1. Introduction

Let $F$ be a $p$-adic field of characteristic zero and $\eta : F^\times \to \{\pm 1\}$ be a nontrivial quadratic character. Let $G = \text{GL}_{2n,F}$ and $H = \text{GL}_{n,F} \times \text{GL}_{n,F}$ with an embedding

$$(h_1, h_2) \mapsto \begin{pmatrix} h_1 & \cr & h_2 \end{pmatrix}, \quad h_1, h_2 \in \text{GL}_{n,F}.$$ 

Put

$$\theta(g) = \begin{pmatrix} 1_n & \cr & -1_n \end{pmatrix} g \begin{pmatrix} 1_n & \cr & -1_n \end{pmatrix}.$$ 

Then $H = \{g \in G \mid \theta(g) = g\}$. Let

$$S = \{g^{-1}\theta(g) \mid g \in G\} \subset G$$

This is a closed subvariety of $G$ over $F$ and $H$ acts on $S$ by conjugation. The goal of this note is to prove some standard harmonic analysis results on $S$, e.g. density of regular semisimple orbital integrals, representability of Fourier transform of orbital integrals, representability of spherical characters, etc. Note that these results are not expected for general symmetric spaces, as indicated by various counterexamples of Rader and Rallis [RR96]. This means that the symmetric space $S$ is of a particular good shape in this regard. Some of these results will be used in a subsequent paper.

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of the author on factorization of linear periods. We do not discuss the motivation from automorphic forms here but simply refer the readers to that paper. Our argument follows closely the traditional route. The new ingredient is a detailed study of the nilpotent orbital integrals, which is needed in verifying the homogeneity properties of the nilpotent orbital integrals. This study leads to some very interesting linear algebra problems. One of them is the following: classify pairs of \( n \times n \) matrices \((A, B)\) with \( AB\) being nilpotent, up to the equivalence relation

\[
(A, B) \sim (A', B') \iff \exists h_1, h_2 \in \text{GL}_n(F), \text{ s.t. } A' = h_1^{-1}Ah_2, B' = h_2^{-1}Bh_1.
\]

This innocent looking problem is in fact equivalent to the classification of nilpotent orbits and is (surprisingly) not easy, c.f. Section 3 for a solution.

We now describe our results more precisely. Elements of \( S \) are all of the form

\[
\begin{pmatrix}
a & b \\
c & d
\end{pmatrix}, \quad a^2 = d^2 = 1_n + bc, \quad ab = bd, \quad dc = ca.
\]

We say that an element \( x \in S \) is \( \theta \)-semisimple (resp. \( \theta \)-regular semisimple) if it is semisimple (resp. regular semisimple) in \( \text{GL}_n(F) \) (in the usual sense) and \( \det(a^2 - 1_n) \neq 0 \). We say that an element \( x \in G \) is \( \theta \)-semisimple (resp. \( \theta \)-regular) if its image in \( S \) is so.

Let \( f \in C_\infty^\infty(G) \) and \( g \in G \) be a \( \theta \)-semisimple element. We define the \( \theta \)-semisimple orbital integral

\[
O(g, \eta, f) = \int_{(H \times H)_g \backslash H \times H} f(h_1gh_2)\eta(\det h_2)dh_1dh_2,
\]

where \( (H \times H)_g = \{(h, h') \in H \times H \mid hgh' = g\} \). This integral is absolutely convergent. Let \( D(G)^{H \times H, \eta} \) be the space of left \( H \)-invariant and right \( (H, \eta) \)-invariant distributions on \( G \). Then \( O(g, \eta, \cdot) \in D(G)^{H \times H, \eta} \) for all \( \theta \)-regular semisimple \( g \in G \).

**Theorem 1.1.** The set \( \{O(g, \eta, \cdot) \mid g \in G \text{ is } \theta \text{-regular semisimple}\} \) is weakly dense in \( D(G)^{H \times H, \eta} \).

We also consider the “Lie algebra” of \( S \). Let \( \mathfrak{s} = M_{n,F} \times M_{n,F} \), considered as a subspace of \( M_{2n,F} \) consisting of matrices of the form

\[
\begin{pmatrix}
0 & X \\
Y & 0
\end{pmatrix}, \quad X, Y \in M_{n,F}.
\]

The group \( H \) acts on \( \mathfrak{s} \) by conjugation. An element in \( \mathfrak{s} \) is \( \theta \)-semisimple or \( \theta \)-regular semisimple if it is so in \( M_{2n,F} \). The locus of \( \theta \)-semisimple and \( \theta \)-regular semisimple elements in \( \mathfrak{s} \) are denoted by \( \mathfrak{s}_{\theta-ss} \) and \( \mathfrak{s}_{\theta-reg} \) respectively.

Let \( \gamma \in \mathfrak{s}_{\theta-ss} \) and \( f \in C_\infty^\infty(\mathfrak{s}) \), we define an orbital integral

\[
O(\gamma, \eta, f) = \int_{H_\gamma \backslash H} f(h^{-1}\gamma h)\eta(\det h)dh,
\]

where \( H_\gamma = \{h \in H \mid h^{-1}\gamma h = \gamma\} \). The integral is absolutely convergent.

Let \( D(\mathfrak{s})^{H, \eta} \) be the \((H, \eta)\)-invariant distributions on \( \mathfrak{s} \). Then \( O(\gamma, \eta, \cdot) \in D(\mathfrak{s})^{H, \eta} \) for all \( \theta \)-regular semisimple \( \gamma \) in \( \mathfrak{s} \).
Theorem 1.2. The set \( \{ O(\gamma, \eta, \cdot) \mid \gamma \in \mathfrak{s}_{\theta-ss} \} \) is weakly dense in \( D(\mathfrak{s})^{H, \eta} \).

Let us fix an \( H \)-invariant inner product on \( \mathfrak{s} \) by \( \langle \gamma, \delta \rangle = \text{Tr} \gamma \delta \), where on the right hand side the product and the trace are taken in \( M_{2n,F} \). Thus we can speak of the Fourier transform of elements in \( C_c^\infty(\mathfrak{s}) \) and hence the Fourier transform of distributions on \( \mathfrak{s} \). The following result is proved in [Zha15, Theorem 6.1]

Proposition 1.3. Let \( \gamma \in \mathfrak{s} \) be \( \theta \)-regular semisimple. Then the Fourier transform of the distribution \( O(\gamma, \eta, \cdot) \) is represented by a locally integrable \( (H, \eta) \)-conjugation invariant function on \( \mathfrak{s} \). This function is locally constant on \( \mathfrak{s}_{\theta-reg} \).

We will define “\( \theta \)-nilpotent orbital integrals” in this note and prove the following result.

Proposition 1.4. The Fourier transform of \( \theta \)-nilpotent orbital integrals are represented by locally integrable functions on \( \mathfrak{s} \). This function is locally constant on \( \mathfrak{s}_{\theta-reg} \).

This proposition is the technical heart of the technical part of this note. The hard part is that, as opposed to the case of the classical orbital integrals or the nonsplit analogue of this paper treating orbital integrals on \( GL_n(E) \setminus GL_{2n}(F) \) [Guo98], the naive integration on the \( \theta \)-nilpotent orbits is not absolutely convergent in our case and some subtle regularization process is needed to define “\( \theta \)-nilpotent orbital integrals”.

A standard consequence of this proposition is the representability of the relative spherical characters. It immediately follows from Proposition 1.4 and the germ expansion of the relative spherical characters [Guo98, Theorem 2.3]. As this is a direct and well-known consequence of Proposition 1.4, we will not mention this result nor its proof in the main body of the paper again. Let \( \pi \) be an irreducible unitary (or generally class one) representation of \( G \). Assume that \( \text{Hom}_H(\pi, \mathbb{C}) \neq 0 \) and \( \text{Hom}_H(\pi \otimes \eta, \mathbb{C}) \neq 0 \). Fix nonzero elements \( l \) and \( l_\eta \) in \( \text{Hom}_H(\pi, \mathbb{C}) \) and \( \text{Hom}_H(\pi, \eta) \) respectively. Define a distribution on \( G \) by

\[
J_\pi(f) = \sum_\varphi l(\pi(f)\varphi)l_\eta(\varphi), \quad f \in C_c^\infty(G).
\]

Then \( J_\pi \in D(G)^{H \times H, \eta} \).

Theorem 1.5. The distribution \( J_\pi \) is represented by a left \( H \)-invariant and right \( (H, \eta) \)-invariant locally integrable function on \( G \).

We end this introduction with a question. Let \( (G, H) \) be a general symmetric space in the sense that \( G \) is a reductive group over \( F \) and \( H \) is the fixed point in \( G \) of an involution. Rader and Rallis [RR96] showed using many counterexamples that the results in this note in general do not hold for \( (G, H) \). That is, regular semisimple orbital integrals might not be weakly dense in the space of all invariant distributions; the spherical characters might not be representable by a locally integrable functions. Apart from the case treated in this note, we only know that these good properties hold for the following pairs:
– the classical group case: \((H \times H, H)\);
– the Galois case: \((\text{Res}_{E/F} H, H)\) where \(E/F\) is a quadratic field extension;
– the linear case: \((A^\times, B^\times)\) where \(E/F\) is a quadratic field extension and \(A\) is a central simple algebra over \(F\) containing \(E\) and \(B\) the centralizer of \(E\) in \(A\).

The question is: Can you characterize symmetric spaces with these good properties in terms of their geometric properties or combinatorial invariants?

The note is organized as follows. We start with the semisimple descent of orbital integrals in Section 2. In Sections 3–7 we are going to work on the Lie algebra \(s\). We study \(\theta\)-nilpotent orbital integrals in Sections 3 and 4. We define all orbital integrals in Section 5. Then we establish the Shalika germ expansion in Section 6 and prove that they are linearly independent in Section 7. Theorem 1.2 and Proposition 1.4 are also proved simultaneously with linear independence of Shalika germs. Finally in the last section, we deduce the results on the level of groups from the results on the Lie algebras.

**Notation.** Let \(F\) be a field, \(G\) be an algebraic group over \(F\) and \(V\) be a \(G\)-variety over \(F\), i.e. \(V\) admits an action of \(G\). This action is sometimes denoted by \(g \cdot v\) where \(g \in G\) and \(v \in V\). If \(x \in V(F)\), we denote by \(G_x\) the stabilizer of \(x\) in \(G\). If \(C\) is a subset of \(V(F)\) and \(g \in G(F)\), then we let \(C^g\) the subset consisting of all elements of the form \(g \cdot v\) where \(v \in C\), and we let \(C^G = \cup_{g \in G} C^g\). Thus if \(x \in V\), then \(x^G\) stands for the orbit of \(x\). The adjoint action of \(G\) on its Lie algebra (or subgroup of \(G\) acting on subspaces of the Lie algebra of \(G\)) is denoted by \(\text{Ad}\).

