{\mathrm{U}(V)}(\pi \mathbin{\widehat{\otimes}} \sigma \mathbin{\widehat{\otimes}} \omega, \mathbb{C})$$

be a nonzero linear form.

Lemma 2.6. *For all $e \in \pi$, $f \in \sigma$, and $\phi \in \mathcal{S}$, there are constants C and d such that we have*

$$|\ell(\pi(g)e, f, \omega(h(v,0))\phi)| \leq C \Xi(g) \varsigma(g)^d (1 + \|v\|)^d$$

for all $g \in \mathrm{U}(V)$ and $v \in V$.

Proof. Let $Y_1, \cdots, Y_{n+1}$ be a basis of $\mathfrak{h}(V)_{\mathbb{R}}$, and put

$$\Delta = 1 - Y_1^2 - \cdots - Y_{n+1}^2 \in \mathcal{U}(\mathfrak{h}(V)).$$

Let $k > n+1$ be an integer. By elliptic regularity, cf. [BP20, Section 2.1], we can find $\varphi_1 \in C_c^{k-n}(H(V))$ and $\varphi_2 \in C_c^\infty(H(V))$ such that

$$\omega(\varphi_1)\omega(\Delta^k) + \omega(\varphi_2) = \mathbf{1}_\omega$$

where $\mathbf{1}_\omega$ stands for the identify automorphism of ω. Therefore

$$\begin{aligned} &\ell(\pi(g)e, f, \omega(h(v,0))\phi) \\ =&\ell(\pi(g)e, f, \omega(\varphi_1)\omega(\Delta^k)\omega(h(v,0))\phi) + \ell(\pi(g)e, f, \omega(\varphi_2)\omega(h(v,0))\phi). \end{aligned} \tag{2.11}$$

We estimate the first term $\ell(\pi(g)e, f, \omega(\varphi_1)\omega(\Delta^k)\omega(h(v,0))\phi)$, the other term

$$\ell(\pi(g)e, f, \omega_{\psi,\mu}(\varphi_2)\omega(h(v,0))\phi)$$

can be estimated similarly.

Note that

$$h(v,0)^{-1}\Delta^k h(v,0)-\Delta^k$$

lies in the center of $\mathfrak{h}(V)$, and there is a polynomial of degree at most $2k$ on V (viewed as real vector space) such that

$$\omega(h(v,0)^{-1}\Delta^k h(v,0))\phi = p(v)\omega(\Delta^k)\phi.$$

We can find functions $\varphi_3, \varphi_4 \in C_c^\infty(\mathrm{U}(V))$, and an $f_0 \in \sigma$ such that

$$\begin{aligned}&\ell(\pi(g)e, f, \omega(\varphi_1)\omega(\Delta^k)\omega(h(v,0))\phi)\\ =&p(v)\ell(\pi(\varphi_3)\pi(g)e, \sigma(\varphi_3)\sigma(\varphi_4)f_0, \omega(\varphi_3)\omega(\varphi_1)\omega(h(v,0))\omega(\Delta^k)\phi).\end{aligned}$$

We denote by $\varphi_3 * \varphi_4$ the usual convolution of φ_3 and φ_4, and $\varphi_3 * \varphi_1$ the function on $J(V)$ given by

$$\varphi_3 * \varphi_1(gh) = \varphi_3(g)\varphi_1(h), \quad g \in \mathrm{U}(V), \quad h \in H(V).$$

Then

$$(\varphi_3 * \varphi_4) \otimes (\varphi_3 * \varphi_1)$$

gives a function in $C_c^{k-n}(\mathrm{U}(V) \times J(V))$, and

$$(\sigma \mathbin{\widehat{\otimes}} (\pi \mathbin{\widehat{\otimes}} \omega))^\vee((\varphi_3 * \varphi_4) \otimes (\varphi_3 * \varphi_1))\ell \in \sigma^{(k-n-k_1)} \mathbin{\widehat{\otimes}}_{\mathrm{h}} (\pi \mathbin{\widehat{\otimes}} \omega)^{(k-n-k_1)},$$

where k_1 is a fixed integer depending only on ℓ. It follows that

$$\ell(\pi(\varphi_3)\pi(g)e, \sigma(\varphi_3 * \varphi_4)f_0, \omega(\varphi_3)\omega(\varphi_1)\omega(h(v,0))\omega(\Delta^k)\phi)$$

equals

$$\langle(\sigma \mathbin{\widehat{\otimes}} (\pi \mathbin{\widehat{\otimes}} \omega))^\vee((\varphi_3 * \varphi_4) \otimes (\varphi_3 * \varphi_1))\ell, f \otimes (e \otimes \phi)\rangle$$

We now fix a positive integer l such that the weak inequality holds for elements in $\pi^{(l)}$ and $\sigma^{(l)}$, and the estimate in Lemma 2.1 holds for elements in $\mathcal{S}^{(l)}$. By (2.5) we may then fix a large k, such that

$$(\pi \mathbin{\widehat{\otimes}} \omega)^{(k-n-k_1)} \subset \pi^{(l)} \mathbin{\widehat{\otimes}}_{\mathrm{h}} \omega^{(l)},$$

and hence

$$(\pi^J \otimes \sigma)^\vee((\varphi_3 * \varphi_1) \otimes (\varphi_3 * \varphi_4))\ell \in \pi^{(l)} \mathbin{\widehat{\otimes}}_{\mathrm{h}} \omega^{(l)} \mathbin{\widehat{\otimes}}_{\mathrm{h}} \sigma^{(l)}.$$

Therefore

$$\langle(\pi^J \otimes \sigma)^\vee((\varphi_3 * \varphi_1) \otimes (\varphi_3 * \varphi_4))\ell, f \otimes e \otimes \phi\rangle$$

is bounded by a constant multiple of $\Xi(g)\varsigma(g)^d$ for some d. □

This lemma leads to the following proposition.

Proposition 2.7. *Let the notation be as in the previous lemma. Take $f_1, f_2 \in \mathcal{C}(\mathrm{U}(V))$, and $\phi_1, \phi_2 \in \omega$. For all $e \in \pi$, $f \in \sigma$ and $\phi \in \mathcal{S}$, the integral*

$$\int_{\mathrm{U}(V)\times\mathrm{U}(V)} \int_{H(V)/Z} \ell(\pi(g_1)e, \sigma(g_2)f, \omega(g_2h)\phi) f_1(g_1) f_2(g_2) \overline{\langle\omega(g_2h)\phi_1, \phi_2\rangle} \mathrm{d}h\mathrm{d}g_1\mathrm{d}g_2$$

is absolutely convergent.

Proof. Let us make a change of variable $g_1 \mapsto g_2 g_1$. We need to prove that

$$\int_{\mathrm{U}(V)} \int_V \int_{\mathrm{U}(V)} \ell(\pi(g_1)e, f, \omega(h(v,0))\phi) f_1(g_2 g_1) f_2(g_2) \overline{\langle \omega(g_2 h)\phi_1, \phi_2\rangle} \mathrm{d}g_2 \mathrm{d}h \mathrm{d}g_1$$

is absolutely convergent.

By Lemma 2.5 and Lemma 2.6, we need to show that for any d_1 we can find a sufficiently large $d_2 > 0$ we have

$$\int_{\mathrm{U}(V)} \int_{\mathrm{U}(V)} \int_V \Xi(g_1)(1+\|v\|^2)^{d_1} \Xi(g_2 g_1)\Xi(g_2) |\langle \omega(g_2 h(v,0))\phi_1, \phi_2\rangle| \varsigma(g_2)^{-d_2} \varsigma(g_1)^{-d_2} \mathrm{d}v \mathrm{d}g_2 \mathrm{d}g_1$$

is convergent.

We use the Cartan decomposition for g_2, and integrate it first. The integral becomes

$$\int_{\mathrm{U}(V)} \int_V \int_{A^+} \int_K \int_K \varphi(a)\Xi(g_1)(1+\|v\|^2)^{d_1} \Xi(a k_2 g_1)\Xi(a)$$
$$|\langle \omega(k_1 a k_2 h(v,0))\phi_1, \phi_2\rangle| \varsigma(a)^{-d_2} \varsigma(g_1)^{-d_2} \mathrm{d}v \mathrm{d}g_2 \mathrm{d}g_1$$

Make another change of variable $v \mapsto k_2^{-1} v$ to obtain

$$\int_{\mathrm{U}(V)} \int_V \int_{A^+} \int_K \int_K \varphi(a)\Xi(g_1)(1+\|k_2^{-1}v\|^2)^{d_1} \Xi(a k_2 g_1)\Xi(a)$$
$$|\langle \omega(a h(v,0) k_2)\phi_1, \omega(k_1^{-1})\phi_2\rangle| \varsigma(a)^{-d_2} \varsigma(g_1)^{-d_2} \mathrm{d}k_1 \mathrm{d}k_2 \mathrm{d}a \mathrm{d}v \mathrm{d}g_1$$

Since K is compact and $\|v\|^2$ is a homogenenous polynomial, the ratio

$$\frac{\sup_{k\in K} \|kv\|^2}{\|v\|^2}$$

is bounded above by a constant independent of v. Moreover by the uniform boundedness principle for any seminorm ν on $\mathcal{S}$

$$\sup_{k\in K} \nu(\omega(k)\phi)$$

is again a seminorm on $\mathcal{S}$. Thus using the estimate from Lemma 2.5, we only need to prove the absolute convergence of

$$\int_{\mathrm{U}(V)} \int_V \int_{A^+} \int_K \varphi(a)\Xi(g_1)(1+\|v\|^2)^{d_1} \Xi(a k_2 g_1)\Xi(a)$$
$$(a_1 \cdots a_r)^{-1}(1+\|la^{-1}+l'+v_0\|^2)^{-d} \varsigma(a)^{-d_2} \varsigma(g_1)^{-d_2} \mathrm{d}k_2 \mathrm{d}a \mathrm{d}v \mathrm{d}g_1 \tag{2.12}$$

for sufficiently large d. Here we follow the notation of Lemma 2.5, $a = (a_1, \cdots, a_r) \in A^+$, $v = l + v_0 + l'$, $l \in X$, $l' \in X^\vee$, $v_0 \in V_0$.