We denote by \(q: V \to V/\!/G\), or simply \(V/\!/G\), the categorical quotient. A subset of \(U\) of \(V(F)\) is called saturated if \(U = q^{-1}(q(U))\).

Let \(X\) be a scheme over \(F\). Usually we simply write \(X\) for \(X(F)\) unless there are ambiguities. One notable exception is with the categorical quotient in which case we always distinguish the notation of the scheme from its set of \(F\)-points. On the scheme \(X\) we always use the Zariski topology while on the set of \(F\)-points \(X(F)\) we always use the analytic topology.

We use capital letters to denote various groups and symmetric spaces. We use the corresponding Gothic letters to denote their Lie algebras, e.g. if \(G\) is an algebraic group, then without saying to the contrary, \(g\) stands for the Lie algebra of \(G\). Elements in the groups or symmetric spaces are usually denoted using lower case Latin letters, while elements in the Lie algebras are usually denoted by lower case Greek letters.

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2. **Semisimple descent**

First we consider some general setup. Let \(G\) be a reductive group over \(F\), \(X\) be a \(G\)-variety over \(F\) and \(x \in X(F)\) be \(G\)-semisimple point, i.e. the orbit of \(x\) is closed. We let \(N^X_x\) be the normal space of \(x^G\) at \(x\). It admits a natural action of \(G_x\) and we call \((H_x, N^X_x)\) the sliced representation.
there exist the following data which we refer to as the analytic slice at $x$. We use analytic topology throughout.

1. An $G(F)$-invariant open neighbourhood $U$ of $x^{G(F)}$ in $X(F)$ with an $G(F)$-equivariant retraction map $p: U \rightarrow x^{G(F)}$.

2. An $G_x(F)$-equivariant embedding $\psi: p^{-1}(x) \rightarrow N^X_x$ with an open and saturated image such that $\psi(x) = 0$.

If $y \in p^{-1}(x)$ and $z = \psi(y)$, then we have

1. $(G_x)_z \simeq G_y$ and $N^N_z \simeq N^N_y$ as representations of $(G_x)_z$ and $G_y$;

2. $y$ is $G$-semisimple in $X$ if and only if $z$ is $G_x$-semisimple in $N^X_x$.

The analytic slice at $x$ is denoted by $(U, p, \psi)$.

Let us now specialize to the case $X = s$ or $S$ with the conjugation action of $H$. In these cases an element being $H$-semisimple is the same as being $\theta$-semisimple.

First consider the case $X = s$. Let $\gamma \in g_{\theta-ss}$ and $G_\gamma = \{g \in G \mid g^{-1}\gamma g = \gamma\}$ be its stabilizer in $G$ and then $H_\gamma = H \cap G_\gamma$. Let $g_\gamma, h_\gamma$ be the Lie algebras of them respectively. The involution $\theta$ preserves $G_\gamma$, and hence $g_\gamma$. Let $s_\gamma$ be the $(-1)$-eigenspace of $\theta$ in $g_\gamma$. Then $g_\gamma = h_\gamma \oplus s_\gamma$ and $H_\gamma$ acts on $s_\gamma$. By [AG09, Proposition 7.2.1], the sliced representation at $x$ is isomorphic to $(H_\gamma, s_\gamma)$. By [JR96], up to conjugation by $H$, the $\theta$-semisimple element $\gamma$ takes the following form

$$\gamma = \begin{pmatrix} X & 0_r \\ 1_{n-r} & 0_r \\ 0_r & 0_r \end{pmatrix},$$

where $X \in GL_{n-r}(E)$. It is not hard to check that the symmetric pair $(G_\gamma, H_\gamma)$ is of the form $(G_1, H_1) \times (G_2, H_2)$, where

$$G_1 \simeq \left\{ x = \begin{pmatrix} a & Xc \\ c & a \end{pmatrix} \in GL_{2n-2r}(E) \mid aX = Xa, \ Xc = cX \right\},$$

and

$$H_1 \simeq \left\{ h = \begin{pmatrix} a \\ a \end{pmatrix} \mid aX = Xa \right\}.$$
Then we have $g_x = h_x \oplus s_x$. Again by [AG09, Proposition 7.2.1], the sliced representation at $x$ is isomorphic to $(H_x, s_x)$. According to [JR96, Proposition 4.1], $x$ is $H$-conjugate to an element of the form

$$
\begin{pmatrix}
  a & a - 1_r \\
  1_s & a \\
  -1_{n-r-s} & 1_s \\
 a + 1_r & a \\
-1_{n-r-r} & -1_{n-r-r}
\end{pmatrix},
$$

where $a \in \text{GL}_r(F)$ is semisimple in the usual sense and $\det(a^2 - 1_r) \neq 0$. Then it follows that the symmetric space $(G_x, H_x)$ is a product

$$(G_1, H_1) \times (G_2, H_2) \times (G_3, H_3),$$

where $(G_2, H_2)$ and $(G_3, H_3)$ are of the same shape of $(G, H)$ but of smaller sizes and

$$G \simeq \left\{ \begin{pmatrix} b & (a + 1_r)c \\ (a - 1_r)c & b \end{pmatrix} \bigg| ab = ba, \ ac = ca \right\}, \quad H \simeq \left\{ \begin{pmatrix} b \\ b \end{pmatrix} \bigg| ab = ba \right\}.$$

The sliced representation $s_x$ is isomorphic to $s_1 \times s_2 \times s_3$ where $H_1 \times H_2 \times H_3$ acts componentwise. Here $(H_1, s_1)$ is isomorphic to the adjoint action of $H_1$ on its Lie algebra, and $(H_2, s_2), (H_3, s_3)$ are of the same shape as $(H, s)$ but of smaller sizes.

The following proposition connects the orbital integrals on $S$ or $s$ near a $\theta$-semisimple point $x$ to the orbital integrals on the sliced representation at $x$. This procedure will be referred to as semisimple descent.

**Proposition 2.1.** Let $X = s$ or $S$ and $x \in X$ be $\theta$-semisimple. There exists an open neighbourhood $\omega_x \subset \psi(p^{-1}(x))$ of $0 \in N_x^X$ with the following property: if $f \in C_c^\infty(X)$, then there is an $f_x \in C_c^\infty(N_x^X)$ so that for all $\theta$-regular semisimple $z \in \omega_x$, $z = \psi(y)$ with $y \in p^{-1}(x)$, we have

$$\int_{H_x \backslash H} f(h^{-1}yz)\eta(\det h)dh = \int_{H_x \backslash H_x} f_x(h^{-1}zh)\eta(h)dh. \quad (2.1)$$

**Proof.** This is stated in [Zha15, Proposition 5.20]. We give a short proof here as we will make use of the explicit construction (not merely the existence) of $f_x$ later.

As usual the proof begins with the following compactness result.

**Claim.** Let $\omega_x \subset \psi(p^{-1}(x))$ be a saturated subset whose image in $(X_x/\sim H_x)(F)$ is relatively compact. Let $\omega \subset X$ be a compact subset. Then the closure of

$$\{ h \in H \mid \psi^{-1}(\omega_x)^h \cap \omega \neq \emptyset \}$$

is compact in $H_x \backslash H$. 

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The proof of this claim is clear. We consider the diagram

\[
\begin{array}{ccc}
H \times H & \xrightarrow{i} & X \times (N^X_x // H_x) \\
\downarrow{j} & & \downarrow{j} \\
H_x \setminus H & & 
\end{array}
\]

The horizontal arrow is a closed embedding. The set in the claim is contained in the compact set \(j i^{-1}(\omega \times \omega_\gamma)\).

With this claim at hand, we proceed as follows. Let \(f \in C^\infty_c(X)\) and \(\omega = \text{supp } f\). Let \(C\) be an open compact subset of \(H_x \setminus H\) which contains the closure of the set in the claim. Choose a function \(\alpha \in C^\infty_c(H)\) such that

\[
\int_{H_x} \alpha(hg)dh = 1_{C}(g).
\]

Put

\[
f_x(z) = \int_{H} f(h^{-1}\psi^{-1}(z)h)\eta(\det h)\alpha(h)dh, \quad z \in \omega_\gamma.
\]

Then \(f_x \in C^\infty_c(\omega_x)\) and we view \(f_x\) as a function on \(N^X_x\). Let \(z \in \omega_x\) be \(\theta\)-regular semisimple and \(y = \psi^{-1}(z) \in p^{-1}(x)\). Then \(y \in X\) is \(\theta\)-regular semisimple and \((H_x)_z = H_y\). A little computation gives

\[
\int_{(H_x)_z \setminus H_x} f_x(h^{-1}zh)\eta(\det h)dh = \int_{H_y \setminus H} f(h^{-1}yh)\eta(\det h)dh.
\]

This proves the proposition. \(\square\)

3. THE NILPOTENT CONE

Let \(\mathcal{N} \subset \mathfrak{s}\) be the nilpotent cone, i.e. the closed subvariety of \(\mathfrak{s}\) consisting of all elements whose orbit closure contains \(0 \in \mathfrak{s}\). This is equivalent to saying that the element is nilpotent in \(\mathfrak{g}\) in the usual sense. Elements or orbits contained in \(\mathcal{N}\) are called \(\theta\)-nilpotent.

To analyse the \(\theta\)-nilpotent orbits, it is better to use a more canonical formulation. Let \(V = V^+ \oplus V^-\) be a \(\mathbb{Z}/2\mathbb{Z}\)-graded vector space with homogeneous components \(V^\pm\) and \(\dim V^\pm = n\). Then we have

\[
\mathfrak{s} \simeq \text{Hom}(V^+, V^-) \oplus \text{Hom}(V^-, V^+), \quad H \simeq \text{GL}(V^+) \times \text{GL}(V^-).
\]

The nilpotent cone in \(\mathfrak{s}\) consists of pairs of endomorphism \(\xi = (X, Y) \in \text{End}(V),\ X \in \text{Hom}(V^+, V^-)\) and \(Y \in \text{Hom}(V^-, V^+)\) such that \(XY\) and hence \(YX\) are both nilpotent. This condition is equivalent to saying that \(\xi = (X, Y) \in \text{End}(V)\) is nilpotent.