Now by the doubling principle, cf. [BP20, Proposition 1.5.1(vi)], integrating over $k_2 \in K$ gives

$$\int_K \Xi(a k_2 g_1) \mathrm{d}k = \Xi(a)\Xi(g_1).$$

Thus the integral (2.12) is a product of

$$\int_{\mathrm{U}(V)} \Xi(g_1)^2 \varsigma(g_1)^{-d_2} \mathrm{d}g_1$$

and

$$\int_V \int_{A^+} \varphi(a)(1+\|v\|^2)^{d_1}\Xi(a)^2(a_1\cdots a_r)^{-1}(1+\|la^{-1}+l'+v_0\|)^{-d}\varsigma(a)^{-d_2}\mathrm{d}a\mathrm{d}v$$

It is well-known that the first integral is absolutely convergent when d_2 is large, cf. [BP20, Proposition 1.5.1(v)]. For the second one, we make a change of variable and $l \mapsto la$ and observe that since $0 < a_1 \leq \cdots \leq a_r \leq 1$ we have

$$\|la + l' + v_0\| \leq \|v\|.$$

Then the second integral is bounded above by

$$\int_V (1+\|v\|^2)^{d_1-d}\mathrm{d}v \times \int_{A^+} \varphi(a)\Xi(a)^2(a_1\cdots a_r)^{-1}\varsigma(a)^{-d_2}\mathrm{d}a$$

The first term is absolutely convergent if d is sufficiently large. To see the second term, we apply the standard estimates (2.9) and (2.10) to reduce it to the absolute convergence of

$$\int_{0<a_1\leq\cdots\leq a_r\leq 1} (a_1\cdots a_r)(-\log a_1 - \cdots - \log a_r)^{-d_2}\mathrm{d}a_1\cdots\mathrm{d}a_r$$

for sufficiently large d_2, which is clear. $\square$

3. The basic case: codimension zero and tempered

The goal of this section is to prove Theorem 1.1 under the assumptions that

- $t = 0$, so $W = V$,
- π and σ are tempered, and in particular they are unitary.

We keep these assumptions throughout this section. We also slightly change notation for the ease of exposition. We write π for a representation of $\mathrm{U}(V) \times \mathrm{U}(V)$, instead of $\pi \widehat{\otimes} \sigma$. The Weil representation ω is realized on the mixed model $\mathcal{S}$.

3.1. Tempered intertwining.

3.1. Tempered intertwining. Let $G = \mathrm{U}(V) \times \mathrm{U}(V)$ and $G^J = \mathrm{U}(V) \times J(V)$. Let $H = \mathrm{U}(V)$, which embeds in G and G^J diagonally.

Let π be a finite length tempered representation of G. We put $\pi^J = \pi \widehat{\otimes} \overline{\omega}$, which is a finite length tempered representation of G^J. Let $\mathrm{End}(\pi^J)$ be the algebra of (continuous) endomorphism of π^J, which has an action of $G^J \times G^J$ by left and right multiplication. Let $\mathrm{End}(\pi^J)^\infty$ be the smooth vector in $\mathrm{End}(\pi^J)$, which is identified with $\pi^J \widehat{\otimes} \overline{\pi^J}$.

We define

$$\mathcal{L}_{\pi^J} : \mathrm{End}(\pi^J)^\infty = \pi^J \widehat{\otimes} \overline{\pi^J} \to \mathbb{C}, \quad T \mapsto \mathcal{L}_{\pi^J}(T) = \int_H \mathrm{Trace}(\pi(h)T)\mathrm{d}h.$$

By [Xuea, Lemma 3.3] the integral is absolutely convergent and $\mathcal{L}_{\pi^J}$ is a continuous linear form. It then follows that $\mathcal{L}_{\pi^J}$ defines a continuous linear map

$$L_{\pi^J} : \pi^J \mapsto \overline{(\pi^J)^\vee}, \quad L_{\pi^J}(e) = (f \mapsto \mathcal{L}_{\pi^J}(e \otimes f)).$$

The image of L_{π^J} lies in $\mathrm{Hom}_H(\overline{\pi^J}, \mathbb{C})$, which is finite dimensional. Therefore if $T \in \mathrm{End}(\pi^J)^\infty$ then we have the compositions

$$L_{\pi^J} T : \overline{(\pi^J)^\vee} \to \overline{(\pi^J)^\vee}, \quad T L_{\pi^J} : \pi^J \to \pi^J,$$

which are both finite rank operators and their traces make sense. It follows immediately from the definition that

$$\mathrm{Trace}\, L_{\pi^J} T = \mathrm{Trace}\, T L_{\pi^J} = \mathcal{L}_{\pi^J}(T).$$

The importance of $\mathcal{L}_{\pi^J}$ is manifested in the following proposition, which we proved in [Xuea, Proposition 3.6].

Proposition 3.1. *Theorem 1.1 holds if the condition $m(\pi) \neq 0$ is replaced by $\mathcal{L}_{\pi^J} \neq 0$.*

Thus to prove Theorem 1.1 in the case $t = 0$ in the tempered case, it is enough to prove the following theorem.

Theorem 3.2. *Assume π is irreducible, then $m(\pi) \neq 0$ if and only if $\mathcal{L}_{\pi^J} \neq 0$.*

The proof of this theorem occupies the rest of this section.

3.2. **Induction and tempered intertwining.** Let $P = MN$ be a parabolic subgroup of G, and τ be a tempered representation of M. Let $\alpha \in \sqrt{-1}\mathcal{A}_M^*$, then we have the induced representation

$$I_\alpha = \mathrm{Ind}_P^G \tau_\alpha.$$

Let K be a maximal compact subgroup of G. The representation I_α is realized on the space

$$\{f : C^\infty(K, \tau) \mid f(pk) = \tau(p) f(k),\ p \in P \cap K\}.$$

We denote this space by $\mathcal{V}$, which is independent of α. With this realization,

$$\mathcal{L}_{(I_\alpha)^J} \in (\mathcal{V} \otimes \overline{\mathcal{V}})^\vee, \quad L_{(I_\alpha)^J} \in \mathrm{Hom}(\mathcal{V}, \overline{\mathcal{V}^\vee}).$$

As $(\mathcal{V} \otimes \overline{\mathcal{V}})^\vee$ and $\mathrm{Hom}(\mathcal{V}, \overline{\mathcal{V}^\vee})$ are topological vector spaces independent of α, it make sense to speak of the smoothness of $\mathcal{L}_{(I_\alpha)^J}$ and $L_{(I_\alpha)^J}$, cf. [BP20, Appendix A.3]. The same argument as in [BP20, Lemma 7.2.2(i)] gives the following lemma.

Lemma 3.3. *The maps*

$$\alpha \mapsto \mathcal{L}_{(I_\alpha)^J}, \quad \alpha \mapsto L_{(I_\alpha)^J}, \quad \alpha \in \sqrt{-1}\mathcal{A}_M^*$$

are smooth.

The main result of this subsection is the following.

Proposition 3.4. *If $\mathcal{L}_{(I_\alpha)^J} \neq 0$ for some $\alpha \in \sqrt{-1}\mathcal{A}_M^*$, then it is not zero for all $\alpha \in \sqrt{-1}\mathcal{A}_M^*$.*

Proof. In theory, one could proves this directly by relating it to the tempered intertwining for σ (should we have defined it). But we take a shortcut and transport the results of [BP20, Proposition 7.4.1] to our current situation by using theta lifts.

We introduce more notation. The parabolic subgroup P of G also has a decomposition $P = P_1 \times P_2$, where $P_i = M_i N_i$, $i = 1, 2$ is a parabolic subgroup of $\mathrm{U}(V)$. We have $M_i = \mathrm{U}(V_i) \times \mathrm{GL}_{a_i}(\mathbb{C})$, $i = 1, 2$, where $V_i \subset V$ is of codimension $2a_i$. There is an irreducible tempered representation σ_i of $\mathrm{U}(V_i)$, an irreducible tempered representation τ_i of $\mathrm{GL}_{a_i}(\mathbb{C})$. We also have $\sqrt{-1}\mathcal{A}^*_M = \sqrt{-1}\mathcal{A}^*_{M_1} \times \sqrt{-1}\mathcal{A}^*_{M_2}$ and $\alpha = (\alpha_1, \alpha_2)$ where $\alpha_i \in \sqrt{-1}\mathcal{A}^*_{M_i}$, $i = 1, 2$. Then

$$I_{i,\alpha_i} = \mathrm{Ind}_{P_i}^{\mathrm{U}(V)} \sigma_i \mathbin{\widehat{\otimes}} \tau_{i,\alpha_i}, \quad i = 1, 2,$$

and $I_\alpha = I_{1,\alpha_1} \mathbin{\widehat{\otimes}} I_{2,\alpha_2}$. As $\mathcal{L}_{(\pi_\alpha)^J} \neq 0$, we may find subrepresentations

$$\pi_i \subset I_{\alpha_i}, \quad i = 1, 2,$$

such that $\mathcal{L}_{(\pi_1 \mathbin{\widehat{\otimes}} \pi_2)^J} \neq 0$.

The setup of theta lifts is explained in [Xuea, Section 3] in detail. We do not need all the details and will only summarize what we need. For simplicity we will assume from now on that n is even. The odd case requires only a slight modification of notation, e.g. taking dual at various places. Let V' be a hermitian space of dimension $n+1$ and W' a hermitian space of dimension n, such that $V' = W' \oplus^\perp L_+$ where L_+ is a hermitian line of sign $+1$ (in the case of odd n, we need sign -1). We consider the theta lifts from $\mathrm{U}(V)$ to $\mathrm{U}(V')$ and from $\mathrm{U}(W')$ to $\mathrm{U}(V)$, which we denote by $\theta_{V,V'}$ and $\theta_{W',V}$ respectively. The theta lifts involve the choices of various characters, and we fix them as in [Xueb, Subsection 3.4].

We may find a (unique) W' such that there is an irreducible representation σ' of $\mathrm{U}(W')$ such that $\theta_{W',V}(\overline{\sigma'}) = \overline{\pi_2}$. Let $\pi' = \theta_{V,V'}(\pi_1)$. By the induction principle of theta lifts, cf. [Pau98, Theorem 4.5.5], we have the following description of π' and σ'. There is a parabolic subgroup $P' = M'N'$ of $\mathrm{U}(V')$ with $M' = \mathrm{U}(V'_0) \times \mathrm{GL}_{a_1}(\mathbb{C})$, $\pi'_0 = \theta_{V_1,V'_0}(\sigma_1)$, such that π' is a subrepresentation of

$$I'_{\alpha_1} = \mathrm{Ind}_{P'}^{\mathrm{U}(V')} \pi'_0 \mathbin{\widehat{\otimes}} \overline{\tau_{1,\alpha_1}}.$$

There is a parabolic subgroup $Q' = L'U'$ of $\mathrm{U}(W')$ with $L' = \mathrm{U}(W'_0) \times \mathrm{GL}_{a_2}(\mathbb{C})$ and an irreducible tempered representation σ'_0 of $\mathrm{U}(W'_0)$ such that $\theta_{W'_0,V_0}(\overline{\sigma'_0}) = \overline{\sigma_2}$, and σ' is a subrepresentation of

$$I'_{\alpha_2} = \mathrm{Ind}_{Q'}^{\mathrm{U}(Q')} \sigma'_0 \mathbin{\widehat{\otimes}} \overline{\tau_{2,\alpha_1}}.$$

For any representation ρ' of $\mathrm{U}(V') \times \mathrm{U}(W')$, a tempered intertwining linear form $\mathcal{L}_{\rho'}$ was introduced in [BP20]. It defines an element in

$$\mathrm{Hom}_{\mathrm{U}(W')}(\rho', \mathbb{C}) \otimes \overline{\mathrm{Hom}_{\mathrm{U}(W')}(\rho', \mathbb{C})},$$

and if ρ' is irreducible and this Hom space is not zero, then $\mathcal{L}_{\rho'} \neq 0$, cf. [BP20, Theorem 7.2.1].

Since $\mathcal{L}_{(\pi_1 \widehat{\otimes} \pi_2)^J} \neq 0$, by [Xueb, Lemma 3.8], we have $\mathcal{L}_{\pi' \widehat{\otimes} \sigma'} \neq 0$. Note that this is where n being even or odd makes a slight difference, as [Xueb, Lemma 3.8] involves the characters $\psi^{(-1)^n}$ and $\mu^{(-1)^n}$. Therefore $\mathcal{L}_{I'_{\alpha_1} \widehat{\otimes} I'_{\alpha_2}} \neq 0$ for this α_1 and α_2. By [BP20, Proposition 7.4.1] we have

$$\mathcal{L}_{I'_{\alpha_1} \widehat{\otimes} I'_{\alpha_2}} \neq 0.$$

for all α_1 and α_2. Therefore if $\beta_i \in \sqrt{-1}\mathcal{A}^*_{M_i}$, $i = 1, 2$, there are irreducible subrepresentation

$$\pi'_{\beta_1} \subset I'_{\beta_1}, \quad \sigma'_{\beta_2} \subset I'_{\beta_2}$$

such that $\mathcal{L}_{\pi'_{\beta_1} \widehat{\otimes} \sigma'_{\beta_2}} \neq 0$. Again by the induction principle there is an irreducible subrepresentation $\pi_{1,\beta_1} \subset I_{\beta_1}$ such that $\theta_{V,V'}(\pi_{\beta_1}) = \pi'_{\beta_1}$ and $\pi_{2,\beta_2} = \theta_{W',V}(\sigma'_{\beta_2})$ is an irreducible subrepresentation of I_{β_2}. Using [Xueb, Lemma 3.8] again we conclude that $\mathcal{L}_{(\pi_{1,\beta_1} \widehat{\otimes} \pi_{2,\beta_2})^J} \neq 0$ and hence $\mathcal{L}_{(I_{\beta_1} \widehat{\otimes} I_{\beta_2})^J} \neq 0$. This proves the propositions. □

3.3. **Proof of Theorem 3.2.** We denote by $\mathrm{Temp}(G)$ the set of irreducible tempered representations of G. Put

$$\mathcal{X}_{\mathrm{temp}}(G) = \bigcup_{P,\sigma} \left\{ \mathrm{Ind}_P^G \sigma_\alpha \mid \alpha \in \sqrt{-1}\mathcal{A}^*_M \right\}$$

where $P = MN$ ranges over all parabolic subgroups of G and σ ranges over all irreducible square integrable representation of M, up to the conjugation by G. Thus $\mathcal{X}_{\mathrm{temp}}$ has a structure of a manifold with infinitely many connected components, and each

$$\{\mathrm{Ind}_P^G \sigma_\alpha \mid \alpha \in \sqrt{-1}\mathcal{A}^*_M\}$$

is a connected component. There is a Plancherel measure on $\mathcal{X}_{\mathrm{temp}}(G)$, which we denote by $\mathrm{d}\mu$, cf. [BP20, Section 2.6].

Let

$$\mathcal{O} = \left\{ \mathrm{Ind}_P^G \sigma_\alpha \mid \alpha \in \sqrt{-1}\mathcal{A}^*_M \right\}$$

be a connected component of $\mathcal{X}_{\mathrm{temp}}$. We realize representations in $\mathcal{O}$ on a vector space $\mathcal{V}$ that is independent of α, as in Subsection 3.2. Define

$$C_c^\infty(\mathcal{O}, \mathcal{V} \widehat{\otimes} \overline{\mathcal{V}})$$

to be the compactly supported smooth functions on $\mathcal{O}$ valued in $\mathcal{V} \widehat{\otimes} \overline{\mathcal{V}}$. We view each $T_\alpha \in C_c^\infty(\mathcal{O}, \mathcal{V} \widehat{\otimes} \overline{\mathcal{V}})$ as a family of endomorphisms in $\mathrm{End}(\mathrm{Ind}_P^G \sigma_\alpha)^\infty$. By the matrical Paley–Wiener Theorem as stated in [BP20, Theorem 2.6.1], for any T_α there is a unique $f \in \mathcal{C}(G)$ such that the map

$$\mathcal{X}_{\mathrm{temp}}(G) \ni \pi \mapsto \pi(f)$$

is supported on $\mathcal{O}$ and equals T_α on $\mathcal{O}$.

Lemma 3.5. *Let $\pi \in \mathrm{Temp}(G)$ or $\mathcal{X}_{\mathrm{temp}}(G)$, and $S, T \in \mathrm{End}(\pi^J)^\infty$. Then*

$$\mathcal{L}_{\pi^J}(S)\mathcal{L}_{\pi^J}(T) = \mathcal{L}_{\pi^J}(S L_{\pi^J} T). \tag{3.1}$$

Proof. We will assume that $\mathrm{Hom}_H(\pi^J, \mathbb{C}) \neq 0$ for otherwise $\mathcal{L}_\pi = 0$ and the lemma is void.