Let \(\theta \in H\) be the element which acts on \(V^\pm\) by \(\pm 1\). Then \(\theta\) acts on \(\mathfrak{gl}(V)\) by sending \(Z \in \mathfrak{gl}(V)\) to \(\text{Ad}(\theta)Z = \theta Z \theta\). It is clear that \(\mathfrak{h}\) and \(\mathfrak{s}\) are eigenspaces of \(\text{Ad}(\theta)\) of eigenvalue \(1\) and \(-1\) respectively.

Let \(\xi = (X, Y) \in \mathcal{N}\). Then we have a filtration on \(V\) given by

\[
0 = W_0 \subset W_1 \subset W_2 \subset \cdots \subset W_r \subset W_{r+1} = V, \quad W_i = \text{Ker} \xi^i.
\]

(3.1)
We may view $V$ as an $F[\xi]$-module and $V$ is a direct sum of indecomposable $F[\xi]$-submodules. By [KP79, Section 4], one can choose the generators of these submodules to be homogeneous. More concretely, let $U$ be such an indecomposable submodule of dimension $a$ over $F$. Then we can choose a homogeneous element $u \in U$ so that

$$u, \xi u, \xi^2 u, \ldots \xi^{a-1} u$$

form a $F$-basis of $U$. It follows that for each $i$, we have

$$W_i = W_i^+ \oplus W_i^-,$$

$$W_i^\pm = W_i \cap V^\pm.$$  

Therefore we have two filtrations

$$0 = W_0^\pm \subset W_1^\pm \subset W_2^\pm \subset \cdots \subset W_{s-1}^\pm \subset W_s^\pm = V^\pm.$$

Note that while the filtration (3.1) is strictly increasing, these two filtration might not be strictly increasing.

We put $r_i^\pm = \dim W_i^\pm/W_{i-1}^\pm$ where ? stands for $+$, $-$, or empty. Note that $\xi$ induces an injective map $W_{i+1}/W_i \to W_i/W_{i-1}$ for $i = 1, \ldots, s - 1$. It follows that $r_i \geq r_{i+1}$ for all $i$. Moreover since $\xi$ induces injective maps $W_{i+1}/W_i \to W_i^\mp/W_{i-1}^\mp$, we conclude that $r_i^\pm \geq r_{i+1}^\mp$ for all $i$. By suitably choosing bases of these successive quotients and lifting them to $V^\pm$, we may assume that the maps $W_{i+1}/W_i \to W_i^\mp/W_{i-1}^\mp$ induced by $\xi$ are all of the form $\begin{pmatrix} 1 & r_{i+1}^\mp \\ 0 & 0 \end{pmatrix}$, where $0$ stands for the zero matrix of size $(r_{i+1}^\mp - r_{i+1}^\pm) \times r_{i+1}^\pm$.

Let $P = MN$ be the parabolic subgroup of $GL(V)$ stabilizing the filtration (3.1), and $P^+ = M^+N^+$ be the parabolic subgroup of $H$ stabilizing both filtrations (3.2). We have

$$M^+ \simeq \prod_{i=0}^{s-1} GL(W_{i+1}^+ / W_i^+),$$

$$\prod_{i=0}^{s-1} GL(W_{i+1}^- / W_i^-).$$

**Lemma 3.1.** We have

$$P \cap H = P^+, \quad M \cap H = M^+, \quad N \cap H = N^+.$$

*Proof.* It follows from the definition that $P \cap H \supset P^+$. If $h \in H \cap P$, then $h(W_i^\pm) \subset W_i$. But $h(W_i^\pm) \subset V^\pm$. It follows that $h(W_i^\pm) \subset W_i \cap V^\pm = W_i^\pm$. This proves $P \cap H = P^+$. One can similarly prove the other two equalities. \qed

**Lemma 3.2.** The following assertions hold.

1. We have

$$\text{Ad}(N^+) \xi = \xi + [n, n] \cap s,$$

where $[-, -]$ stands for the Lie algebra bracket of $n$.

2. For any $h \in H$, if $\text{Ad}(h)(n \cap s) \subset n \cap s$, then $h \in P^+$. 

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Proof. By [How74, Lemma 2(b)], $\text{Ad}(N)\xi = \xi + [n, n]$. Note that $\text{Ad}(\theta)\xi = -\xi$. Then both sides of (3.3) are $(-1)$-eigenspaces of $\text{Ad}(\theta)$. This proves the first assertion.

By [How74, Lemma 2(d)], if $g \in G$ and $\text{Ad}(g)\xi \subset n$, then $g \in P$. Note that $\xi \in n \cap s$. Then the second assertion follows from Lemma 3.1. \qed

Lemma 3.3. The $P^+$-orbit of $\xi$ in $s$ is an (Zariski) open subset of $n \cap s$ consisting of elements $Z$ with the properties that

$$Z|_{W_{i+1}^+/W_i^+} : W_{i+1}^+/W_i^+ \rightarrow W_{i+1}^+/W_{i-1}^+, \quad i = 1, \ldots, s - 1$$

is injective.

Proof. Since $\text{Ad}(N^+)\xi$ is the coset $\xi + [n, n] \cap s$ in $n \cap s$, it is enough to consider the image of $\text{Ad}(M^+)\xi$ in

$$n \cap s/([n, n] \cap s),$$

which is isomorphic to

$$\bigoplus_{i=1}^{s-1} \text{Hom}(W_{i+1}^+/W_i^+, W_{i-1}^+/W_{i-2}^+) \oplus \bigoplus_{i=1}^{s-1} \text{Hom}(W_{i+1}^-/W_i^-, W_{i-1}^+/W_{i-2}^-).$$

As explained before, $\xi$ induces an injective map $W_{i+1}^+/W_i^+ \rightarrow W_{i+1}^+/W_{i-1}^+$ for all $i$ and with suitable choice of basis, this map is represented by the matrix \[ \begin{pmatrix} 1_{r_{i+1}}^- \end{pmatrix}. \] Moreover by choosing suitable bases, any injective map $W_{i+1}^+/W_i^+ \rightarrow W_{i+1}^+/W_{i-1}^+$ can be represented by a matrix of this form. It follows that the image of $\text{Ad}(P^+)\xi$ in $\text{Hom}(W_{i+1}^+/W_i^+, W_{i-1}^+/W_{i-2}^+)$ is the subset of all injective maps. This proves the lemma. \qed

We thus have the following classification of $\theta$-nilpotent orbits.

Lemma 3.4. The set of $\theta$-nilpotent orbits in $\mathcal{N}$ is in one-to-one correspondence with the set of two sequences of integers $r_i^+, i = 1, \ldots, s$, such that

$$n = r_1^+ + \cdots + r_s^+, \quad r_1^+ \geq r_2^+ \geq r_3^+ \geq \cdots, \quad r_1^- + r_1^+ > r_2^- + r_2^+ > \cdots > r_s^- + r_s^+ > 0. \quad (3.4)$$

Proof. To each $\xi \in \mathcal{N}$, we have constructed as above two sequences of vector space $W_i^\pm, i = 1, \ldots, s$. We simply put $r_i^\pm = \text{dim} W_i^\pm/W_{i-1}^\pm$ and they satisfy (3.4).

Conversely, given any two sequences of integers $r_i^\pm$ satisfying (3.4), one can find an element $\xi \in \mathcal{N}$ so that $\text{dim} W_i^\pm/W_{i-1}^\pm = r_i^\pm$. This can be achieved as follows. We are going to write $s'$ explicitly as matrices of the form \[ \begin{pmatrix} X & Y \end{pmatrix} \] as before. First write $X$ as a blocked matrix where rows correspond to the partition $n = r_1^+ \cdots + r_s^+$ and columns correspond to the partition $n = r_1^- + \cdots + r_s^-$. Similarly write $Y$ as a blocked matrix where rows correspond to the partition $n = r_1^- + \cdots + r_s^-$ and columns correspond to the partition $n = r_1^+ \cdots + r_s^+$. Then $\xi$ is the matrix of following form. All the block entries of $X$ and $Y$ are zero except for the $(i, i+1)$ entry. The $(i, i+1)$ entry of $X$ and $Y$ are of
size $r_i^+ \times r_{i+1}^-$ and $r_i^- \times r_{i+1}^+$ respectively and we have $r_i^+ \geq r_{i+1}^-$. The $(i, i + 1)$ entry of $X$ and $Y$ are of the form $\begin{pmatrix} 1_{r_{i+1}^+} \pm 1 & \ast \\ 0 & 1_{r_{i+1}^-} \end{pmatrix}$ where $1_{r_{i+1}^+}$ stands for the identity matrix of size $r_{i+1}^+$ in $X$ and size $r_{i+1}^-$ in $Y$, and $0$ stands for the zero matrix. It is not hard to check that this $\xi$ is the desired $\theta$-nilpotent matrix.

We now study the stabilizer $M_\xi^+$ of $\xi$ in $M^+$. If the $H$-orbit represented by $\xi$ were to support an $(H, \eta)$-invariant distribution, then $\eta \circ \det$ would have to be trivial on $M_\xi^+$.

We have two chains of injective maps induced by the element $\xi$:

\begin{equation}
W_s^\epsilon/W_{s-1}^\epsilon \hookrightarrow \cdots \hookrightarrow W_3^\epsilon/W_2^\epsilon \hookrightarrow W_2^\epsilon/W_1^\epsilon \hookrightarrow W_1^\epsilon,
\end{equation}

where $\epsilon = +$ or $-$ according to the parity of $r$. For each $i$, the map $W_i^+/W_{i-1}^\epsilon \to W_i^+/W_{i-1}^\epsilon$ is either an isomorphism or (genuine) injective and it is an isomorphism if and only if $\dim W_i^+/W_{i-1}^\epsilon = \dim W_i^+/W_{i-1}^\epsilon$. We call the integer $i$ a jump if $\dim W_i^+/W_{i-1}^\epsilon < \dim W_i^+/W_{i-1}^\pm$ (either the + one or the − one, the inequality does not have to hold for both filtrations). To unify treatment, we call $s$ a jump if $\dim W_s^\epsilon/W_{s-1}^\epsilon \neq 0$.

**Lemma 3.5.** Suppose that the orbit represented by $\xi$ supports an $(H, \eta)$-invariant distribution. Then all jumps are even integers, i.e. we have the strict inequality $r_i^\epsilon > r_{i+1}^- (\epsilon = +$ or $-)$ in (3.4) only when $i$ is even.

**Proof.** To facilitate understanding, we suggest an example after the proof. Let $i$ be the smallest jump in one of the chains of injective maps (3.5), say the one ends with $W_i^+$. Assume that $i$ is odd. The last a few terms in the filtration looks like

\[ W_{i+2}^-/W_{i+1}^- \hookrightarrow W_{i+1}^+/W_{i-1}^- \cong W_{i+1}^-/W_{i-2}^- \cong \cdots \cong W_1^+, \]

where the leftmost arrow is injective but not an isomorphism. We construct a basis of $V$ as follows. Choose linearly independent elements in $W_s^\pm$ so that its image in $W_s^+/W_{s-1}^\pm$ is a basis. Then the image under $\xi$ of these elements in $W_{s-1}^\pm$ are also linearly independent. We extend them to a set of linearly independent elements in $W_{s-1}^\pm$ so that the image in $W_{s-1}^+/W_{s-2}^\pm$ forms a basis. We repeat this process for all $W_j^\pm$’s. Then we get a basis of $V$. Among elements in this basis, we can find $w_j \in W_j^{(-1)^{j-1}}$ so that $\xi(w_j) = w_{j-1}$, $j = 1, \cdots, i$, and $w_i$ is not in the image of $W_{i+1}^-$ under $\xi$. Choose $\lambda \in F^\times$ with $\eta(\lambda) = -1$ and let $h \in \text{GL}(M^+)$ be an element such that it acts as multiplication by $\lambda$ on $w_1, \cdots, w_i$ and trivially on all elements of the basis of $V$. Then by construction $h \in M_\xi^+$. But $\eta(\det h) = (-1)^i = -1$ since $i$ is odd, which contradicts the fact that $\eta \circ \det$ is trivial on $M_\xi^+$. Thus $i$ is even. We may repeat this process for all other jumps. \qed
Example 3.6. Let us consider the case \( n = 4 \) and the nilpotent element \( \xi \in \mathfrak{s} \) given by the following matrix.

\[
\xi = \begin{pmatrix}
0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\end{pmatrix}, \quad V^+ = \begin{pmatrix}
* \\
* \\
* \\
0 \\
\end{pmatrix}, \quad V^- = \begin{pmatrix}
0 \\
0 \\
* \\
* \\
\end{pmatrix}.
\]

Simple computation gives

\[
W_1^+ = \begin{pmatrix}
* \\
* \\
0 \\
0 \\
\end{pmatrix}, \quad W_2^+ = \begin{pmatrix}
* \\
* \\
0 \\
0 \\
\end{pmatrix}, \quad W_3^+ = \begin{pmatrix}
* \\
* \\
0 \\
0 \\
\end{pmatrix}; \quad W_1^- = \begin{pmatrix}
0 \\
0 \\
* \\
0 \\
\end{pmatrix}, \quad W_2^- = \begin{pmatrix}
0 \\
0 \\
* \\
0 \\
\end{pmatrix}, \quad W_3^- = \begin{pmatrix}
0 \\
0 \\
* \\
* \\
\end{pmatrix}.
\]

Moreover \((r_1^+, r_2^-, r_3^+) = (2, 2, 1)\) and \((r_1^-, r_2^+, r_3^-) = (1, 1, 1)\). The elements \(w\) and \(h\) that we chose in the proof of Lemma 3.5 is

\[
w = \begin{pmatrix}
0 \\
0 \\
0 \\
0 \\
\end{pmatrix}, \quad h = \begin{pmatrix}
1 & \lambda \\
\lambda & 1 \\
1 & \lambda \\
\lambda & 1 \\
\end{pmatrix}, \quad \det h = \lambda^3.
\]

It is straightforward to check that \(h\) commutes with \(\xi\). According to our terminology, in the sequence \(r_1^+ = r_2^- > r_3^+\), 2 and 3 are jumps, which are not all even. The orbit represented by \(\xi\) does not support any \((H, \eta)\) invariant distribution.

4. Nilpotent orbital integrals

In this subsection, we are going to show that the necessary condition in Lemma 3.5 that a nilpotent orbital integral supports an \((H, \eta)\) invariant distribution is also sufficient. Moreover these \((H, \eta)\) invariant distributions extend to an \((H, \eta)\) invariant distribution on \(\mathfrak{s}\).
Let us keep the notation from Section 3. Let $O$ be a $\theta$-nilpotent orbit in $\mathfrak{s}$ represented by an element $\xi$. Then attached to $\xi$ is a parabolic subgroup $P^+ = M^+N^+$ of $H$. We also have two sequences of integers $r_1^+ \geq r_2^+ \geq \cdots$. We assume that all the jumps in these two sequences are even integers. By Lemma 3.5, this is a necessary condition for $O$ to support an $(H, \eta)$ invariant distribution.

Let $2i_1 < \cdots < 2i_a$ be the set of all jumps in the sequence $r_1^+ \geq r_2^+ \geq \cdots$. Let $2j_1 < \cdots < 2j_b$ be the set of all jumps in the sequence $r_1^- \geq r_2^- \geq \cdots$. Note that we either have $2i_a = s + 1$ and $W_{s+1}^-/W_s^+ \neq 0$, or $2i_a < s + 1$ and all $W_{i+1}^-/W_i^+$ if $i \geq 2i_a$ where $\epsilon$ is an appropriate sign. We have a similar assertion for the jump $2j_b$. Then the space $n \cap s/[n, n] \cap s$ is isomorphic to

$$\bigoplus_{i=1}^{2i_a} \text{Hom}(W_i^{(-1)^i}/W_i^{-1}, W_i^{(-1)^{i-1}}/W_i^{-1}) \bigoplus_{i=1}^{2j_b} \text{Hom}(W_i^{(-1)^{i+1}}/W_i^{(-1)^i}, W_i^{(-1)^{i+1}}/W_i^{-1}).$$

Let us define some determinant functions. Let us write an element in $n \cap s/[n, n] \cap s$ as a sequence

$$m = (x_1, \cdots, x_{2j_b}; y_1, \cdots, y_{2j_b}),$$

with

$$x_i \in \text{Hom}(W_i^{(-1)^{i+1}}/W_i^{-1}, W_i^{(-1)^{i}}/W_i^{-1}), \ \ y_i \in \text{Hom}(W_i^{(-1)^{i+1}}/W_i^{(-1)^{i}}, W_i^{(-1)^{i+1}}/W_i^{-1}).$$

Note that if $i$ is odd, then both $r_i^\pm = r_i^\mp$ by the assumption that all jumps are even integers. Moreover

$$\xi|_{W_{i+1}^\pm/W_i^\pm} : W_{i+1}^\pm/W_i^\pm \rightarrow W_{i+1}^\mp/W_i^\mp$$

is an isomorphism. To shorten notation, we put $\xi^\mp = \xi|_{W_{i+1}^\pm/W_i^\pm}$. Define

$$\det_{2i-1}^+(x_{2i-1}) = \det x_{2i-1}(\xi_{2i-1})^{-1}, \ \ \det_{2i-1}^-(y_{2i-1}) = \det y_{2i-1}(\xi_{2i-1})^{-1},$$

and

$$\det_n(m) = \det_1^+(x_1)\det_3^+(x_3)\cdots\det_{2j_b-1}^+(x_{2j_b-1})\det_1^-(y_1)\det_3^-(y_3)\cdots\det_{2j_b-1}^-(y_{2j_b-1}).$$

**Lemma 4.1.** For $p \in P^+$ and $u \in n \cap s$, we have

$$\eta(\det_n(pup^{-1})) = \eta(\det p)\eta(\det_n u).$$

**Proof.** This follows from the definition of $\det_n$. \hspace{1cm} \Box

Let $n'$ be the subspace of $n \cap s$ generated by $[n, n] \cap s$ and

$$\bigoplus_{i \text{ even}} \text{Hom}(W_{i+1}^+/W_i^+, W_i^-/W_{i-1}^-) \bigoplus \text{Hom}(W_{i+1}^-/W_i^-, W_i^+/W_{i-1}^+).$$

Let $f \in C_c^\infty(\mathfrak{s})$, we define a function $\tilde{f} \in C_c^\infty(n \cap s/n')$ as

$$\tilde{f}(m) = \int_{n'} f(m + u)du.$$
Before we proceed, let us recall the following result due to Godement and Jacquet [GJ72, Theorem 3.3] (taking the representation $\pi$ to be $\eta \circ \det$). Note that the holomorphy is a consequence of the fact that $E/F$ is a quadratic extension of nonarchimedean local fields and $\eta$ is nontrivial.

**Lemma 4.2.** Let $\varphi \in C^\infty_c(M_n(F))$. Put
\[
Z(s, \eta, \varphi) = \int_{\text{GL}_n(F)} \varphi(h)|\det h|^s\eta(\det h)dh,
\]
where $dh$ stands for the multiplicative measure on $\text{GL}_n(F)$. Then this integral is convergent if $\Re s \gg 0$ and has a meromorphic continuation to the whole complex plane. It is holomorphic at all $s \in \mathbb{R}$.

The function $\tilde{f}$ is a function in the variables
\[
m = (x_1, x_3, \cdots, x_{2j_a-1}, y_1, y_3, \cdots, y_{2j_b-1}).
\]
Let $\underline{s} = (s_1, s_3, \cdots, s_{2j_a-1})$ and $\underline{t} = (t_1, t_3, \cdots, t_{2j_b-1})$ be complex numbers. Put
\[
\det_{n, \underline{s}, \underline{t}}(m) = |\det_1^1(x_1)|^{s_1} |\det_3^3(x_3)|^{s_3} \cdots |\det_{2j_a-1}^1(x_{2j_a-1})|^{s_{2j_a-1}} |\det_1^{-1}(y_1)|^{t_1} |\det_3^{-1}(y_3)|^{t_3} \cdots |\det_{2j_b-1}^{-1}(y_{2j_b-1})|^{t_{2j_b-1}}.
\]
Consider the integral
\[
Z(\underline{s}, \underline{t}, \eta, \tilde{f}) = \int \tilde{f}(m)\eta(\det_n(m))\det_{n, \underline{s}, \underline{t}}(m)dm,
\]
where the domain of integration is $n \cap s'/n'$, which is identified with
\[
\bigoplus_{i \text{ odd}} \text{Hom}(W_{i+1}^-/W_i^- , W_i^+/W_{i-1}^+) \oplus \bigoplus_{i \text{ odd}} \text{Hom}(W_{i+1}^+/W_i^+ , W_i^-/W_{i-1}^-).
\]

By Lemma 4.2, the integral $Z(\underline{s}, \underline{t}, \eta, \tilde{f})$ is convergent when the real part of $s_i$ and $t_i$'s are large enough and $Z(\underline{s}, \underline{t}, \eta, \tilde{f})$ has meromorphic continuation to $\mathbb{C}^{j_a+j_b}$, which is holomorphic at the points where all $s_i$ and $t_i$'s are integers. We define
\[
\tilde{\mu}_\sigma(f) = Z(\underline{s}, \underline{t}, \eta, \tilde{f}) \Big|_{s_i = r_i^-, \text{ for all } i} \quad t_i = r_i^+, \text{ for all } i.
\]
The point is that for the variable coming from one of the decreasing sequences, we evaluate this integral at the point given by the corresponding integer in the other sequence.