The fact that $SL_{\pi^J}T \in \mathrm{End}(\pi^J)^\infty$ and the smoothness of $\pi \mapsto SL_{\pi^J}T$ is proved in the same way as [BP20, Lemma 7.2.2(iii)]. Moreover both sides of (3.1) are continuous linear forms on both variables S and T. Thus to prove (3.1) it is enough to assume that

$$S = (e_1 \otimes f_1)^{\phi_1,\phi_2}, \quad T = (e_2 \otimes f_2)^{\phi_3,\phi_4}, \quad e_1, e_2, f_1, f_2 \in \pi, \quad \phi_1, \phi_2, \phi_3, \phi_4 \in \mathcal{S}.$$

If $e \in \pi$ and $\phi \in \mathcal{S}$ then

$$SL_{\pi^J}T(e \otimes \phi) = \langle \phi_4, \phi \rangle \langle e, f_2 \rangle \mathcal{L}_{\pi^J}(e_2 \otimes \phi_3 \otimes f_1 \otimes \phi_2) e_1 \otimes \phi_1,$$

and hence as an element in $\pi^J \mathbin{\widehat{\otimes}} \overline{\pi^J}$, it equals

$$\mathcal{L}_{\pi^J}(e_2 \otimes \phi_3 \otimes f_1 \otimes \phi_2) \left((e_1 \otimes \phi_1) \otimes (f_2 \otimes \phi_4) \right).$$

Therefore

$$\mathcal{L}_{\pi^J}(SL_{\pi^J}T) = \mathcal{L}_{\pi^J}(e_2 \otimes \phi_3 \otimes f_1 \otimes \phi_2) \mathcal{L}_{\pi^J}(e_1 \otimes \phi_1 \otimes f_2 \otimes \phi_4).$$

We also have by definition that

$$\mathcal{L}_{\pi^J}(S) \mathcal{L}_{\pi^J}(T) = \mathcal{L}_{\pi^J}(e_1 \otimes \phi_1 \otimes f_1 \otimes \phi_2) \mathcal{L}_{\pi^J}(e_2 \otimes \phi_3 \otimes f_2 \otimes \phi_4).$$

Assume $\pi \in \mathrm{Temp}(G)$. Then $\mathrm{Hom}_H(\pi^J, \mathbb{C})$ is one dimensional and we fix a nonzero element ℓ in it. Thus there is a constant a depending only on π such that

$$\mathcal{L}_\pi = a(\ell \otimes \overline{\ell}).$$

We conclude that

$$\mathcal{L}_{\pi^J}(S) \mathcal{L}_{\pi^J}(T) = \mathcal{L}_{\pi^J}(SL_{\pi^J}T)$$

as they both equal

$$a^2 \ell(e_2 \otimes \phi_3) \overline{\ell(f_1 \otimes \phi_2)} \ell(e_1 \otimes \phi_1) \overline{\ell(f_2 \otimes \phi_4)}.$$

Now assume that $\pi \in \mathcal{X}_{\mathrm{temp}}(G)$. We can find a parabolic subgroup $P = MN$ and an irreducible square integrable representation σ of M such that

$$\pi = \mathrm{Ind}_P^G \sigma.$$

We put

$$I_\alpha = \mathrm{Ind}_P^G \sigma_\alpha, \quad \alpha \in \sqrt{-1}\mathcal{A}_M^*.$$

Then I_α is irreducible for almost all α, and hence

$$\mathcal{L}_{(I_\alpha)^J}(S) \mathcal{L}_{(I_\alpha)^J}(T) = \mathcal{L}_{(I_\alpha)^J}(SL_{(I_\alpha)^J}T)$$

for almost all α. Then since both sides are continuous functions in α, we conclude that it holds for all α and in particular $\alpha = 0$, i.e. it holds for π. $\square$

If $T \in \mathrm{End}(\pi)^\infty$ and $\phi_1, \phi_2 \in \mathcal{S}$, then we define $T^{\phi_1,\phi_2} \in \mathrm{End}(\pi^J)^\infty$ to be

$$T^{\phi_1,\phi_2}(e \otimes \phi) = \langle \phi_2, \phi \rangle (T(e) \otimes \phi_1), \quad e \in \pi, \ \phi \in \mathcal{S}.$$

We define

$$L_\pi^{\phi_1,\phi_2} : \pi \to \overline{\pi^\vee}, \quad e \mapsto (f \mapsto \mathcal{L}_{\pi^J}(e \otimes \phi_1 \otimes f \otimes \phi_2)).$$

Lemma 3.6. *Let $S = A^{\phi_1,\phi_2}$, $T = B^{\phi_3,\phi_4}$ where $A, B \in \mathrm{End}(\pi)^\infty$ and $\phi_1, \phi_2, \phi_3, \phi_4 \in \mathcal{S}$. Then*

$$SL_{\pi^J}T = (AL_\pi^{\phi_3,\phi_2}B)^{\phi_1,\phi_4}, \tag{3.2}$$

Proof. Observe that in the variables A and B, both sides are continuous linear maps

$$\mathrm{End}(\pi)^\infty \times \mathrm{End}(\pi)^\infty \to \mathrm{End}(\pi^J)^\infty.$$

Thus we only need to check this when $A = e_1 \otimes f_1$ and $B = e_2 \otimes f_2$, $e_1, e_2, f_1, f_2 \in \pi$. Then the proof of Lemma 3.5 gives that the left hand side of (3.2) equals

$$\mathcal{L}_{\pi^J}(e_2 \otimes \phi_3 \otimes f_1 \otimes \phi_2)((e_1 \otimes \phi_1) \otimes (f_2 \otimes \phi_4)).$$

The right hand side also equals this by definition. □

If $f \in \mathcal{C}(G)$ and $\phi_1, \phi_2 \in \mathcal{S}$, then

$$gh \mapsto f(g)\overline{\langle \phi_1, \omega(gh)\phi_2 \rangle}, \quad g \in G, \ h \in H(V)$$

is a smooth function on G^J. We let $\mathcal{C}(G^J)^\circ$ be the space of functions that are finite linear combinations of this functions of this form. If $f \in \mathcal{C}(G)$ and $\phi_1, \phi_2 \in \mathcal{S}$, then

$$\pi^J(f^{\phi_1,\phi_2}) = \pi(f)^{\phi_1,\phi_2}.$$

Lemma 3.7. *For all $f \in \mathcal{C}(G^J)^\circ$, such that the map*

$$\mathcal{X}_{\mathrm{temp}}(G) \to \mathrm{End}(\pi^J)^\infty, \quad \pi \to \pi^J(f)$$

is compactly supported, we have

$$\int_H f(h)\mathrm{d}h = \int_{\mathcal{X}_{\mathrm{temp}}(G)} \mathcal{L}_{\pi^J}(\pi^J(f))\mathrm{d}\mu(\pi),$$

Proof. We may assume that $f = f_1^{\phi_1,\phi_2}$ where $f_1 \in \mathcal{C}(G)$, $\phi_1, \phi_2 \in \mathcal{S}$. Then

$$\mathrm{Trace}\, \pi^J(f_1^{\phi_1,\phi_2}) = \overline{\langle \phi_1, \phi_2 \rangle}\, \mathrm{Trace}\, \pi(f)$$

The right hand side equals

$$\int_{\mathcal{X}_{\mathrm{temp}}(G)} \int_H \overline{\langle \omega(h)\phi_1, \phi_2 \rangle}\, \mathrm{Trace}(\pi(h)\pi(f))\mathrm{d}h\mathrm{d}\mu(\pi).$$

Since the map $\pi \mapsto \pi^J(f)$ is compactly supported we conclude that the double integral is absolutely convergent and hence we can change the order. It follows that the right hand side equals

$$\int_H \int_{\mathcal{X}_{\mathrm{temp}}(G)} \overline{\langle \omega(h)\phi_1, \phi_2 \rangle}\, \mathrm{Trace}(\pi(h)\pi(f))\mathrm{d}\mu(\pi)\mathrm{d}h.$$

By the Plancherel formula for G, cf. [BP20, Theorem 2.6.1], the inner integral equals $f(h)$. The lemma then follows. $\square$

We are now ready to prove the main result.

Proof of Theorem 3.2. We choose an $\ell \in \mathrm{Hom}_H(\pi^J, \mathbb{C})$ and $v \in \pi^J$ such that $\ell(v) \neq 0$. We may assume that $v = e \otimes \phi$ where $e \in \pi$ and $\phi \in \mathcal{S}$ and $\langle \phi, \phi \rangle = 1$.

We can find a connected component $\mathcal{O}$ of $\mathcal{X}_{\mathrm{temp}}(G)$ such that π is a subrepresentation of a representation in $\mathcal{O}$. More precisely $P = MN$ is a parabolic subgroup of G, and

$$\mathcal{O} = \{\mathrm{Ind}_P^G \sigma_\alpha \mid \alpha \in \sqrt{-1}\mathcal{A}_M^*\}$$

such that $\pi \subset I_0$. To shorten notation, put $I_\alpha = \mathrm{Ind}_P^G \sigma_\alpha$. As before we realize I_α on a space $\mathcal{V}$ independent of α.