**Lemma 4.3.** For any $f \in C^\infty_c(s)$, and any $p \in P^+$, we have
\[
(4.1) \quad \tilde{\mu}_\sigma(\text{Ad}(p)f) = \delta_{P^+}(p)\eta(\det p)\tilde{\mu}_\sigma(f).
\]

**Proof.** The invariance by elements in $N^+$ is straightforward to check. One has to prove (4.1) for elements in $M^+$. We may even assume that $m \in \text{GL}(W_{i+1}^+/W_i^+)$. The other cases can be derived from this one or follow from the same argument.
Elementary computation shows that
\[ \delta_{p^+}(m) = |\det m|^{-(r_1^+ + \cdots + r_1^- + r_{i+2}^+ + \cdots + r_s^+)}. \]

If \( i \) is odd, then in computing the integration over \( n' \), after changing variables, we obtain
\[ |\det m|^{-(r_{i-1}^- + \cdots + r_1^- + r_{i+2}^+ + \cdots + r_s^-)}. \]

In computing the integration \( Z(s, \xi, \eta, \tilde{f}) \), by changing the variable, we obtain another term
\[ |\det m|^{-r_i^+} \eta(\det m). \]

Note that we have \( r_1^+ = r_2^-, r_3^- = r_4^+ \), etc. Thus we conclude
\[ -(r_1^+ \cdots + r_1^-) + r_{i+2}^+ + \cdots + r_s^+ = -(r_{i-1}^- + \cdots + r_1^-) + r_{i+2}^- + \cdots + r_s^- + (-r_i^+). \]

This proves (4.1) when \( i \) is odd. The case \( i \) being even is similar. \( \square \)

Let us now choose an open compact subgroup \( K \) of \( H \) so that \( H = P^+K \). Let us put
\[ (4.2) \quad f_K(\gamma) = \int_K f(\gamma k) \eta(\det k) dk, \quad \mu_O(f) = \tilde{\mu}_O(f_K). \]

It follows from [How74, Proposition 4] that the distribution on \( C_\infty^c(s) \) given by \( f \mapsto \mu_O(f) \) is \((H, \eta)\) invariant. Even though the statement of [How74, Proposition 4] does not involve the extra character \( \eta \), the same argument goes through without change. It is also clear that the linear form \( \mu_O \) extends the \((H, \eta)\)-invariant distribution on \( O \) to an \((H, \eta)\)-invariant distribution on \( s \) supported on \( O \).

To summarize, for any \( \theta \)-nilpotent orbit \( O \) satisfying the necessary condition in Lemma 3.5, we have constructed an \((H, \eta)\) invariant distribution \( \mu_O \) on \( s \) supported on \( O \). Moreover this distribution, by their very construction, when restricted to \( O \), equals the \((H, \eta)\)-invariant distribution on \( O \). In the following, we call such a \( \theta \)-nilpotent orbit or any element contained in it \textit{visible}. We let \( N_0 \) be the subset of \( N \) consisting of visible \( \theta \)-nilpotent orbits. Of course from the discussion above, the set
\[ \{ \mu_O \mid O \subset N_0 \} \]

is a natural basis of the space of \((H, \eta)\) invariant distributions supported on \( N \).

Let us put \( d_O = \dim N^+ \).

**Lemma 4.4.** Let \( f \in C_\infty^c(s) \) and for any \( t \in F^\times \) we put \( f_t(X) = f(t^{-1}X) \). Let \( O \subset N_0 \) then we have
\[ \mu_O(f_t) = |t|^{d_O} \eta(t)^n \mu_O(f), \quad \mu_O(f_t) = |t|^{2n^2-d_O} \eta(t)^n \mu_O(f) \]

**Proof.** We just need to prove the first equality. The second one on the Fourier transform follows from the first one easily. Suppose that \( O \) is represented by \( \xi \) and gives rise to the sequences of integers \( r_1^+ \geq r_2^- \geq \cdots \). It follows from the definition of \( \mu_O \) that
\[ \mu_O(f_t) = |t|^{d_O n + 2(r_1^+ r_1^- + r_2^+ r_3^- + \cdots) \eta(t)^n \mu_O(f)}. \]
It is thus enough to prove that
\[(4.3) \quad \dim N^+ = \dim n' + 2(r_1^+ r_1^- + r_3^+ r_3^- + \cdots).\]

We have
\[(4.4) \quad \dim N^+ = \sum_{i=1}^{n} \sum_{j \geq i+1} r_i^+ r_j^- + r_i^- r_j^-.\]

To organize the terms on the right hand side of (4.3) into a better form, let us write
\[2(r_1^+ r_1^- + r_3^+ r_3^- + \cdots)\]

as
\[r_1^+ r_2^- + r_3^+ r_4^- + \cdots + r_1^- r_2^- + r_3^- r_4^- + \cdots\]

Then the right hand side becomes
\[(4.5) \quad \sum_{i \text{ odd}} \left(r_i^+ r_{i+1}^- + r_i^- r_{i+1}^+ + \sum_{j \geq i+2} (r_i^+ r_j^- + r_i^- r_j^+)\right) + \sum_{i \text{ even}} \sum_{j \geq i+1} (r_i^+ r_j^- + r_i^- r_j^+).\]

Let \(i\) be an integer. In computing the dimension of \(N^+\), the terms involving \(r_i^+\) are \(r_i^+(r_{i+1}^- + r_{i+2}^+ r_{i+3}^- + \cdots)\). If \(i\) is odd, then on the right hand side of (4.3), the terms involving \(r_i^+\) are
\[r_i^+ r_{i+1}^- + r_i^- (r_{i+2}^- + r_{i+3}^+ + \cdots).\]

If \(i\) is even, then we have
\[r_i^+(r_{i+1}^- + r_{i+2}^+ + \cdots).\]

Note that we have \(r_1^+ = r_2^+, r_3^+ = r_4^+\) etc. So we conclude that for a fixed \(i\), the terms in (4.4) and in (4.5) involving \(r_i^+\) coincide. Similarly we can conclude that the terms involving \(r_i^-\) coincide. Thus we conclude that (4.4) and (4.5) are the same, i.e. the identity (4.3) holds. This proves the lemma. \(\square\)

Again to facilitate understanding, we suggest the following example.

**Example 4.5.** Let \(\mathcal{O}\) be the nilpotent orbit represented by
\[
\xi = \left(\begin{array}{cccc}
0 & 1 & & \\
0 & 1 & & \\
& & & \\
& & & \\
0 & 1 & & \\
& & & \\
& & & \\
& & & \\
0 & 1 & & \\
\end{array}\right)
\]
We have \( r_1^+ = r_2^- = r_3^+ = r_4^- = 1 \) and \( r_1^- = r_2^+ = 2 > r_3^- = r_4^+ = 0 \). So this orbit is visible. The spaces \([n, n] \cap s, n \cap s/[n, n] \cap s\), and \(n'\) look like the following respectively

\[
\begin{pmatrix}
0 & * \\
0 & * \\
0 & *
\end{pmatrix}
\begin{pmatrix}
0 & * \\
0 & 0 \\
* & 
\end{pmatrix}
\begin{pmatrix}
0 & * \\
0 & * \\
0 & 0
\end{pmatrix}
\]

In this case, direct computation shows that we have \( \mu_O(f_t) = |t|^{10} \mu_O(f) \). This is compatible with Lemma 4.4.

5. Orbital integrals

In this section, we define all orbital integrals on \( s \), not necessarily \( \theta \)-semisimple or \( \theta \)-nilpotent.

Let \( \gamma \in s \) and \( \gamma = \gamma_s + \gamma_n \) be the Jordan decomposition of \( \gamma \) in \( g \), \( \gamma_s \) being semisimple and \( \gamma_n \) being nilpotent (in the usual sense). Since \( \theta(\gamma_s) \) and \( \theta(\gamma_n) \) are still semisimple and nilpotent respectively in \( g \) and \( \theta(\gamma) = -\gamma \), we conclude that \( \gamma_s, \gamma_n \in s \). Note that \( \gamma_s \gamma_n = \gamma_n \gamma_s \), we conclude that \( \gamma_n \in s_{\gamma_s} \) and is \( \theta \)-nilpotent in \( s_{\gamma_s} \). Assume that \( \gamma_n \) is visible in \( s_{\gamma_s} \) and its orbit is denoted by \( O_{\gamma_n} \). Let \( f \in C^\infty_c(s) \) and \( h \in H \). Let us define a function

\[
f_1(h) = \mu_{O_{\gamma_n}}(f(h^{-1}(\gamma_s + \cdot)h)).
\]

We claim that as a function in \( h \in H \), it is compactly supported on \( H_{\gamma_s} \setminus H \). Indeed, if for some \( h \in H \), \( f_1(h) \neq 0 \). Then there is some \( y \in H_{\gamma_s} \) such that \( h^{-1}(\gamma_s + y^{-1} \gamma_n y)h \in \text{supp} f \) which is a compact set. Note that \( h^{-1} \gamma_s h \) is \( \theta \)-semisimple in \( s \) and \( h^{-1}y^{-1} \gamma_n y h \) is \( \theta \)-nilpotent in \( s \). So \( h \gamma_s h^{-1} \) is the semisimple part of \( h^{-1}(\gamma_s + y^{-1} \gamma_n y)h \) and hence lies in some compact subset \( C \) of \( s \). As the orbit of \( \gamma_s \) is closed, it follows that \( y \) lies in some compact subset of \( H_{\gamma_s} \setminus H \). This proves the claim.

It follows from the definition that \( f_1(yh) = \eta(\det y)f_1(h) \) if \( y \in H_{\gamma_s} \). We then put

\[
O(\gamma, \eta, f) = \int_{H_{\gamma_s} \setminus H} f_1(h) \eta(\det h)dh.
\]

This integral is absolutely convergent. It is not hard to check that if the restriction \( f \) to the orbit of \( \gamma \) is compactly supported, then \( O(\gamma, \eta, f) \) agrees with the integral on the orbit of \( \gamma \).

We now connect the orbital integral on \( s \) with the orbital integral on \( s_{\gamma_s} \). We keep the notation from Section 2 (the proof of) Proposition 2.1. We have the analytic slice \((U, p, \psi)\) at \( \gamma \). Let \( f \in C^\infty_c(s) \) and we have constructed an \( f_{\gamma_s} \in C^\infty_c(s_{\gamma_s}) \). According to the definition, we have

\[
f_{\gamma}(\xi) = \int_{H} f(h^{-1}\psi^{-1}(\xi)h)\eta(\det h)\alpha(h)dh, \quad \xi \in \omega_{\gamma}.
\]
When we restrict it to the nilpotent cone in $s_{\gamma_n}$, it equals

$$\int_H f(h^{-1}(\gamma_s + \cdot)h)\eta(\det h)\alpha(h)dh.$$  

From this and the definition of $O(\gamma, \eta, f)$ we conclude that

$$(5.1) \quad \mu_{O_{\gamma_n}}(f_{\gamma}) = \int_H f_1(h)\eta(\det h)\alpha(h)dh = \int_{H_{\gamma_n}\setminus H} f_1(h)\eta(\det h)1_C(h)dh = O(\gamma, \eta, f).$$

We finish the definition of orbital integrals with the following lemma.

**Lemma 5.1.** If $\gamma_n$ is not visible in $s_{\gamma_s}$, then the orbit if $\gamma$ in $s$ does not support any $(H, \eta)$-invariant distribution.

**Proof.** An obvious necessary condition that the orbit represented by $\gamma$ supports an $(H, \eta)$-invariant distribution is $\eta(\det h) = 1$ if $h \in H_\gamma$. If $h \in H_\gamma$, i.e. $h^{-1}\gamma h = \gamma$, then $h^{-1}\gamma_s h + h^{-1}\gamma_n h = \gamma_s + \gamma_n$. As $h^{-1}\gamma_s h$ is $\theta$-semisimple and $h^{-1}\gamma_n h$ is $\theta$-nilpotent, we conclude $h^{-1}\gamma_s h = \gamma_s$ and $h^{-1}\gamma_n h = \gamma_n$ by the uniqueness of the Jordan decomposition. Therefore $H_\gamma$ is a subgroup of $H_{\gamma_s}$ that stabilizes $\gamma_n$. Then the condition $\eta(\det h) = 1$ if $h \in H_\gamma$ is precisely that $\gamma_n$ represents a visible $\theta$-nilpotent orbit in $s_{\gamma_s}$.

\[\square\]

6. THE GERM EXPANSION

We study an analogue of the Shalika germ expansion in this section.

**Proposition 6.1.** There is a unique $(H, \eta)$-invariant real valued function $\Gamma_\mathcal{O}$ on $s_{\theta-\text{reg}}$ for each nilpotent orbit $\mathcal{O} \subset \mathcal{N}_0$ with the following properties.

1. For any $f \in C^\infty_c(s)$, there is an $H$-invariant neighbourhood $U_f$ of $0 \in s$ such that

$$(6.1) \quad O(\gamma, f) = \sum_{\mathcal{O} \subset \mathcal{N}_0} \Gamma_{\mathcal{O}}(\gamma)\mu_{\mathcal{O}}(f).$$

for all $\theta$-regular $\gamma \in U_f$.

2. For all $t \in F^\times$ and all $\xi \in s_{\theta-\text{reg}}$, we have

$$\Gamma_\mathcal{O}(t\gamma) = |t|^{-d_{\mathcal{O}}}\eta(t)^n\Gamma_\mathcal{O}(\gamma).$$

**Proof.** It follows from [RR96, Proposition 1.2] that there are functions $\Gamma'_\mathcal{O}$ on $s_{\theta-\text{reg}}$ for each $\mathcal{O} \subset \mathcal{N}_0$ with property (1). Note that if $\Gamma''_\mathcal{O}$ is another set of functions satisfying (1), then $\Gamma'_\mathcal{O}$ and $\Gamma''_\mathcal{O}$ have the same germs at $0 \in s$ (i.e. they equal in a small neighbourhood of $0$). We first explain that $\Gamma'_\mathcal{O}$ can be chosen to be real valued, at least when $\gamma$ is close to $0 \in s$. In fact, since $\mu_{\mathcal{O}}$‘s form a basis of $(H, \eta)$-invariant distributions on $s$ that are supported on $\mathcal{N}$, for each $\mathcal{O} \subset \mathcal{N}_0$ we can find a function $f_{\mathcal{O}}$ so that $\mu_{\mathcal{O}}(f_{\mathcal{O}'}) = \delta_{\mathcal{O},\mathcal{O}'}$ (Kronecker delta). It is obvious that $f_{\mathcal{O}}$‘s can be chosen to be real valued. For this particular choice, we have $O(\gamma, f_{\mathcal{O}}) = \Gamma'_\mathcal{O}(\gamma)$ when $\gamma$ lies in a small neighbourhood of $0$. Indeed, this can be taken as the definition of $\Gamma'_\mathcal{O}(\gamma)$. As $f_{\mathcal{O}}$ is real, it follows
that $\Gamma'_O(\gamma)$ can be taken to be real. We need to prove that among these functions, we can choose a unique $\Gamma_O$ for each $O \subset N$ with property (2).

Let $t \in F^\times$ be fixed. We claim that as a function of $\gamma$, $\Gamma_O(t\gamma)$ and $|t|^{-d_O}\eta(t)^n\Gamma_O(\gamma)$ have the same germs at 0. Indeed, on the one hand, we have

$$O(\gamma, f) = \sum_{\mathcal{O} \subset N_0} \Gamma'_O(\gamma)|t|^{-d_O}\eta(t)^n\mu_O(f)$$

when $\gamma$ lies in a small neighbourhood (depending on $f$ and $t$) of $0 \in \mathfrak{s}$. On the other hand,

$$O(\gamma, f) = O(t^{-1}\gamma, f) = \sum_{\mathcal{O} \subset N_0} \Gamma'_O(t^{-1}\gamma)\mu_O(f).$$

when $\gamma$ lies in a small neighbourhood (depending on $f$ and $t$) of $0 \in \mathfrak{s}$. Comparing these two, we conclude that $\Gamma'_O(t\gamma)$ and $|t|^{-d_O}\eta(t)^n\Gamma'_O(\gamma)$ have the same germs at $\gamma = 0$.

Thus we put

$$\Gamma_O(\gamma) = \lim_{t \to 0} |t|^{-d_O}\eta(t)^n\Gamma'_O(t\gamma).$$

It is straightforward to check that $\Gamma_O(\gamma)$ does satisfy property (2). Of course, in order that $\Gamma_O$ satisfies property (2), it has to be of this form. Thus this function is unique. \hfill \Box

The function $\Gamma_O$ in the lemma is called the Shalika germ indexed by $O$.

We now consider the Shalika germ expansion around an arbitrary $\theta$-semisimple element $\gamma \in \mathfrak{s}$. We keep the notation from Section 2. The space $\mathfrak{s}_\gamma$, with an action of $H_\gamma$, is isomorphic to $\mathfrak{s}_1 \times \mathfrak{s}_2$ with an action of $H_1 \times H_2$, where the action of $H_1$ on $\mathfrak{s}_1$ is isomorphic to the conjugation of $H_1$ on its Lie algebra and the action of $H_2$ on $\mathfrak{s}_2$ is of the same shape as the action of $H$ on $\mathfrak{s}$ but of a smaller size. Note that according to the decomposition $\mathfrak{s} = \mathfrak{s}_1 \times \mathfrak{s}_2$, $\gamma = (\gamma^{(1)}, 0)$ where $\gamma^{(1)} \in \mathfrak{s}_1$ is a central element in $\mathfrak{s}_1$. A $\theta$-nilpotent orbit in $\mathfrak{s}_\gamma$, is of the form $O^{(1)} \times O^{(2)}$ where $O^{(1)}$ is a nilpotent orbit in $\mathfrak{s}_1$ (in the usual sense) and $O^{(2)}$ is a $\theta$-nilpotent orbit in $\mathfrak{s}_2$. The orbit $O$ is visible in $\mathfrak{s}_\gamma$ if and only if $O^{(2)}$ is visible in $\mathfrak{s}_2$. Let $\{O_1, \ldots, O_r\}$ be the set of nilpotent orbits in $\mathfrak{s}_\gamma$. We thus have the Shalika germs on $\mathfrak{s}_\gamma$, indexed by the $\theta$-nilpotent orbits in $\mathfrak{s}_\gamma$, which on $\mathfrak{s}_1$ is given by the one defined in [Kot05, Section 17] and on $\mathfrak{s}_2$ is given by the one we have just defined. Let $\{\xi_1, \ldots, \xi_r\}$ be a complete set of representatives of $\theta$-nilpotent elements in $\mathfrak{s}_\gamma$ and $\xi_i \in O_i$. We denote the Shalika germ on $\mathfrak{s}_\gamma$ indexed by $O_i$ by $\Gamma'_O(\gamma)$.\hfill \Box

**Corollary 6.2.** Let $f \in C_c^\infty(\mathfrak{s})$. Then there is a neighbourhood $U_f$ of $\gamma$ in $\mathfrak{s}_\gamma$ so that for any $\xi \in U_f \cap \mathfrak{s}_{\theta-reg}$, we have

$$O(\xi, \eta, f) = \sum_{i=r}^r \Gamma'_i(\xi)O(\gamma + \xi_i, \eta, f).$$

**Proof.** Let us keep the notation from Section 2 Proposition 2.1. We have constructed an $f_{\gamma_{\mathfrak{s}_1}} \in C_c^\infty(\mathfrak{s}_{\gamma_{\mathfrak{s}_1}})$. Apply Proposition 6.1 (germ expansion on $\mathfrak{s}_2$ near 0) and [Kot05, Theorem 17.5] (germ
expansion on $\mathfrak{s}_1$ near a central element), we have
\[ O^\gamma(\xi, \eta, f_\gamma) = \sum_{i=1}^{r} \Gamma_i^\gamma(\xi) \mu_{O_i}^\gamma(f_\gamma), \]

Applying Proposition 2.1 to the left hand side and the equality (5.1) to the right hand side, we obtain the desired result in the corollary.