Let $f \in \mathcal{C}(G^J)^\circ$. By Proposition 2.7, the integral

$$\int_{G^J} \ell(\pi^J(x)e) f(x) \mathrm{d}x, \tag{3.3}$$

is absolutely convergent. Moveover it equals $\ell(\pi^J(f)e)$, cf. the argument in [BP20, (7.5.4)]. Assume the function f is of the form $f_1^{\phi_1,\phi_2}$, $f_1 \in \mathcal{C}(G)$ and $\phi_1, \phi_2 \in \mathcal{S}$, and the function

$$\mathcal{X}_{\mathrm{temp}}(G) \ni \pi \mapsto \pi(f_1)$$

is compactly supported in $\mathcal{O}$. By Lemma 3.7 we have

$$\ell(\pi^J(f)v) = \int_{H\backslash G^J} \ell(\pi^J(x)v) \left(\int_{\mathcal{X}_{\mathrm{temp}}(G)} \mathcal{L}_{(I_\alpha)^J}((I_\alpha)^J(f)(I_\alpha)^J(x^{-1}))\mathrm{d}\mu(\alpha) \right) \mathrm{d}x.$$

Since any $T_\alpha \in C_c^\infty(\mathcal{O}, \mathcal{V} \widehat{\otimes} \overline{\mathcal{V}})$ is of the form $I_\alpha(f_1)$ for some $f_1 \in \mathcal{C}(G)$, therefore we conclude that for any $T_\alpha \in \mathrm{End}((I_\alpha)^J)^\infty$ of the form $A_\alpha^{\phi_1,\phi_2}$ where $A_\alpha \in C_c^\infty(\mathcal{O}, \mathcal{V} \widehat{\otimes} \overline{\mathcal{V}})$ and $\phi_1, \phi_2 \in \mathcal{S}$ we have

$$\ell(T_0 v) = \int_{H\backslash G^J} \ell(\pi^J(x)v) \left(\int_{\mathcal{X}_{\mathrm{temp}}(G)} \mathcal{L}_{(I_\alpha)^J}(T_\alpha (I_\alpha)^J(x^{-1}))\mathrm{d}\mu(\alpha) \right) \mathrm{d}x. \tag{3.4}$$

We first find a $T_\alpha = A_\alpha^{\phi,\phi}$ such that $A_0 e = e$ and $A_0|_{\pi'} = 0$ for all $\pi' \subset I_0$ and $\pi' \neq \pi$. Then $\ell(T_0 v) \neq 0$. This implies that there is an α such that $\mathcal{L}_{(I_\alpha)^J} \neq 0$, and by Proposition 3.4 we conclude that $\mathcal{L}_{(I_0)^J} \neq 0$.

We can therefore find an $S_\alpha = B_\alpha^{\phi_1,\phi_2}$ with $B_\alpha \in C_c^\infty(\mathcal{O}, \mathcal{V} \otimes \overline{\mathcal{V}})$ and $\phi_1, \phi_2 \in \mathcal{S}$ such that

$$\mathcal{L}_{(I_0)^J}(S_0) \neq 0.$$

Consider a family of endomorphisms $\mathcal{L}_{(I_\alpha)^J}(S_\alpha)T_\alpha$, $\alpha \in \sqrt{-1}\mathcal{A}_M^*$. Since $T_\alpha = A_\alpha^{\phi,\phi}$ we have $\mathcal{L}_{(I_\alpha)^J}(S_\alpha)T_\alpha = (\mathcal{L}_{(I_\alpha)^J}(S_\alpha)A_\alpha)^{\phi,\phi}$ and $\mathcal{L}_{(I_\alpha)^J}(S_\alpha)A_\alpha \in C_c^\infty(\mathcal{O}, \mathcal{V} \widehat{\otimes} \overline{\mathcal{V}})$. We now apply (3.4) to $\mathcal{L}_{(I_\alpha)^J}(S_\alpha)T_\alpha$ we obtain

$$\mathcal{L}_{(I_0)^J}(S_0)\ell(T_0 v) = \int_{H\backslash G^J} \ell(\pi^J(x)v) \left(\int_{\mathcal{X}_{\mathrm{temp}}(G)} \mathcal{L}_{(I_\alpha)^J}(S_\alpha)\mathcal{L}_{(I_\alpha)^J}(T_\alpha (I_\alpha)^J(x^{-1}))\mathrm{d}\mu(\alpha) \right) \mathrm{d}x \tag{3.5}$$

By Lemma 3.5 the right hand side of (3.5) equals

$$\int_{H\backslash G^J} \ell(\pi^J(x)v)\left(\int_{\mathcal{X}_{\mathrm{temp}}(G)} \mathcal{L}_{(I_\alpha)^J}((S_\alpha L_{(I_\alpha)^J} T_\alpha)(I_\alpha)^J(x^{-1}))\mathrm{d}\mu(\alpha)\right)\mathrm{d}x$$

By Lemma 3.6 we have

$$S_\alpha L_{(I_\alpha)^J} T_\alpha = \left(B_\alpha L_{I_\alpha}^{\phi,\phi_2} A_\alpha\right)^{\phi_1,\phi}$$

and $B_\alpha L_{I_\alpha}^{\phi,\phi_2} A_\alpha \in C_c^\infty(\mathcal{O}, \mathcal{V}\widehat{\otimes}\overline{\mathcal{V}})$. Apply once again (3.4) to $S_\alpha L_{(I_\alpha)^J} T_\alpha$ have

$$0 \neq \mathcal{L}_{(I_0)^J}(S_0)\ell(T_0 v) = \ell(S_0 L_{(I_0)^J} T_0(v)).$$

Since $T_0 = A_0^{\phi,\phi}$ and $A_0|_{\pi'} = 0$ if $\pi' \subset I_0$ and $\pi' \neq \pi$, we conclude that $L_{(I_0)^J}$ is not zero when restricted to π^J, i.e. $L_{\pi^J} \neq 0$ or equivalently $\mathcal{L}_{\pi^J} \neq 0$. □

4. Reduction to the basic case

We make extensive use of the Schwartz homology theory developed in [CS21, Xueb] in this section. If G is a almost linear Nash group we denote the Schwartz homology of G by $\mathrm{H}_i(G, -)$, $i = 0, 1, 2, \cdots$.

4.1. **Reduction to the tempered case.** We still assume $t = 0$ in this section, so $W = V$. We return to the notation from the Introduction, and denote by π and σ representations of $\mathrm{U}(V)$. In the Weil representation we fix $\mu = \xi_1$.

Let σ be a representation of $\mathrm{U}(V)$ of the form

$$\mathrm{Ind}_{Q_b}^{\mathrm{U}(V)}\left(\xi_{m_1}|\cdot|_{\mathbb{C}}^{t_1} \otimes \cdots \otimes \xi_{m_b}|\cdot|_{\mathbb{C}}^{t_b} \otimes \sigma_b\right), \tag{4.1}$$

where

- $V_b \subset V$ be a hermitian space so that its orthogonal complement is a split hermitian space of dimension $2b$,
- Q_b is a parabolic subgroup of $\mathrm{U}(V)$ so that its Levi component is isomorphic to $(\mathbb{C}^\times)^b \times \mathrm{U}(V_b)$,
- $m_1, \cdots, m_b \in \mathbb{Z}$ and $t_1, \cdots, t_b$ are complex numbers with nonnegative real parts,
- σ_b is an irreducible limit of discrete series representation of $\mathrm{U}(V_b)$.

Assume that V has a decomposition

$$V = V_0 \oplus^\perp \langle z_1, z_{-1}\rangle$$

such that $q_V(z_1, z_{-1}) = 1$ and z_1, z_{-1} are isotropic vectors. Let $P = MN$ be the parabolic subgroup stabilizing the line generated by z_1. Then $M = \mathbb{C}^\times \times \mathrm{U}(V_0)$. Let π_0 be an irreducible representation of $\mathrm{U}(V_0)$, $\chi = \xi_l|\cdot|_{\mathbb{C}}^u$ and

$$\pi = \mathrm{Ind}_P^{\mathrm{U}(V)} \chi \otimes \pi_0.$$

Proposition 4.1. *Assume that $l+2u$ is not an integer with the same parity as $n+1$. Assume that for any nonnegative integer j, either*

$$l+2u+2j+2 \neq -m_i \pm 2t_i, \quad i=1,\cdots,b$$

or

$$l-2u-2j \neq -m_i \pm 2t_i, \quad i=1,\cdots,b.$$

Then

$$m(\pi,\sigma)=m(\sigma,\pi_0).$$

Proof. First by [CS21, Proposition 7.4], we have

$$\pi \widehat{\otimes} \overline{\omega} = \mathrm{Ind}_P^{\mathrm{U}(V)} \left((\chi \otimes \pi_0) \widehat{\otimes} \overline{\omega}|_P \right),$$

and it follows that

$$\mathrm{Hom}_{\mathrm{U}(V)}(\pi \widehat{\otimes} \sigma \widehat{\otimes} \overline{\omega}, \mathbb{C}) = \mathrm{Hom}_{\mathrm{U}(V)} \left(\mathrm{Ind}_P^{\mathrm{U}(V)} \left((\chi \otimes \pi_0) \widehat{\otimes} \overline{\omega}|_P \right) \widehat{\otimes} \sigma, \mathbb{C} \right).$$