7. Linear independence of Shalika germs

The goal of this section is to prove the linear independence of Shalika germs that we defined in the last section and the density of $\theta$-regular semisimple integrals on $\mathfrak{s}$ simultaneously. We follow the argument of [Kot05, Section 27] closely. We will be sketchy at various points, referring the readers to the argument in [Kot05].

The first step is the following lemma.

**Lemma 7.1.** The set of all orbital integrals is weakly dense in $D(\mathfrak{s})^H$.

**Proof.** The proof is identical to [Kot05, Proposition 27.1], making use of the finiteness of the $\theta$-nilpotent orbits and the fact that all visible $\theta$-nilpotent orbits support $(H, \eta)$-invariant distributions that extend to all $\mathfrak{s}$. We omit the detailed arguments.

The second step is to prove the following.

**Lemma 7.2.** The functions $\Gamma_{O}$’s for $O \subset N_0$ are linearly independent if and only if their restrictions to an arbitrary small neighbourhood of $0 \in \mathfrak{s}$ are still linearly independent.

**Proof.** The proof is identical to [Kot05, Lemma 27.2], making use of the homogeneity property of $\Gamma_{O}$’s. We omit the details.

The next step is to relate the linear independence of the Shalika germs to the density of $\theta$-regular semisimple orbital integrals.

**Lemma 7.3.** The following assertions are equivalent.

1. The Shalika germs $\Gamma_{O}$, $O \subset N_0$ are linearly independent.

2. The $\theta$-nilpotent orbital integrals $\mu_{O}$’s lie in the weak closure of the set of $\theta$-regular orbital integrals in $D(\mathfrak{s})^{H, \eta}$.

**Proof.** (1) $\Rightarrow$ (2). Let $f \in C^\infty_c(\mathfrak{s})$ and assume that the $\theta$-regular orbital integrals $O(\gamma, \eta, f)$ are all zero. Then it follows from the Shalika germ expansion that
\[ \sum_{O \subset N} \mu_{O}(f) \Gamma_{O}(\gamma) = 0 \]
for any $\theta$-regular $\gamma \in U_f$ where $U_f$ is a small neighbourhood of $0 \in \mathfrak{s}$. Since $\Gamma_{O}$’s are linear independent, by the previous lemma, they are linearly independent even when restricted to $U_f$. Thus we conclude that $\mu_{O}(f) = 0$ for all $O$. 

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Suppose that we have a linear relation
\[ \sum_{\mathcal{O} \subset N_0} a_{\mathcal{O}} \Gamma_{\mathcal{O}}(\gamma) = 0, \quad \text{for all } \gamma \in \mathfrak{s}_{\theta-\text{reg}}. \]
As \( \mu_{\mathcal{O}} \)'s form a basis of the space of \((H, \eta)\)-invariant distributions on \( \mathfrak{s} \) supported on \( N \), we may choose a test function \( f \in C^\infty_c(\mathfrak{s}) \) so that \( \mu_{\mathcal{O}}(f) = a_{\mathcal{O}} \) for all \( \mathcal{O} \subset N \). Thus using the Shalika germ expansion, we conclude that there is a small neighbourhood \( U_f \) of \( 0 \in \mathfrak{s} \) so that
\[ O(\gamma, \eta, f) = \sum_{\mathcal{O} \subset N_0} \mu_{\mathcal{O}}(f) \Gamma_{\mathcal{O}}(\gamma) = \sum_{\mathcal{O} \subset N_0} a_{\mathcal{O}} \Gamma_{\mathcal{O}}(\gamma) = 0 \]
for all \( \theta \)-regular \( \gamma \in U_f \). The set \( U_f^H \) contains an open and closed neighbourhood \( V \) of \( N \). Let \( f' = f 1_V \). Then we have that \( O(\gamma, \eta, f') = 0 \) for all \( \theta \)-regular \( \gamma \). Moreover, since \( V \) is an open and closed neighbourhood of \( N \), we have that \( \mu_{\mathcal{O}}(f) = \mu_{\mathcal{O}}(f') \) for all \( \mathcal{O} \subset N \). Now by assertion (2), the \( \theta \)-nilpotent orbital integrals \( \mu_{\mathcal{O}} \)'s all lie in the weak closure of the \( \theta \)-regular orbital integrals. Since \( O(\gamma, \eta, f') = 0 \) for \( \theta \)-regular \( \gamma \), we conclude that \( a_{\mathcal{O}} = \mu_{\mathcal{O}}(f) = \mu_{\mathcal{O}}(f') = 0 \). This proves (1). \( \square \)

The following lemma allows us to use induction.

Lemma 7.4. Let \( \gamma \in \mathfrak{s}_{\theta-\text{ss}} \). Suppose that \( \Gamma^\gamma_{\mathcal{O}} \)'s are linearly independent. Then for all \( \xi \) whose \( \theta \)-semisimple part is \( \gamma \) the orbital integral \( O(\xi, \eta, f) \) lies in the weak closure of the set of all \( \theta \)-regular orbital integrals.

Proof. Let \( N_\gamma \) be the nilpotent cone of \( \mathfrak{s}_\gamma \) and for each nilpotent orbit \( \mathcal{O} \subset N_\gamma \), we fix an element \( \xi_{\mathcal{O}} \in \mathcal{O} \). Then by the Shalika germ expansion at \( \gamma \), there is a small neighbourhood \( U_f \) of \( \gamma \) in \( \mathfrak{s}_\gamma \), so that for all \( \theta \)-regular \( \xi \in U_f \),
\[ O(\xi, \eta, f) = \sum_{\mathcal{O} \subset N_\gamma, 0} \Gamma^\gamma_{\mathcal{O}}(\xi) O(\gamma + \xi_{\mathcal{O}}, \eta, f). \]
As \( \Gamma^\gamma_{\mathcal{O}} \)'s are linearly independent and they remain linearly independent when restricted to \( U_f \), we conclude that if \( O(\xi, \eta, f) = 0 \) for all \( \theta \)-regular \( \xi \in U_f \), we have \( O(\gamma + \xi_{\mathcal{O}}, \eta, f) = 0 \) for all \( \mathcal{O} \subset N_\gamma, 0 \).
This proves the lemma. \( \square \)

We now prove the linear independence of Shalika germs and the density of \( \theta \)-regular semisimple orbital integrals simultaneously.

Theorem 7.5. The following assertions hold.

1. The Shalika germs \( \Gamma_{\mathcal{O}} \)'s, \( \mathcal{O} \subset N_0 \), are linearly independent.
2. The set of \( \theta \)-regular orbital integrals are weakly dense in \( D(\mathfrak{s})^{H, \eta} \).

Proof. We argue by induction on \( n \), i.e. the size of \( \mathfrak{s} \).

First we show that, under the inductive hypothesis, the two assertions in the proposition are equivalent. In fact, if the second assertion holds, then the first holds by Lemma 7.3. If the first assertion holds, then \( \theta \)-nilpotent orbital integrals lie in the weak closure of \( \theta \)-regular orbital
integrals. When combined with the induction hypothesis and Lemma 7.4, this implies that all orbital integrals lie in the weak closure of the \( \theta \)-regular orbital integrals. This proves that two assertions in the proposition are equivalent. We will prove the second assertion under the induction hypothesis.

Put
\[
C_1 = \{ f \in C_c^\infty(s) \mid \text{all orbital integrals of } f \text{ vanish} \};
\]
\[
C_2 = \{ f \in C_c^\infty(s) \mid \text{all } \theta\text{-regular orbital integrals of } f \text{ vanish} \}.
\]

By Lemma 7.4 and the induction hypothesis, the set \( C_2 \) consists of all functions \( f \in C_c^\infty(s) \) such that all orbital integrals, except the \( \theta \)-nilpotent orbital integrals, vanish. Thus \( C_2/C_1 \) is spanned by all \( \mu_O \)'s, \( O \subset \mathcal{N}_0 \).

By Lemma 7.1, \( C_1 \) consists of all \( f \in C_c^\infty(s) \) such that \( I(f) = 0 \) for all \( (H, \eta) \)-invariant distribution \( I \). Thus it is clear that \( C_1 \) is closed under the Fourier transform. Since the Fourier transform of \( \theta \)-regular orbital integrals are represented by \( (H, \eta) \)-invariant locally integrable functions on \( s_{\theta-\text{reg}} \) by Proposition 1.3, we conclude that \( C_2 \) is also preserved under the Fourier transform. Thus Fourier transform induces an isomorphism of \( C_2/C_1 \) onto itself. Therefore \( C_2/C_1 = 0 \) and this proves the proposition.

Corollary 7.6. The Fourier transform of \( \mu_O \) is represented by a locally integrable function in \( s \) for all \( O \subset \mathcal{N}_0 \).

Proof. We need to make use of Howe’s finiteness theorem for \( s \), established by Rader and Rallis in [RR96, Theorem 6.7]. We do not need the statement this theorem, but rather a standard consequence of it, i.e. the so-called uniformity of the germ expansion. Let \( L \subset s \) be a lattice, i.e. an open compact subgroup of \( s \). Then Howe’s finiteness theorem implies that there is a neighbourhood \( U_L \) such that the germ expansion
\[
O(\gamma, \eta, f) = \sum_{O \subset \mathcal{N}_0} \Gamma_O(\gamma) \mu_O(f)
\]
holds for all \( f \in C_c^\infty(s/L) \) and all \( \theta \)-regular semisimple \( \gamma \in U_L \).

Now let \( L \subset s \) be a lattice. There is a lattice \( L' \) in \( s \) (in fact the dual lattice of \( L \)) so that \( \hat{f} \in C_c^\infty(s/L') \) for all \( f \in C_c^\infty(L) \). Therefore there is a neighbourhood \( U_L \) of \( 0 \in s \) so that
\[
O(\gamma, \eta, \hat{f}) = \sum_{O \subset \mathcal{N}_0} \Gamma_O(\gamma) \mu_O(\hat{f})
\]
holds for all \( \theta \)-regular semisimple \( \gamma \in U_L \) and all \( f \in C_c^\infty(L) \). By Theorem 7.5 and Lemma 7.2, \( \Gamma_O \)'s, \( O \subset \mathcal{N}_0 \), when restricted to \( U_{L'} \), are linearly independent. Therefore we can choose a \( \theta \)-regular
semisimple $\gamma_\mathcal{O}$ for each $\mathcal{O} \subset \mathcal{N}_0$ so that matrix
\[(\Gamma_\mathcal{O}(\gamma_\mathcal{O}'))_{\mathcal{O}, \mathcal{O}' \subset \mathcal{N}_0}\]
is invertible. We then conclude that there are constants $c_{\mathcal{O}}$, so that
\[\mu_\mathcal{O}(\tilde{f}) = \sum_{\mathcal{O}' \subset \mathcal{N}_0} c_{\mathcal{O}} O(\gamma_\mathcal{O}', \eta, \tilde{f})\]
holds for all $f \in C^\infty_c(L)$.

By Proposition 1.3 there is a locally constant function $K_{\gamma_\mathcal{O}'}$ on $\mathfrak{s}_{\theta-reg}$ which is locally integrable on $\mathfrak{s}$ so that the distribution on $\mathfrak{s}$ given by $f \mapsto O(\gamma_\mathcal{O}', \eta, \tilde{f})$ is represented by $K_{\gamma_\mathcal{O}'}$. It follows that for all $f \in C^\infty_c(L)$ we have
\[\mu_\mathcal{O}(\tilde{f}) = \int_\mathfrak{s} f(\gamma) \left( \sum_{\mathcal{O}' \subset \mathcal{N}_0} c_{\mathcal{O}} K_{\gamma_\mathcal{O}'}(\gamma) \right) d\gamma.\]
We put $K_{\mathcal{O}, L}(\gamma) = \sum_{\mathcal{O}' \subset \mathcal{N}_0} c_{\mathcal{O}} K_{\gamma_\mathcal{O}'}(\gamma)$ for $\gamma \in \mathfrak{s}_{\theta-reg}$. This function is locally constant on $\mathfrak{s}_{\theta-reg}$ and is locally integrable on $\mathfrak{s}$.

We now choose another lattice $L_1$ so that $L \subset L_1$. Then we get another function $K_{\mathcal{O}, L'}$. We claim that $K_{\mathcal{O}, L_1}(\gamma) = K_{\mathcal{O}, L}(\gamma)$ if $\gamma \in L$ and is $\theta$-regular semisimple. In fact both functions, when restricted to $L$, represent the distribution $f \mapsto \mu_\mathcal{O}(\tilde{f})$. Then we conclude by the local constancy of them.

It follows that there is a well-defined function $K_\mathcal{O}$ on $\mathfrak{s}$, which is locally constant on $\mathfrak{s}_{\theta-reg}$ and locally integrable on $\mathfrak{s}$, so that $K_\mathcal{O}(\gamma) = K_{\mathcal{O}, L}(\gamma)$ if $L$ is a lattice in $\mathfrak{s}$ and $\gamma \in L$. It is then clear that $K_\mathcal{O}$ represents the distribution $f \mapsto \mu_\mathcal{O}(\tilde{f})$ on $\mathfrak{s}$. \[\square\]

8. DENSITY OF REGULAR SEMISIMPLE ORBITAL INTEGRALS

We explain how to establish the results on the level of $G$ in this section.

We fix an $H$-invariant neighbourhood $\omega$ of $0 \in \mathfrak{s}$ and a neighbourhood $\Omega$ of $1 \in S$ so that the exponential (rational) map $\exp : \mathfrak{s} \to \Omega$ is defined and is a diffeomorphism. Let $f \in C^\infty_c(G)$. We put $\tilde{f}(g^{-1}\theta(g)) = \int_H f(hg) dh$ and $f_\omega \in C^\infty_c(\omega)$ given by $f_\omega(\gamma) = \tilde{f}(\exp(\gamma))$. We extend $f_\omega$ to a function on $\mathfrak{s}$ via extension by zero.

We consider the $H \times H$ action on $G$ by left and right multiplication and the conjugation action of $H$ on $S$. We say that an element $x \in S$ or rather the $H$ orbit of $x$ is $\theta$-unipotent if it is nilpotent in $G$. We say that $g \in G$ is $\theta$-unipotent if $x = g^{-1}\theta(g)$ is so in $S$. Let $\mathcal{Y} \subset S$ be the variety of $\theta$-unipotent elements in $S$. By [JR96, Lemma 4.1], the exponential map induces an $H$-equivariant isomorphism $Y \to \mathcal{N}$ and thus induces a bijection on the set of $H$-orbits in $Y$ and that in $\mathcal{N}$. Let $u_1, \cdots, u_r, u_{r+1}, \cdots, u_s$ be a complete set of representatives of $\theta$-unipotent orbits in $G$. Let $\mathcal{O}_i$ be the $\theta$-nilpotent orbits in $\mathfrak{s}$ represented by $\exp(u_i^{-1}\theta(u_i))$ and we may label these $u_i$’s so that $\mathcal{O}_i$ is visible precisely when $1 \leq i \leq r$. Therefore $u_i$ represents a $\theta$-unipotent orbit in $G$ which supports a left $H$-invariant and right $(H, \eta)$-invariant distribution precisely when $1 \leq i \leq r$. We call these
\( \theta \)-unipotent elements or their orbits visible. If \( f \in C_c^\infty(G) \), we simply define \( O(u_i, \eta, f) = \mu_\mathcal{O}(f) \). This is the \( \theta \)-unipotent orbital integral on \( G \).

**Proposition 8.1.** Let \( f \in C_c^\infty(G) \). There is a neighbourhood \( U_f \subset \Omega \) of \( 1 \in S \) so that if \( g \in G \) is a \( \theta \)-regular element in \( G \) with \( g^{-1}\theta(g) \in U_f \), \( g^{-1}\theta(g) = \exp(X) \) where \( X \in \omega \), then

\[
O(g, \eta, f) = \sum_{i=1}^{r} \Gamma_{\mathcal{O}_i}(X) O(u_i, \eta, f).
\]

**Proof.** This follows from the corresponding identity on the Lie algebra. \( \square \)

**Corollary 8.2.** Let \( f \in C_c^\infty(G) \). Assume that there is a neighbourhood \( U \) of \( 1 \in S \) so that if \( g \in G \) and \( g^{-1}\theta(g) \in U \), then \( O(g, \eta, f) = 0 \). Then all unipotent orbital integrals of \( f \) vanish.

**Proof.** This follows from the Proposition and the linear independence of Shalika germs. \( \square \)

Let \( x \in S \) be \( \theta \)-semisimple. Let \( N_x \subset \mathfrak{s}_x \) be the nilpotent cone and the map \( N_x \to S_x \), \( \xi \mapsto x \exp(\xi) \) is \( H_x \)-equivariant and induces a bijection between the \( \theta \)-nilpotent orbits in \( \mathfrak{s}_x \) and the orbits in \( S_x \). We have constructed an analytic slice \((U, p, \psi)\) at \( x \) in Section 2. With the explicit construction of the analytic slice given in [Zha15, Section 5.3], in the notation of Section 2, we can take \( p^{-1}(x) \) to be a small neighbourhood of \( x \in S_x \) and identify \( N_x^S \) with \( \mathfrak{s}_x \). The map \( \xi \mapsto x \exp(\xi) \) define an \( H_x \)-equivariant homeomorphism from a neighbourhood of \( 0 \in N_x^S \) to \( p^{-1}(x) \) and we can and will take \( \psi \) to be the inverse of this map.

Let \( g \in G \) and \( x = g^{-1}\theta(g) \). Let \( x = x_s x_n = x_n x_s \) be the Jordan decomposition of \( x \) in \( G \) (with obvious notation). Then one checks readily that \( x_s, x_n \in S \). Let \( \mathcal{O} \subset N_{x_s} \) be a visible \( \theta \)-nilpotent orbit and assume that \( x_n \) is contained in the image of \( \mathcal{O} \) under the exponential map. Let \( f \in C_c^\infty(G) \) and \( \tilde{f} \in C_c^\infty(S) \). We define \( f_1 \in C^\infty(H) \) by

\[
f_1(h) = \mu_\mathcal{O}^\mathfrak{s}_s(\tilde{f}(h^{-1}(x_s \exp(\cdot))h)).
\]

As in the case of \( \mathfrak{s} \), the image of \( \text{supp} f_1 \) in \( H_{x_s} \backslash H \) is compact. We put

\[
O(g, \eta, f) = \int_{H_{x_s} \backslash H} f_1(h) \eta(\det h) dh.
\]

The same argument as in the case of \( \mathfrak{s} \) gives that

\[
O(g, \eta, f) = \mu_\mathcal{O}^\mathfrak{s}_s(\tilde{f}_{x_s}),
\]

where \( \tilde{f}_{x_s} \) is the function constructed in Proposition 2.1 from \( \tilde{f} \). As in the case of orbital integrals on \( \mathfrak{s} \), if \( x_n \) is not contained in the image of a visible \( \theta \)-nilpotent orbit, then the orbit of \( g \) does not support any distribution that is left \( H \)-invariant and right \( (H, \eta) \)-invariant.

**Theorem 8.3.** The set of \( \theta \)-regular semisimple orbital integrals is weakly dense in the set of left \( H \)-invariant and right \( (H, \eta) \)-invariant distributions on \( G \).
Proof. As in the case of invariant distributions on \( \mathfrak{s} \), we need to prove that if \( f \in C_c ^\infty (G) \), and \( O(g, \eta, f) = 0 \) for all \( \theta \)-regular semisimple \( g \in G \), then all orbital integrals of \( f \) vanish. Let \( x \in S \) be \( \theta \)-semisimple. We have a function \( \tilde{f} \in C_c ^\infty (S) \) and we let \( \tilde{f}_x \in C_c ^\infty (\mathfrak{s}_x) \) be the function constructed in Proposition 2.1. Then by Proposition 2.1, all \( \theta \)-regular semisimple orbital integrals of \( \tilde{f}_x \) near \( 0 \in \mathfrak{s}_x \) vanish. Thus by Corollary 8.2 we conclude that all \( \theta \)-nilpotent orbital integrals of \( \tilde{f}_x \) vanish. By the equality \((8.2)\), we conclude that \( O(g, \eta, f) = 0 \) if the \( \theta \)-semisimple part of \( g^{-1} \theta(g) = x \). This shows that all orbital integrals of \( f \) vanish and proves the corollary. \( \square \)

REFERENCES


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