We make use of the mixed model

$$\mathcal{S} = \mathcal{S}(\mathbb{C}) \widehat{\otimes} \mathcal{S}_0,$$

as described in Subsection 2.2, where $(\omega_0, \mathcal{S}_0)$ is the Weil representation of the Jacobi group $S = N \rtimes \mathrm{U}(V_0)$. Then $\mathcal{S}$ has a closed P-invariant subspace

$$\mathcal{S}(\mathbb{C}^\times) \widehat{\otimes} \mathcal{S}_0$$

whose quotient is isomorphic (as vector spaces) to

$$\mathbb{C}[[z,\overline{z}]] \widehat{\otimes} \mathcal{S}_0.$$

The space $\mathbb{C}[[z,\overline{z}]]$ is filtered by the degree which gives a filtration on $\mathbb{C}[[z,\overline{z}]] \widehat{\otimes} \mathcal{S}_0$. By Lemma 2.4, the group N preserves this filtration and acts trivially on the graded pieces. Moreover $\mathbb{C}^\times$ acts on $\mathbb{C}[[z,\overline{z}]]$ by multiplication by

$$z^i \overline{z}^j \mapsto \mu(a)|a|_{\mathbb{C}}^{\frac{1}{2}} a^i \overline{a}^j z^i \overline{z}^j = \mu(a)|a|_{\mathbb{C}}^{\frac{i+j+1}{2}} \xi_{i-j}(a) z^i \overline{z}^j.$$

In conclusion,

$$\mathrm{Ind}_P^{\mathrm{U}(V)} (\chi \otimes \pi_0) \widehat{\otimes} \overline{\omega}|_P$$

has a subrepresentation

$$\mathrm{Ind}_P^{\mathrm{U}(V)} (\chi \otimes \pi_0) \widehat{\otimes} \overline{\mathcal{S}^0}$$

and the quotient has a filtration whose graded pieces (indexed by k) are direct sums of

$$\rho_k^j = \mathrm{Ind}_P^{\mathrm{U}(V)} \xi_{l+2j-k+1} |\cdot|_{\mathbb{C}}^{u+\frac{k+1}{2}} \otimes \pi_0, \quad j=0,1,\cdots,k.$$

By [Xueb, Lemma 4.2], our assumptions on l and u imply that

$$\mathrm{H}_i(\mathrm{U}(V), \rho_k^j \widehat{\otimes} \sigma) = 0$$

for all i. Therefore by [Xueb, Corollary 2.14] we have

$$\mathrm{Hom}_{\mathrm{U}(V)}\left(\mathrm{Ind}_P^{\mathrm{U}(V)}\left((\chi\otimes\pi_0)\,\widehat{\otimes}\,\overline{\mathcal{S}}|_P\right)\widehat{\otimes}\,\sigma,\mathbb{C}\right)=\mathrm{Hom}_{\mathrm{U}(V)}\left(\mathrm{Ind}_P^{\mathrm{U}(V)}\left((\chi\otimes\pi_0)\,\widehat{\otimes}\,\overline{\mathcal{S}^0}\right)\widehat{\otimes}\,\sigma,\mathbb{C}\right).$$

By Lemma 2.2, as a representation P we have $\mathcal{S}^0=\mathrm{ind}_S^P\,\mu|\cdot|_{\mathbb{C}}^{\frac{1}{2}}\otimes\mathcal{S}_0$. Thus the above Hom-space is

$$\mathrm{Hom}_{\mathrm{U}(V)}\left(\mathrm{ind}_P^{\mathrm{U}(V)}\left(\chi\delta_P^{\frac{1}{2}}\otimes\pi_0\,\widehat{\otimes}\,\overline{\mathrm{ind}_S^P\left(\mu|\cdot|_{\mathbb{C}}^{\frac{1}{2}}\otimes\mathcal{S}_0\right)}\right)\widehat{\otimes}\,\sigma,\mathbb{C}\right),$$

which, by [CS21, Proposition 7.4] and the induction by stages [CS21, Proposition 7.2], equals

$$\mathrm{Hom}_{\mathrm{U}(V)}\left(\mathrm{ind}_S^{\mathrm{U}(V)}(\pi_0\,\widehat{\otimes}\,\overline{\mathcal{S}_0})\,\widehat{\otimes}\,\sigma,\mathbb{C}\right).$$

Finally Frobenius reciprocity, cf. [CS21, Theorem 6.8], gives the proposition (note that S is unimodular). □

Proof of Theorem 1.1 assuming $t=0$. Let π be an irreducible representation of $\mathrm{U}(V)$ and assume that π lies in a generic packet. Then π can be written as an irreducible parabolic induction

$$\xi_{l_1}|\cdot|_{\mathbb{C}}^{s_1}\times\cdots\times\xi_{l_a}|\cdot|_{\mathbb{C}}^{s_a}\times\pi_0, \tag{4.2}$$

where

- $l_1,\cdots,l_a\in\mathbb{Z}$,
- $s_1,\cdots,s_a\in\mathbb{C}$ with nonnegative real part,
- $\xi_{l_i}|\cdot|_{\mathbb{C}}^{s_i}$ is not conjugate self-dual of sign $(-1)^n$,
- $V_a\subset V$ is a hermitian subspace such that $V_a^{\perp}$ is a split hermitian space of dimension $2a$,
- π_0 is a limit of discrete series representation of $\mathrm{U}(V_a)$.

By [Xueb, Lemma 4.4], $l_i\pm 2s_i$, $i=1,\cdots,a$, and $m_j\pm 2t_j$, $j=1,\cdots,b$, are not integers of the same parity as $n+1$. By relabeling, we may assume that

$$\mathrm{Re}\,\frac{l_i+2s_i}{2}\geq\mathrm{Re}\,\frac{l_{i+1}+2s_{i+1}}{2},\quad i=1,\cdots,a-1$$

and

$$\mathrm{Re}\,\frac{-m_i+2t_i}{2}\geq\mathrm{Re}\,\frac{-m_{i+1}+2t_{i+1}}{2},\quad i=1,\cdots,b-1.$$

If $a=b=0$, then we are in the tempered case so the theorem is proved.

Assume that

$$\mathrm{Re}\,\frac{l_1+2s_1+1}{2}\geq\mathrm{Re}\,\frac{-m_1+2t_1}{2},$$

or $b=0$. Then $\mathrm{Re}\,(l_1+2s_1+2j+2)>\mathrm{Re}\,(-m_i\pm 2t_i)$ for all $i=1,\cdots,b$ and all nonnegative integer j (this is a vacuum statement if $b=0$). This is because by our ordering, $-m_1+2t_1$ has the maximal real part among $-m_i\pm 2t_i$'s, $i=1,\cdots,b$. The conditions in Proposition 4.1 are verified. Put

$$\pi_1^-=\xi_{l_2}|\cdot|^{s_2}\times\cdots\times\xi_{l_a}|\cdot|^{s_a}\times\pi_0.$$

Then by Proposition 4.1, we have

$$m(\pi,\sigma)=m(\sigma,\pi_1^-).$$

Choose $s_1' \in \sqrt{-1}\mathbb{R}$ such that $\mathrm{Im}\, s_1' \neq 0, \pm \mathrm{Im}\, t_i$, $i = 1, \cdots, b$, and

$$\pi_1 = |\cdot|^{s_1'} \times \xi_{l_2}|\cdot|^{s_2} \times \cdots \times \xi_{l_a}|\cdot|^{s_a} \times \pi_0.$$

still lies in the generic packet. The conditions in Proposition 4.1 are verified again and we conclude

$$m(\pi_1, \sigma) = m(\sigma, \pi_1^-).$$

The net effect is that we replace a possibly nonunitary quasi-character $\xi_{l_1}|\cdot|^{s_1}$ by a unitary character $|\cdot|^{s_1'}$. We can of course repeat this process for $\xi_{l_i}|\cdot|_{\mathbb{C}}^{s_i}$, $i = 1, 2, \cdots, c$, as long as

$$\mathrm{Re}\, \frac{l_i + 2s_i + 1}{2} \geq \mathrm{Re}\, \frac{-m_1 + 2t_1}{2}, \quad i = 1, 2, \cdots, c,$$

or $b = 0$. The net effect is that we replace $\xi_{l_i}|\cdot|_{\mathbb{C}}^{s_i}$ by $|\cdot|_{\mathbb{C}}^{s_i'}$, $i = 1, 2, \cdots, c$, where $s_i' \in \sqrt{-1}\mathbb{R}$ is a generic purely imaginary number. Put

$$\pi_i = |\cdot|_{\mathbb{C}}^{s_1'} \times \cdots |\cdot|_{\mathbb{C}}^{s_i'} \times \xi_{l_{i+1}}|\cdot|_{\mathbb{C}}^{s_{i+1}} \times \cdots \times \xi_{l_a}|\cdot|_{\mathbb{C}}^{s_a} \times \pi_0, \quad i = 1, 2, \cdots, c.$$

We have

$$m(\pi, \sigma) = m(\pi_1, \sigma) = \cdots = m(\pi_c, \sigma),$$

Suppose now that we have

$$\mathrm{Re}\, \frac{l_{c+1} + 2s_{c+1} + 1}{2} < \mathrm{Re}\, \frac{-m_1 + 2t_1}{2}, \tag{4.3}$$

or we are in the case either $c = 0$ or $a = 0$. The case $c = 0$ or $a = 0$ simply means that we have

$$\mathrm{Re}\, \frac{l_1 + 2s_1 + 1}{2} < \mathrm{Re}\, \frac{-m_1 + 2t_1}{2}$$

or $a = 0$ to begin with, and thus did not implement the procedures as described above. We let $\pi_c = \pi$ if this is the case.

The condition (4.3) is equivalent to

$$\mathrm{Re}\, \frac{m_1 - 2t_1 + 1}{2} < \mathrm{Re}\, \frac{-l_{c+1} - 2u_{c_1}}{2},$$

which implies $\mathrm{Re}\,(m_1 - 2t_1 - 2j) < \mathrm{Re}\,(-l_i - 2u_i)$ for all nonnegative integer j and all $i = c+1, \cdots, b$, since by our ordering, $\mathrm{Re}\,(-l_{c+1} - 2u_{c+1})$ is the smallest among all $\mathrm{Re}\,(-l_i - 2u_i)$, $i = c+1, \cdots, b$. Moreover by our choice of the s_i', $i = 1, \cdots, c$, we conclude that

$$m_1 - 2t_1 - 2j \neq \pm 2s_i',$$

for all nonnegative integer j and all $i = 1, \cdots, c$. Thus we can apply Proposition 4.1 to σ and π_c and argue as in the previous step. Let us put

$$\sigma_1^- = \omega_{m_2}|\cdot|^{t_2} \times \cdots \times \omega_{m_b}|\cdot|^{t_b} \times \sigma_0.$$

By Proposition 4.1 we have

$$m(\pi_c, \sigma) = m(\pi_c, \sigma_1^-).$$

Let $t_1' \in \sqrt{-1}\mathbb{R}$ be a generic purely imaginary number, and put

$$\sigma_1 = |\cdot|^{t_1'} \times \omega_{m_2}|\cdot|^{t_2} \times \cdots \times \omega_{m_b}|\cdot|^{t_b} \times \sigma_0.$$

Then by Proposition 4.1 again we have

$$m(\pi_c, \sigma_1^-) = m(\pi_c, \sigma_1).$$

The net effect of this process is to replace a possibly nonunitary quasi-character $\omega_{m_1}|\cdot|^{t_1}$ by a unitary one $|\cdot|^{t_1'}$. We may repeat this process for $\xi_{m_1}|\cdot|_{\mathbb{C}}^{t_1}, \cdots, \xi_{m_d}|\cdot|_{\mathbb{C}}^{t_d}$ as long as

$$\operatorname{Re}\frac{l_{c+1}+2s_{c+1}}{2} < \operatorname{Re}\frac{-m_i+2t_i}{2}, \quad i = 1, \cdots, d.$$

The net effect is that we replace $\xi_{m_i}|\cdot|_{\mathbb{C}}^{t_i}$ by $|\cdot|_{\mathbb{C}}^{t_i'}$, $i = 1, \cdots, d$, where $t_i' \in \sqrt{-1}\mathbb{R}$ is a generic purely imaginary number. Put

$$\sigma_i = |\cdot|_{\mathbb{C}}^{t_1'} \times \cdots \times |\cdot|_{\mathbb{C}}^{t_i'} \times \xi_{m_{i+1}}|\cdot|_{\mathbb{C}}^{t_{i+1}} \times \cdots \times \xi_{m_b}|\cdot|_{\mathbb{C}}^{t_b} \times \sigma_0.$$

We have

$$m(\pi_c, \sigma) = m(\pi_c, \sigma_1) = \cdots = m(\pi_c, \sigma_d).$$

Suppose that we have

$$\operatorname{Re}\frac{l_{c+1}+2s_{c+1}}{2} \geq \operatorname{Re}\frac{-m_{d+1}+2t_{d+1}}{2}.$$

Then we can switch back to π_c and make modifications of it in the same way as to π. We do the modification to the characters $\xi_{c+1}|\cdot|_{\mathbb{C}}^{s_{c+1}}$, $\xi_{c+2}|\cdot|_{\mathbb{C}}^{s_{c+2}}$ and so on in π_c as π until we are not able to, and then switch to σ_d and make modifications to it in the same way as σ. We keep repeating this process and switching back and forth between π and σ. The process terminates after $a+b$ steps and the ultimate effect is that we find generic purely imaginary numbers

$$s_1', \cdots, s_a', t_1', \cdots, t_b' \in \sqrt{-1}\mathbb{R}$$

so that we have

$$\pi_a = |\cdot|^{s_1'} \times \cdots \times |\cdot|^{s_a'} \times \pi_0, \quad \sigma_b = |\cdot|^{t_1'} \times \cdots \times |\cdot|^{t_b'} \times \sigma_0,$$

with

$$m(\pi, \sigma) = m(\pi_a, \sigma_b).$$

Since π_a and σ_b are both tempered, Theorem 1.1 holds for (π_a, σ_b). By the description of the generic packets, we have $A_{\phi_\pi} = A_{\phi_{\pi_a}}$, $A_{\phi_\sigma} = A_{\phi_{\sigma_b}}$ and $\eta_\pi = \eta_{\pi_a}$, $\eta_\sigma = \eta_{\sigma_b}$. Theorem 1.1 thus holds for (π, σ). □

4.2. **Reduction to the codimension zero case.** Let us recall the following setup from the Introduction. Let $W \subset V$ be skew-hermitian spaces so that $V = W \oplus^{\perp} Z$ where Z is a split skew-hermitian space of dimention $2t$. We fix a basis $z_{\pm 1}, \cdots, z_{\pm t}$ of Z so that

$$q_V(z_i, z_{-j}) = \delta_{ij}, \quad i, j = \pm 1, \cdots, \pm t.$$

Let U be the unipotent radical of the parabolic subgroup of $\mathrm{U}(V)$ stabilizing the flag of completely isotropic subspaces

$$\langle z_t \rangle \subset \langle z_t, z_{t-1} \rangle \subset \cdots \subset \langle z_t, \cdots, z_1 \rangle,$$

We define a character of U by

$$\psi_U(u) = \psi\left(-\mathrm{Tr}_{\mathbb{C}/\mathbb{R}} \sum_{i=1}^{t-1} q_V(z_{-i-1}, u z_i) \right), \quad u \in U.$$

If $t = 0$ or 1 we take ψ_U to be the trivial character. Let $S_V = U \rtimes \mathrm{U}(W)$ be a Fourier–Jacobi subgroup of $\mathrm{U}(V)$. Then the character ψ_U inflates to a character of S_V.

The same construction also applies to $W^+ = W \oplus^{\perp} \langle z_1, z_{-1} \rangle$ and we obtain the Jacobi subgroup S_{W^+} of $\mathrm{U}(W^+)$. Let ω be the Weil representation of S_{W^+}. There is a projection

$$S_V \to S_{W^+}$$

and ω inflates to a representation to S_V which we also denote by ω. Since ψ_U is invariant under the S_{W^+} conjugation action, $\nu = \psi_U \otimes \omega$ is a representation of S_V.

Let π and σ be irreducible representations of $\mathrm{U}(V)$ and $\mathrm{U}(W)$ respectively and assume that π and σ lie in generic packets. Let $s_1, \cdots, s_t$ be complex numbers. Let τ be the principal series representation of $\mathrm{GL}_t(\mathbb{C})$ induced from the characters $|\cdot|_{\mathbb{C}}^{s_1}, \cdots, |\cdot|_{\mathbb{C}}^{s_t}$. Let $P = MN$ be the parabolic subgroup of $\mathrm{U}(V)$ stabilizing $\langle z_1, \cdots, z_t \rangle$. Put

$$\sigma^+ = \mathrm{Ind}_P^{\mathrm{U}(V)} \tau \mathbin{\widehat{\otimes}} \sigma.$$

Proposition 4.2. *Assume that $s_1, \cdots, s_t$ are in general position, i.e. they avoid the zero set of countably many polynomials in t variables. Then*

$$m(\pi, \sigma) = m(\pi, \sigma^+).$$

Proof of Theorem 1.1 assuming Proposition 4.2. We have already proved Theorem 1.1 in the case $t = 0$. Thus the theorem holds for (σ^+, π). Theorem 1.1 then holds for (π, σ) as $A_{\phi_{\sigma^+}} = A_{\phi_\sigma}, \quad \eta_{\sigma^+} = \eta_\sigma$. This finishes the proof of Theorem 1.1. □

Proof of Proposition 4.2. The first step is similar to the proof of Proposition 4.1. The Weil representation is realized on the mixed model $\mathcal{S} = \mathcal{S}(\mathbb{C}^t) \mathbin{\widehat{\otimes}} \mathcal{S}_0$ where $\mathcal{S}_0$ is a realization of the Weil representation ω of S_{W^+}. There is a P-stable subspace $\mathcal{S}^0 = \mathcal{S}(\mathbb{C}^t \backslash \{0\}) \mathbin{\widehat{\otimes}} \mathcal{S}_0$ whose quotient is isomorphic (as vector spaces) to

$$\mathbb{C}[[z_1, \cdots, z_t, \overline{z_1}, \cdots, \overline{z_t}]] \mathbin{\widehat{\otimes}} \mathcal{S}_0.$$

The quotient has a filtration by the degree of the power series. The group N acts trivially on the graded pieces, and the graded piece as representations of $\mathrm{GL}_r(\mathbb{C})$ are

$$\rho_k = \left(\mu|\cdot|_{\mathbb{C}}^{\frac{1}{2}} \mathrm{Sym}^k(\mathbb{C}^r \oplus \overline{\mathbb{C}^r})\right) \widehat{\otimes}\, \mathcal{S}_0, \quad k = 0, 1, \cdots,$$

where $\mathbb{C}^r$ is the standard representation of $\mathrm{GL}_r(\mathbb{C})$, and $\mathrm{GL}_r(\mathbb{C})$ acts trivially on $\mathcal{S}_0$.

We have

$$\mathrm{Hom}_{\mathrm{U}(V)}(\pi \,\widehat{\otimes}\, \sigma^+ \,\widehat{\otimes}\, \overline{\mathcal{S}}, \mathbb{C}) = \mathrm{Hom}_{\mathrm{U}(V)}\left(\pi \,\widehat{\otimes} \left(\mathrm{Ind}_P^{\mathrm{U}(V)} \tau \,\widehat{\otimes}\, \sigma \,\widehat{\otimes}\, \overline{\mathcal{S}}|_P\right), \mathbb{C}\right).$$

We claim that if $s_1, \cdots, s_t$ are in general position, then

$$\mathrm{Hom}_{\mathrm{U}(V)}\left(\pi \,\widehat{\otimes}\, \sigma^+ \,\widehat{\otimes}\, \overline{\mathcal{S}}, \mathbb{C}\right) = \mathrm{Hom}_{\mathrm{U}(V)}\left(\pi \,\widehat{\otimes} \left(\mathrm{Ind}_P^{\mathrm{U}(V)} \tau \,\widehat{\otimes}\, \sigma \,\widehat{\otimes}\, \overline{\mathcal{S}^0}\right), \mathbb{C}\right).$$

Indeed this follows from

$$\mathrm{H}_i\left(\mathrm{U}(V), \pi \,\widehat{\otimes}\, \mathrm{Ind}_P^{\mathrm{U}(V)}(\tau \,\widehat{\otimes}\, \rho_k) \,\widehat{\otimes} (\sigma \,\widehat{\otimes}\, \overline{\mathcal{S}_0})\right) = 0 \tag{4.4}$$

for all i and all k. To see this we use the technique developed in [Xueb, Subsection 3.2]. We have constructed elements in $\mathcal{Z}(\mathfrak{u}(V))$ which annihilates $\mathrm{Ind}_P^{\mathrm{U}(V)}(\tau \,\widehat{\otimes}\, \rho_k) \,\widehat{\otimes} (\sigma \,\widehat{\otimes}\, \overline{\mathcal{S}_0})$. Let z be one of them given in [Xueb, Lemma 3.5]. Since π is irreducible, the element z acts on $\pi^\vee$ by a constant λ_π which is a nonzero polynomial function in $s_1, \cdots, s_{t+1}$. Thus if $(s_1, \cdots, s_{t+1})$ avoids the zeros of this polynomial, we have $\lambda_\pi \neq 0$ and thus obtain the desired vanishing for H_i from [Xueb, Corollary 2.8] for this k. Since there are only countably many k's, we conclude that if $s_1, \cdots, s_{t+1}$ are in general position, then $\mathrm{H}_i = 0$ for all k.

The second step is to understand $\mathcal{S}^0$. This is close to the analysis in [Xueb, Section 6]. Let R be the mirabolic subgroup of $\mathrm{GL}_r(\mathbb{C})$ and put $Q = N \rtimes (R \times \mathrm{U}(W))$, which has a natural quotient isomorphic to S_{W^+}. Then by Lemma 2.2 we have

$$\mathcal{S}^0 = \mathrm{ind}_Q^P \mu|\cdot|_{\mathbb{C}}^{\frac{t}{2}} \,\widehat{\otimes}\, \mathcal{S}_0,$$

where the action of Q on $\mathcal{S}_0$ is via the Weil representation through its quotient S_W^+. Using induction by stages, we conclude that

$$\mathrm{Hom}_{\mathrm{U}(V)}\left(\pi \,\widehat{\otimes} \left(\mathrm{Ind}_P^{\mathrm{U}(V)} \tau \,\widehat{\otimes}\, \sigma \,\widehat{\otimes}\, \overline{\mathcal{S}^0}\right), \mathbb{C}\right) = \mathrm{Hom}_{\mathrm{U}(V)}\left(\pi \,\widehat{\otimes} \left(\mathrm{ind}_Q^{\mathrm{U}(V)} \tau|_R \mu|\cdot|_{\mathbb{C}}^{\frac{t}{2}} \delta_P^{-\frac{1}{2}} \,\widehat{\otimes}\, \sigma \,\widehat{\otimes}\, \overline{\mathcal{S}_0}\right), \mathbb{C}\right).$$

The restriction of τ to the mirabolic subgroup R has been carefully analyzed in [Xueb, Section 5]. It has a subrepresentation τ^0 isomorphic to

$$\mathrm{ind}_{U_t}^R \psi_t$$

where U_t is the usual upper triangular unipotent subgroup of $\mathrm{GL}_t(\mathbb{C})$ and ψ_t is the generic character of U_t given by

$$\psi_t(u) = \psi(u_{12} + \cdots + u_{t-1,t}), \quad u \in U_t.$$

The quotient $(\tau|_R)/\tau^0$ admits a countable filtration whose graded pieces are isomorphic to various induced representations. The point is the same as before. For each graded piece, one can find an element in the center of the universal enveloping algebra which annihilates this piece, and this element acts on $\pi^\vee$ by a scalar, which is a polynomial function in $s_1, \cdots, s_t$. Then the same argument as the proof of (4.4) gives that

$$\mathrm{H}_i \left(\mathrm{U}(V), \pi \mathbin{\widehat{\otimes}} \left(\mathrm{ind}_Q^{\mathrm{U}(V)} \rho \chi_V |\cdot|_{\mathbb{C}}^{\frac{t}{2}} \delta_P^{-\frac{1}{2}} \mathbin{\widehat{\otimes}} \sigma \mathbin{\widehat{\otimes}} \overline{\mathcal{S}_0} \right) \right) = 0$$

for all i and all graded pieces ρ of $(\tau|_R)/\tau^0$ when $s_1, \cdots, s_t$ avoid the zeros of this polynomial function. Since there are only countably many graded piece, we conclude that as before that if $s_1, \cdots, s_t$ are in general position, then

$$\begin{aligned} &\mathrm{Hom}_{\mathrm{U}(V)} \left(\pi \mathbin{\widehat{\otimes}} \left(\mathrm{ind}_Q^{\mathrm{U}(V)} \tau|_R \chi_V |\cdot|_{\mathbb{C}}^{\frac{t}{2}} \delta_P^{-\frac{1}{2}} \mathbin{\widehat{\otimes}} \sigma \mathbin{\widehat{\otimes}} \overline{\mathcal{S}_0} \right), \mathbb{C} \right) \\ = &\mathrm{Hom}_{\mathrm{U}(V)} \left(\pi \mathbin{\widehat{\otimes}} \left(\mathrm{ind}_Q^{\mathrm{U}(V)} \left(\mathrm{ind}_{U_t}^R \psi_t \right) \mathbin{\widehat{\otimes}} \sigma \mathbin{\widehat{\otimes}} \overline{\mathcal{S}_0} \right), \mathbb{C} \right). \end{aligned}$$

Observe that

$$S_V = U_t \rtimes (N \rtimes \mathrm{U}(W))$$

and $\psi_t \otimes \overline{\mathcal{S}_0}$ is precisely the representation ν appearing in the Fourier–Jacobi model. Induction by stages again gives that

$$\mathrm{Hom}_{\mathrm{U}(V)} \left(\pi \mathbin{\widehat{\otimes}} \left(\mathrm{ind}_Q^{\mathrm{U}(V)} \left(\mathrm{ind}_{U_t}^R \psi_t \right) \mathbin{\widehat{\otimes}} \sigma \mathbin{\widehat{\otimes}} \overline{\mathcal{S}_0} \right), \mathbb{C} \right) = \mathrm{Hom}_{\mathrm{U}(V)} \left(\pi \mathbin{\widehat{\otimes}} \left(\mathrm{ind}_{S_V}^{\mathrm{U}(V)} \sigma \mathbin{\widehat{\otimes}} \nu \right), \mathbb{C} \right).$$

Another application of the Frobenius reciprocity gives that this equals (S_V is unimodular)

$$\mathrm{Hom}_{S_V} \left(\pi \mathbin{\widehat{\otimes}} \sigma \mathbin{\widehat{\otimes}} \nu, \mathbb{C} \right).$$

This is what we are after. □

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